ENCYCLOPÉDIE OU *DICTIONNAIRE* RAISONNÉ DES SCIENCES, DES ARTS ET DES MÉTIERS

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COMBINAISON

Combination, (*Mathematics*) should be said properly only of the collection of many things two by two; but one applies it in Mathematics to all the possible ways of taking a number of given quantities.

Father Mersenne has given the *combinations* of all the notes and sounds of Music to the number 64; the sum which comes of it is able to be expressed, according to him, only with 60 digits or figures.

Father Sébastien has shown in the *Mémoires de l'Academie* 1704, that two tiles split each by their diagonals into two triangles of different colors, provide 64 different arrangements of the chessboard: that which must astonish, when one considers that two figures would be able to combine themselves only in two ways. See **Carreau**.

One can make use of this remark by Father Sébastien, in order to tile some apartments.

Doctrine of combinations. A number of quantities being given with the one of the quantities which must enter into each *combination*, to find the number of *combinations*.

A single quantity, as it is evident, does not admit of *combination*; two quantities a & b give one *combination*; three quantities a, b, c, combined two by two, give three *combinations* ab, ac, bc; four of them would give six ab, ac, bc, ad, bd, cd; five of them would give ten ab, ac, bc, ad, bd, cd, ae, be, ce, de.

In general the sequence of the numbers of *combinations* is $1,\ 3,\ 6,\ 10,$ &c. that is to say the sequence of triangular numbers; thus q representing the number of the quantities to combine, $\frac{q-1}{2} \times \frac{q-0}{2}$ will be the number of their *combinations* two by two. *See* **Triangular numbers**.

If one has three quantities $a,\ b,\ c$ to combine three by three, they will provide only a single *combination* abc; if one takes a fourth quantity d, the *combinations* that these four quantities are able to have three by three, will be the four abc, abd, bcd, acd; if one takes a fifth of them, one will have ten *combinations* abc, abd, bcd, acd, abe, bde, bce, ace, ade;[1] if one takes a sixth of them, one will have twenty *combinations*, &c. So that the sequence of *combinations* three by three is that of the pyramidal numbers; & as q expressing always the number of given quantities, $\frac{q-2}{1} \times \frac{q-1}{2} \times \frac{q-0}{3}$, is the one of their *combinations* three by three.

The number of combinations four by four of the same quantities would be found in the same manner $\frac{q-3}{1} \times \frac{q-2}{2} \times \frac{q-1}{3} \times \frac{q-0}{4}$; & in general n expressing the number of letters

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[1]The combination cde is omitted.

which one wishes to introduce into each term of the combination, the quantity $\frac{q-n+1}{1} \times \frac{q-n+2}{2} \times \frac{q-n+3}{3} \times \frac{q-n+4}{4} \times \cdots \frac{q}{n}$ will express the demanded number of *combinations*.

Let one demand, for example, in how many ways six quantities are able to be taken four by four, one will put $q = 6$ & $n = 4$, & one will substitute these numbers into the preceding formula, that which will give $\frac{6-4+1}{1} \times \frac{6-4+2}{2} \times \frac{6-4+3}{3} \times \frac{6-4+4}{4} = 15$.

Corollary. If one wishes to have all the possible *combinations* of any number of letters, taken so many two by two, so many three by three, so many 4 by 4, &c. it will be necessary to add all the preceding formulas $\frac{q-1}{2} \times \frac{q-0}{2}$; $\frac{q-2}{1} \times \frac{q-1}{2} \times \frac{q-0}{3}$; $\frac{q-3}{1} \times \frac{q-2}{2} \times \frac{q-1}{3} \times \frac{q-0}{4}$; &c. that is to say that the number of all these *combinations* will be expressed by $\frac{q \times q-1}{1 \cdot 2} + \frac{q \cdot q-1 \cdot q-2}{2 \cdot 3} + \frac{q \cdot q-1 \cdot q-2 \cdot q-3}{2 \cdot 3 \cdot 4}$ &c.

If one compares presently this sequence with that which represents the raising of any binomial to the power q, one will see that by making equal to unity each of the terms of this binomial, the two sequences are the same in the two first terms near 1, & q, which is lacking in the preceding sequence. From there it follows that instead of this sequence, one can write $2^q - 1 - q$ this which give a very simple way to have all the possible *combinations* from a number q of letters. Let this number be, for example 5, one will have therefore for the total number of its *combinations* $2^5 - 5 - 1 = 32 - 6 = 26$. *See* **Binomial**.

Any number of quantities being given, to find the number of combinations *& of alternations which they are able to receive, by taking them in all possible ways.*

We suppose first that there are only two quantities a, b, one will have first ab & ba, that is to say the number 2; & as each of these quantities can also be combined with itself, one will have again aa & bb, that is to say that the number of *combinations* & alternations is in this case $2 + 2 = 4$. If there are three quantities a, b, c, & if the exponent of their variation be two, one will have three terms for their *combinations*, which are ab, bc, ac: to these three terms one will add again three others ba, cb, ca, for the alternations; & finally three others for the *combinations* aa, bb, cc, of the letters a, b, c, taken each with itself, this which will give $3 + 3 + 3 = 9$. In general it will be easy to see that if the number of the quantities is n, & if the exponent of the variation be 2, n^2 will be the one of all their *combinations* & of their alternations.

If the exponent of the variation is 3, & if one supposes first only three letters a, b, c, one will have for all the *combinations* & alternations *aaa*, *aab*, *aba*, *baa*, *abb*, *aac*, *aca*, *caa*, *abc*, *bac*, *bca*, *acb*, *cab*, *cba*, *acc*, *cac*, *cca*, *bba*, *bab*, *bbb*, *bbc*, *cbb*, *bcb*, *bcc*, *cbc*, *ccb*, *ccc*, that is to say the number 27 or 3^3.

In the same way, if the number of letters were 4, the exponent of the variation 3, 4^3 or 64, would be the number of *combinations* & alternations. And in general if the number of the letters were n, n^3 would be the one of the *combinations* & alternations for the exponent 3. Finally if the exponent is any number, m, n^m will express all the *combinations* & alternations for this exponent.

If one wishes therefore all the *combinations* & alternations of a number n of letters in all the possible varieties, it will be necessary to sum the series $n^n + n^{n-1} + n^{n-2} + n^{n-3} + n^{n-4} + n^{n-5} + n^{n-6} +$&c. until the last term which is n.

Now as all the terms of this sequence are in geometric progression, & as one has the first term n^n, the second n^{n-1}, & the last n, it follows that one will have also the sum of this progression, which will be $\frac{n^{n+1} - \mathrm{n}}{n-1}$.

Let n, for example, be equal to 4, the number of all the possible combinations & alternations will be $\frac{4^5 - 4}{4-1} = \frac{1020}{3} = 340$. Let n be 24, one will have then for all the possible *combinations* & alternations $\frac{24^{25} - 1}{24-1} = \frac{32009658644406818986777955348250600}{23} =$

1391724288887252999425128493402200; & it is this enormous number which expresses the combinations of all the letters of the alphabet among themselves.

See the *Ars Conjectandi* of Jacques Bernoulli, & *L'Analyse des jeux de hasard* of Montmort. These two authors, especially the first, have treated with great care the matter of combinations. This theory is in fact very useful in the calculus of the games of chance; & it is on it that ride all the science of probabilities. *See* **Jeu**, **Pari**, **Avantage**, **Probabilité**, **Certitude**, &c.

It is clear that the science of anagrams (see **Anagramme**) depends on that of combinations. For example, in *Roma* which is composed of four letters, there are twenty-four *combinations* (see **Alternation**); & of these twenty-four *combinations* one will find many which form Latin words, *armo*, *ramo*, *mora*, *amor*, *maro*; one finds also *omar*; likewise in Rome, one finds *more*, *omer*, &c. (*M. d'Alembert*)