“A Problem concerning Chance in Play”

Leonhard Euler∗

Adversaria mathematica† H4, 1740–1750?, pp. 186–187

A has a coins, but B b coins. It is being thrown with dice, and as often as the throw occurs, of which the probability of appearing is \( \frac{1}{m} \), then B surrenders a coin to A; but as often as the different throw happens, of which the probability of appearing is \( \frac{1}{n} \), then A surrenders a coin to B. With this condition they contend so long, until they give up all coins to one or the other, who then gains 1 deposit. It is demanded before the contest begins the expectation of both of them.

**Answer**

The expectation of A will be

\[
\frac{n^b (m^a - n^a)}{m^a+b - n^a+b}
\]

The expectation of B will be

\[
\frac{m^a (m^b - n^b)}{m^a+b - n^a+b}
\]

Therefore the chance of A : the chance of B will be as \( n^b (m^a - n^a) : m^a (m^b - n^b) \).

**Solution**

With regard to the position of A the expectation of the received coin to the expectation of the lost coin is as \( n \) to \( m \). This contest is continuing,

<table>
<thead>
<tr>
<th>as long as A has coins</th>
<th>his expectation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>( \alpha = \frac{n \beta + m \theta}{m + n} )</td>
</tr>
<tr>
<td>2</td>
<td>( \beta = \frac{n \gamma + m \alpha}{m + n} )</td>
</tr>
<tr>
<td>3</td>
<td>( \gamma = \frac{n \delta + m \beta}{m + n} )</td>
</tr>
<tr>
<td>4</td>
<td>( \delta = \frac{n \epsilon + m \gamma}{m + n} )</td>
</tr>
<tr>
<td>5</td>
<td>( \epsilon )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>( a + b )</td>
<td>1</td>
</tr>
</tbody>
</table>

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∗Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. November 22, 2009

†Mathematical notebook
In this sequence three repeating terms are X, Y, Z. There will be
\[ nZ = (m + n)Y - mX \]
and there will be exactly the recurrent sequence; to which let be set
\[ nZZ = (m + n) - m, \]
to be found. Which roots are
\[ Z = 1 \text{ and } z = \frac{m}{n}; \]
therefore the term will be the answer to general index \( x \)
\[ = A + B \left( \frac{m}{n} \right)^x; \]
Let \( x = 0 \) be put; there will be
\[ A + B = 0 \]
Let \( x = a + b \) be put; there will be
\[ A + B \left( \frac{m}{n} \right)^{a+b} = 1 \]
and hence
\[ A = \frac{1}{1 - \left( \frac{m}{n} \right)^{a+b}} = \frac{-n^{a+b}}{m^{a+b} - n^{a+b}} \]
and
\[ B = \frac{n^{a+b}}{m^{a+b} - n^{a+b}} \]
Wherefore as long as A has \( x \) coins, his lot will be
\[ \frac{n^{a+b}}{m^{a+b} - n^{a+b}} \left( \frac{m^x - n^x}{n^x} \right), \]
or the lot of A will be
\[ \frac{n^{a+b - x}(m^x - n^x)}{m^{a+b} - n^{a+b}}. \]
Wherefore from the beginning, provided that A has \( a \) coins, his expectation will be
\[ \frac{n^b(m^a - n^a)}{m^{a+b} - n^{a+b}} \]
Certainly the expectation of B will be
\[ = 1 - \frac{n^b(m^a - n^a)}{m^{a+b} - n^{a+b}} = \frac{m^{a+b} - m^a n^b}{m^{a+b} - n^{a+b}} \]
Q.E.I.
Example

A cast with two dice favorable to A IX, 4 ways,
A cast with two dice favorable to B VII, 6 ways,
therefore $m : n = 3 : 2$. Let $b = 2$, there will be

$$\text{lot of A : lot of B} = 2^2(3^a - 2^a) : 3^a(3^2 - 2^2) = (4 \cdot 3^a - 4 \cdot 2^a) : 5 \cdot 3^a$$

If $b = 2$ and $a = \infty$, there will be

$$\text{chance of A : chance of B} = 4 : 5$$

Therefore, even if A assumes countless coins, nevertheless the lot of B will be better.

Scholion

In order that the expectations of the two are equal, there must be

$$m^a n^b - n^{a+b} = m^{a+b} - m^a n^b$$
or

$$2m^a n^b = m^{a+b} + n^{a+b};$$

and if the ratio $m : n$ is given, there will be

$$a = \frac{b \ln n - \ln(2n^b - m^b)}{lm - ln}$$

Let $m = 9, n = 5, b = 12$; there will be

$$a = \frac{12 \ln 5 - \ln(2 \cdot 5^{12} - 9^{12})}{\ln 9 - \ln 5}$$

Therefore it is understood, in order that the equality of the lots is able to appear, it is necessary, that there be

$$2n^b > m^b \text{ or } 2 \left(\frac{m}{n}\right)^b > 1$$

that is

$$l2 - b \ln \frac{m}{n} > 0$$

and therefore

$$b < \frac{\ln 2}{\ln m - \ln n}$$