VI. ON THE ANALYSIS OF GAMES

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OPUSCULES *MATHÉMATIQUES*
VOLUME IV, TWENTY-THIRD MEMOIR, PP. 79–92

VI. *On the analysis of games*

1. A very simple & very natural consideration to make in the calculus of games, & of which M. de Buffon has given me the first idea, is that the loss is always really greater than the gain that one is able to make. For let x be the sum that the Player can lose or win, & let a be the wealth of this Player; if he wins, his wealth will become $a + x$, & his real gain will be $\frac{x}{a+x}$; instead if he loses, his wealth will be no more than $a - x$, & his real loss will be $\frac{x}{a-x}$; now $\frac{x}{a-x}$ is evidently greater than $\frac{x}{a+x}$.

2. One could deduce from this many consequences. The first is that if x is the expectation of a Player, or the sum which he hopes to win, it will be necessary, in order to find his wager z or the sum which he must put into the game, to assume, not $z = x$, as one ordinarily assumes, but $\frac{x}{a+x} = \frac{z}{a-z}$; whence one deduces $z = \frac{ax}{a+2x}$.

3. The second, is that by changing nothing moreover in the ordinary formulas of the analysis of probabilities, it is necessary perhaps to divide them by the wealth of the Player diminished by the loss, or augmented by the gain; I explain myself.

4. Let, for example, p be the number of cases which win the sum y, q the number of cases which lose the sum x, a the wealth of the Player, is it not necessary to express his expectation (by admitting besides the ordinary formulas of the probabilities) by $\frac{p\times x}{(a+x)(p+q)} + \frac{q\times -y}{(a-y)(p+q)}$? This expression seems to me, I swear it, more correct & more accommodated to the genuine use of the Players, than the one which one uses ordinarily. However it is not necessary, it seems to me, to take this expression for the bet of the Player, or for the sum which he must put into the game before the match; because, following this formula, in order that the loss which he fears was equal to the gain which he hopes, it would be necessary that $\frac{px}{a+x} - \frac{qy}{a-y}$ was $= 0$. Now if $p = q$ for example, & $y = \frac{a}{2}$, it is evident that this is impossible; a consequence which seems at first shocking, but which studying thoroughly, seems very natural; because it is totally simple that a man who will have, for example, 100000 écus of wealth, & who will risk to lose or to win 50000 écus, will be much more damaged if he loses, than be enriched if he wins, since in the first case, he will impoverish himself by the half; & that in the second case, he will enrich himself only by a third.

5. Therefore here is, if I do not fool myself, how one can find in this case the wager of the Player, which I call z; one will consider that after having wagered this sum z, he will win really only $x - z$, & that his wealth will be for then $a + x - z$; & that on the contrary if he loses the sum y, that is to say if after the match he is obligated to give to the other Player this sum, his loss will be really $y + z$, & his wealth will be only $a - y - z$; therefore in order to find the wager z of this Player, it will be necessary that $\frac{p(x-z)}{(a-z+x)(p+q)} +$

Date: 1768.
Translated by Richard J. Pulskamp, Department of Mathematics and Computer Science, Xavier University, Cincinnati, OH .

$\frac{q(-y-z)}{(a-y-z)(p+q)} = 0$; & that which proves the goodness of this formula, or at least its analogy with the received formulas, is that by setting aside the wealth of the Player, one could find $\frac{p(x-z)}{p+q} + \frac{q(-y-z)}{p+q} = 0$, or $z = \frac{px-qy}{p+q}$, conforming to the ordinary formulas.

6. But, I tell you the truth, I do not see with greatest evidence that this last formula, even by setting aside the wealth of the Player, represents necessarily the sum z which he must put into the game; because why would I not make the following reasoning which seems plausible enough? If the Player A who has put z into the game, wins, he will receive from Player B the sum x, & consequently will win $x - z$; if he loses, he must, beyond the sum z which belongs already to Player B, give to him the sum $y-z$ in order to complete the sum y which Player B must have in this case here; therefore Player A will disburse & lose really, not the sum $y + z$, but the sum $z + y - z = y$, thus it is necessary to make $\frac{p(x-z)-qy}{p+q} = 0$, & not $\frac{p(x-z)-q(y+z)}{p+q} = 0$, this which will give $z = \frac{px-qy}{p}$. I do not pretend to give this formula for good; I say only that one can also bring some special reasons in its favor.

7. One can make, I believe, on the ordinary calculus of probabilities & the analysis of games of chance, many other reflections, which I will suggest how from simple doubts, & in the order nearly where they are being presented in my mind.

8. It seems to me first that all the ideas, of *expectation*, of *wager*, of sum which he must give for playing at par, are not very easy to fix in a precise manner.

9. The difficulty comes, if I do not fool myself, from this that the idea of *expectation* encloses two things; the sum which one expects, & the probability that one will win this sum. Now it seems to me that it is *principally the probability* which must rule the expectation; & the expected sum must enter, if I can speak in that manner, only in a manner subordinated to the degree of probability: yet one makes them both enter equally & in the same manner into the calculation.

10. I know not (in consequence of this reflection) if the expectation is well estimated in general, by multiplying the sum to expect by the probability. If one proposes to choose among 100 combinations, of which 99 will win one thousand écus, & the one-hundredth 99 thousand écus; what man will be so foolish to prefer the one which will give 99 thousand écus? The *expectation* in the two cases in not therefore *really* the same, although it is the same following the rules of the probabilities.

11. Does this not prove that it is principally the probability, much more than the expected sum, which constitutes the expectation? Because whatever the expected sum be, what is the expectation if the probability is very small?

12. In the analysis of chances, one regards certitude as 1, & the probability as a fraction of the certitude; is this assumption quite correct in all regards? Because one thousand probabilities will never be a certitude. Moreover, if it is a *certitude* that one will win 500 livres & a *probability* $\frac{1}{2}$ that one will win 1000 livres will one say that the two cases are the same?

13. We suppose that a Player of dice A proposes to a Player B to give him 3 livres, whatever face of the die which comes, the expectation of the Player B will be 3 livres by the rules of probability; he must, in order to play at par, give this sum to Player A; & after the game, neither the one nor the other will have lost; as before the game neither the one nor the other risks anything. But we suppose that the Player A proposes to Player B to give him 18 livres if he brings about 6; the expectation, & consequently the wager of Player B must be 3 livres as in the preceding case. However in this last case he risks, because he is able to lose 3 livres which he has put in the game, & in the first he evidently risks nothing: Can one say again one time that the two cases are the same? One will respond perhaps that in the first case he will neither win nor lose, & in the second he can win; this proves only

that which I advance, that the two cases are different; whence it follows, it seems to me, that these two cases should not be represented by the same formula.

14. *Pierre* says to *Jacques*; we are going to play to heads or to tails on 100 tosses; if I bring heads only at the 100th toss, I will give you 2^{100} écus; if I bring it before, I will give you nothing. One finds that *Jacques* must for his *wager* give an écu to *Pierre*; assuredly *Pierre* would enrich himself playing this game everyday; & there is no person who would not make this bargain; can one therefore believe that when *Jacques* has given or put in the game his écu, his lot becomes equal to that of *Pierre*?

15. Would a man, says Pascal, pass for a fool, if he hesitated to let himself be given death in the case which with three dice one made three sixes twenty times in sequence, or being Emperor if one missed it? I think absolutely as him; but why would this man pass for a *fool*, if the case in question, is *physically* possible? It is necessary therefore to say that he is not; although it must be possible *mathematically*. See for this the second volume of my *Opuscules Mathématiques*, & the fifth of my *Mélanges de Philosophie*.

16. If a Player who I call A has in one hand m pieces, & n in the other, & if he plays against two men who I call B & C, to the one of which he must give that which he has in one hand, & to the other that which he has in the other, it is evident, one says, that this Player A loses $m+n$; therefore the two Players B, C, must give to him $m+n$ in order to play in an equal game; therefore each must give to him $\frac{m+n}{2}$; therefore if he played only against one alone, B or C, this sole Player must give to him $\frac{m+n}{2}$; here is my difficulty on this solution.

1° It is necessary that averaging the wager of both sides, the lot of the Players is equal. Now in the case where there is a Player A against the Players B & C, the Player A will lose absolutely nothing; the one of the two other Players will lose & the other will win; thus there is no equality of lot among the three.

2° In the case where there are two Players B, C against A, the Player A loses nothing; in the case where there is only a Player B or C, who gives him simply $\frac{m+n}{2}$, he is able to lose or to win: the lot of Player A is therefore not the same in the two cases. Therefore the case of the Player A who plays against B & C, or of a Player A who plays against a sole Player B, is not the same; consequently one is not authorized to conclude from the one to the other. In a word, in the first case, as soon as each Player has set a wager, the expectation & the fear of each is null; in the second, the expectation of each is some thing, & the fear is something else also; therefore this is not the same case.

17. In order to know what is the advantage in a Lottery, one supposes ordinarily that one of the interested parties takes all the lottery to himself; under this hypothesis one finds easily the risk which he courts, & one takes this risk for the one of each of the interested parties; it seems to me that this method of estimating the risk is not good; because the lottery is advantageous or disadvantageous to the interested parties, the gain or the loss of the one who takes all the lottery is *certain*, & on the contrary the gain or the loss of the one who takes only a part of the Lottery is *doubtful*; one could not therefore regard the two cases as being the same.

18. M. de Buffon, as I have remarked above elsewhere, estimates differently from other Authors, the probability of the duration of life. If of m persons of the same age, there are dead $\frac{m}{2}$ of them by the end of p years, then there is, he says, odds of one against one for each, that by the end of p years, he will be dead or alive; therefore his expectation of living is p years. This reasoning, although different from those on which one establishes ordinarily this probability, is assuredly very simple & very plausible; now can I not say the same? If there is a Lottery where the half of the tickets carry 20 sols & beyond, the

other half carrying what one will wish, &, if one wishes, nothing at all, there are odds of one against one that the one who will set to this lottery, will win 20 sols; therefore the expectation of that one who has set to the lottery, will be 20 sols. This consequence appears very natural; however this manner of estimating the *expectation* would be quite different from that which the ordinary rule of probabilities could give. Because, according to this rule, it is not sufficient to know in bulk that the half of the tickets carry 20 sols & above, in order to fix the expectation at 20 sols: it is necessary to know what each ticket must give in particular, & to divide the sum of all these sums by the sum of the tickets, which can make much more or much less than 20 sols.

19. If the case, already so much cited, of the Memoirs of Petersburg, where one finds *infinity* for the *wager* in a game of heads & tails, demands a particular solution, different from that which gives the result of the ordinary rules of probability, why does this result give in the other cases some solutions that all the Mathematicians have admitted until here without restriction? Is this not proof that these rules have need of being modified in certain regards?

20. Mr. Daniel Bernoulli says, in the Memoirs of Petersburg, Book V, that in the game in question, there is no person who would give his expectation for twenty écus paid once. Now the twenty écus & quite beyond must be paid to the adversary if *heads* happens only on the sixth toss or beyond. Thus to give his expectation for twenty écus, is this not to suppose tacitly that *tails* will not happen six times in sequence? However this assumption could be too hazardous, & I do not demand so much; I wish only that one accords to me that *tails* not happen (physically speaking) a great number of times in sequence.

21. It is at least certain that to give his expectation for twenty écus, it is supposed tacitly that *heads* will happen infallibly before the fortieth toss; because in supposing that *heads* must happen infallibly by the fortieth toss at the latest, the expectation, according to the ordinary formulas, would be twenty écus; & I much wish in total rigor to hold to this assumption, that *heads* will happen certainly before the fortieth toss; although perhaps it is true to say, (always *physically* speaking) that *heads* will happen often enough.

22. Is it by the probability or by a power of the probability (greater than unity) which it is necessary to multiply the expected sum, in order to have the wager, especially when the probability is small? I ask you to ponder anew the reflections which I have already made above on this subject[1], & from which does there results that when the probability is very small, the power in question, seems to must be greater than unity, at least in the case where the same event is supposed to happen a very great number of times in sequence?

23. According to Mr. Daniel Bernoulli, the odds are nearly 1500000 against 1 that the six Planets, abandoned at random, could not move themselves into one same small zone as that where they themselves move. If they would find themselves in the same plane, Mr. Bernoulli, as I have already remarked elsewhere (See Book V of my *Mélanges de Philosophie*) would find odds of infinity against one, & he would conclude that this arrangement would not be the effect of chance. However, to speak Mathematically, this case is entirely as possible as some other as this is in particular. Why therefore, again one time, distinguish it from the others. It is, one will say, that this uniformity announces a cause. I wish him well: in this case, I will say likewise the uniformity of *heads* happening one hundred times in sequence, announces also a cause, & that consequently if one does not suppose another cause, *heads* would not happen one hundred times in sequence.

24. In general, if it is true that all singular uniformity of events announces a cause, as soon as one will not suppose cause, one must not suppose extraordinary uniformity;

[1] See article V of the present memoir.

therefore all the cases which contain a constant uniformity & out of natural order, must not be regarded as *physically* possible cases.

25. It is so true, that a Player who would have seen *heads* happen one hundred times in sequence, will wager for *heads* on the one hundred first; because it is not probable, & it is perhaps impossible that *heads* happen one hundred times in sequence without some particular cause; & this will be probable because the side *tails* is most heavy, & must be found beneath it.

26. But, one will say, you agree with yourself. You claim here that *heads* will happen on the one-hundred-first toss if it has already happened one hundred times, & in your *Mémoire sur les Probabilities*, Book II of your *Opuscules*, you say that the odds are that *tails* will happen, when *heads* has happened many times in sequence. My response is that it is necessary to distinguish here the different cases; here is my well developed reasoning. If chance alone decided the event, *heads* cannot happen, according to me, a great number of times in sequence; this to me appears proved by the reasons that I have given above & elsewhere. Therefore, if *heads* happens a great number of times in sequence, for example, one hundred times, it is a mark that there is some particular cause which brings about *heads* preferably to *tails*; one thinks therefore that this cause subsists, *heads* will return on the one hundred first toss; but if there is no other supposed cause than pure chance, it is physically impossible that *heads* happen one hundred times, or a very great number of times in sequence. But it is not impossible that it happen in sequence a small number of times. Therefore when *heads* will happen a small enough number of times in sequence, the odds are for *tails* the following toss.

27. One will say without doubt again; that if *heads* has happened two times, why not three? If three, why not four, & thus to infinity?[2]

1° With parallel reasoning, one could prove quite some absurdities. One could say, for example: if it is indifferent to me to lose two sols, why would it not be to lose three sols; if three, why not four; if four, why not five? And thus one would go until a million. One could be able to say likewise: it is very nearly equal to die in one hour than in two, in two than in three, in three than in four, &c. where is it necessary to stop?

2° Let one take a great number of terms of this sequence $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c. the sum can be computed $= 1$; now I ask how many terms it will be necessary to take in order that one can make this assumption? One could make likewise an infinity of other similar questions, on which the calculus cannot take hold; because it could never be determined in a precise & rigorous manner the moral things.

3° The direct response to the objection, is that all the neighboring cases & as infinitely nearby as one compares, are not rigorously the same; there are between each a small difference which is accumulated & becomes sensible after a certain term.

4° Besides when *heads* has happened a certain number of times, one says not that *heads* cannot absolutely happen on the following toss, one says only that it is more probable that *tails* will happen.

28. But, will one say in the end, what is the term where the probability begins to become null? I know nothing of it & it is perhaps one question that the calculus would not resolve. It suffices to me to have exposed the doubts (well founded, it seems to me) that one is able to have on the ordinary theory of probabilities; doubts that I will not push longer as for the present, & which seem to me are not able to be rendered very sensible to the Geometers, in order that they are devoted either to remove them, or to reform the theory after these doubts.

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[2]*Translator's note*: D'Alembert now introduces the paradox of the heap.