THE SOLUTION OF A CURIOUS QUESTION IN THE SCIENCE OF COMBINATIONS

Leonhard Euler

Presented to the assembly 18 October 1779
Memoirs of the academy of sciences of St. Petersburg 3 (1809/10), 1811, p. 57–64

1. The question, which explanation I take up here, is revealed thus: Given a sequence of letters however many a, b, c, d, e etc., let the number of which be n, to find in how many ways the order of them can be changed, so that none of them are found in the same position, which it occupied in the beginning.

This is manifest immediately, if the last condition is omitted and exactly the number of all permutations clearly is sought, it will be the product of all numbers from unity up to n. Moreover, this ought to exclude all the same order, where any letter was occupying its initial position, whence the number of permutations, which I seek, is less than $1 \cdot 2 \cdot 3 \cdots n$.

2. While we should seek the solution of this question, we consider first the simplest case, from which method next we collect the solution for a large number of letters as many as you wish to be drawn. And first indeed, if only the letter a be put forward, it is evident to have no variant position. To put forward two letters ab it has one variation in position, certainly ba. Moreover for three letters abc they are able to give so many as two variations, which are

$$bca, \quad cab.$$

While if four letters $abcd$ be given, there occur three cases, in which either b or c or d obtain the first position; the cases therefore, in which b is located in the first position, admit three variations of the remaining three, which are adc, dac, cda; therefore the same number besides they will be having variations, if all the same the letters c as d are assigned to the first position, and thus they have in all nine variations of position, which are:

$$\begin{array}{ccc} badc & cadb & dabc \\ dbac & cdab & dcab \\ bcda & cdba & dcba \end{array}$$

3. We explain in the same manner the case of five letters $abcde$, where the first position is able to hold either b or c or d or e. Therefore b would occupy the first position, but we dedicate the second position to the letter a and to the remaining three, c, d and e, they admit two variations, which are $badec$, $baecd$. Also since to this a the third seat is assigned, they admit three variations of the remaining three, which therefore we represent $bcaed$, $bdaec$, $beacd$. In like manner if to the letter a is assigned the fourth seat, now too they have three variations of position, which are $bedac$, $bcead$, $bdeac$. Finally if to the letter a is assigned the fifth position, there are three sequential variations: $bcdea$, $bdeca$, $bedca$. Therefore while to the letter b is given the first position, in all it will furnish eleven variations; for indeed just as many occur, if either c or d or e are placed in the first position. Whence we conclude to have in all forty-four or 44 variations of position for the five letters $abcde$.

4. But if however in similar manner we wish to advance to more letters, enumeration of all cases would become too difficult and laborious, moreover hazardous; whence in a certain method to us it will be investigated, because of which the number of variations would be able to be assigned always accurately, let the number of letters be as many as is wished.

To this end it will help greatly to name a proper character to assist, which multitude indicates all variations for whatever number of proposed letters. Therefore let this character

$$\Pi : n$$

be the number of all variations, which n letters admit, and since the cases, in which n is either 1 or 2 or 3 or 4 or 5, we already obtain, now since we know to be

$$\Pi : 1 = 0, \quad \Pi : 2 = 1, \quad \Pi : 3 = 2, \quad \Pi : 4 = 9 \quad \text{and} \quad \Pi : 5 = 44;$$

whence it opens farther in progression the number of variations soon to extend to an immense degree.

5. Now we seek to this appointed character immediately in kind the number of all variations for letters in number $= n$, which therefore will be $\Pi : n$; where the entire work reverts back, as it is investigated, in what way such required number is constructed from the preceding, which are

$$\Pi : (n-1), \quad \Pi : (n-2), \quad \Pi : (n-3) \quad \text{etc.}$$

Indeed we let be instituted reasoning in a similar way, of which previously I made use. At first certainly we will consider the cases, where the letter b is located in the first position; for it is easy to understand, how many variations for this case will be produced, and just as many will be produced, if another different letter is located in the first position; whence we understand, whatever number of variations will be found, as long as the letter b obtains the first position, multiply by $n-1$ the present being the number of all possible variations therefore the value of the character $\Pi : n$.

6. However here meet two cases to be disclosed, to be used the letter a either it holds the second position or some other. Therefore we let a be established in the second position and it will be investigated, how many variations the remaining letters c, d, e, f etc. would admit; of which since there are $n-2$, the number of variations by hypothesis will be $\Pi : (n-2)$. I let hereafter a be in the third position or some different position and now there arises the question, how many variations the letters b, c, d, e, f etc. admit; where it is noted in the variations of them the letter b is not able to occur further, which now obtains the first position, but in the position of it in the variations to be entered the letter a; and thus it will in the same way, and if in rejecting the first position the variations of the letters a, c, d, e, f etc. was sought; since the number of which is $n-1$, the multitude of all variations by hypothesis will be $\Pi : (n-1)$. Consequently, as long as the letter b is constituted in the first position, the number of all variations will be

$$\Pi : (n-2) + \Pi : (n-1).$$

7. Now by them it is manifest each produces just as many variations, if any letter whatever is written in the first position; whereby since all of these letters, the first a

excluded, the number is $n-1$, the number of all variations will be clearly

$$(n-1)\Pi:(n-2)+(n-1)\Pi:(n-1),$$

which therefore is the value of the sought formula $\Pi:n$, thus while there is

$$\Pi:n=(n-1)\Pi:(n-1)+(n-1)\Pi:(n-2)$$

or

$$\Pi:n=(n-1)(\Pi:(n-1)+\Pi:(n-2)).$$

And if the two characters immediately preceding the highest, certainly

$$\Pi:(n-1)+\Pi:(n-2),$$

multiplied by $n-1$, always will give the following character $\Pi:n$, of which help the progression, how the numbers of variations are constituted individually for the numbers of letters, until it will be redeemed, it is able to be continued easily.

8. Because therefore it appears easily, we begin with the easiest cases and we exhibit the values of the character $\Pi:n$ in the following table:

$$\begin{aligned}
\Pi:3&=2(\Pi:2+\Pi:1)=2\cdot(1+0)=2,\\
\Pi:4&=3(\Pi:3+\Pi:2)=3\cdot(2+1)=9,\\
\Pi:5&=4(\Pi:4+\Pi:3)=4\cdot(9+2)=44,\\
\Pi:6&=5(\Pi:5+\Pi:4)=5\cdot(44+9)=265,\\
\Pi:7&=6(\Pi:6+\Pi:5)=6\cdot(265+44)=1854,\\
\Pi:8&=7(\Pi:7+\Pi:6)=7\cdot(1854+265)=14833,\\
\Pi:9&=8(\Pi:8+\Pi:7)=8\cdot(14833+1854)=133496,\\
\Pi:10&=9(\Pi:9+\Pi:8)=9\cdot(133496+14833)=1334961.
\end{aligned}$$

9. We set let these numbers $\Pi:n$ in order, related to its indices n, in the following sequence:

n	1,	2,	3,	4,	5,	6,	7,	8,	9,
$\Pi:n$	0,	1,	2,	9,	44,	265,	1854,	14833,	133496.

And if now we would consider this sequence more carefully, we will discover an exceptional relation, which refers whatever number to the preceding, to the extent that the following table makes clear:

$$\begin{aligned}
2&=3\cdot 1 && -1,\\
9&=4\cdot 2 && +1,\\
44&=5\cdot 9 && -1,\\
265&=6\cdot 44 && +1,\\
1854&=7\cdot 265 && -1,\\
14833&=8\cdot 1854 && +1,\\
133496&=9\cdot 14833 && -1,
\end{aligned}$$

etc.

Therefore of which observations to greatly benefit our progression it is allowed to continue more easily, while whatever end always is a certain multiple of the preceding, either greater or lesser by unity; and so in general it will be

$$\varPi : n + n\varPi : (n-1) \pm 1.$$

Where it is observed the $+$ sign to prevail, if n is an even number, indeed the $-$ sign, when n is an odd number.

10. The surprising will be seen, in what way of these pairs the laws of progressions between them hold together; while from this latter law the prior is derived easily. For putting

$$\varPi : n = n\varPi : (n-1) \pm 1$$

still there will be in a similar way

$$\varPi : (n-1) = (n-1)\varPi : (n-2) \mp 1.$$

Let these two formulas be added, while the ambiguous signs $+$ or $-$ mutually destroy, and the sum will be

$$\varPi : n + \varPi : (n-1) = n\varPi : (n-1) + (n-1)\varPi : (n-2),$$

whence it follows

$$\varPi : n = (n-1)\varPi : (n-1) + (n-1)\varPi : (n-2),$$

which is the prior law of progression itself.

In truth nonetheless it is not easy to derive the latter law from the former; however still the thing will succeed, if certainly we begin in the simplest case, in observing, because $\varPi : 1 = 0$ and $\varPi : 2 = 1$. For hence there will be

$$\varPi : 3 = 2\varPi : 2 = 3\varPi : 2 - 1,$$

whence is

$$3\varPi : 2 = \varPi : 3 + 1.$$

Now since from the prior law there is $\varPi : 4 = 3\varPi : 3 + 3\varPi : 2$, if here in the position $3\varPi : 2$ is substituted the value in the manner invented, there will appear

$$\varPi : 4 = 4\varPi : 3 + 1,$$

whence is

$$4\varPi : 3 = \varPi : 4 - 1.$$

Now the reported sequence will be $\varPi : 5 = 4\varPi : 4 + 4\varPi : 3$; where if in the position the value $4\varPi :\ 3$ is written in the invented manner, it will produce

$$\varPi : 5 = 5\varPi : 4 - 1$$

and therefore

$$5\Pi : 4 = \Pi : 5 + 1.$$

While the reported sequence will be $\Pi : 6 = 5\Pi : 5 + 5\Pi : 4$; in which if in the position of the latter value there is substituted the previously invented, there will be

$$\Pi : 6 = 6\Pi : 5 + 1$$

and therefore

$$6\Pi : 5 = \Pi : 6 - 1,$$

which value in the following relation $\Pi : 7 = 6\Pi : 6 + 6\Pi : 5$ substituting it presents

$$\Pi : 7 = 7\Pi : 6 - 1$$

and therefore further; whence it is clear enough, in what way the latter law is derived from the prior.