Notes to accompany the paper of Fontaine*

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Pierre promises to pay Paul 1 écu if he brings forth heads on the toss of a coin, 2 écus if he brings forth heads only on the second toss of a coin, and in general, 2^{n-1} écus if he brings forth heads only on the nth toss of the coin. The problem is to determine the stake of Paul which makes the game fair when Pierre has finite resources.

Let x be the wealth of Pierre and let y be the stake of Paul. It is assumed that Pierre puts all of his wealth into the game and therefore the total funds in the game is $x + y$.

Fontaine notes that the payoff must be bounded by the total amount in play. Indeed, for this reason there exists an integer N such that $2^N \leq x+y < 2^{N+1}$. Now the payout of 2^N will occur if the first incidence of heads occurs on the $N + 1$st trial.

If $n \leq N + 1$, the payout is 2^{n-1} with probability $\frac{1}{2^n}$. Otherwise, for $n > N + 1$, the payout is $x + y$ with probability $\frac{1}{2^n}$. Therefore the expected payout of Paul is

$$\sum_{n=1}^{N+1} 2^{n-1} \cdot \frac{1}{2^n} + (x+y) \sum_{n=N+2}^{\infty} \frac{1}{2^n}$$

Simplifying, we obtain

$$\frac{N+1}{2} + \frac{x+y}{2^{N+1}}$$

In a fair game this expectation must equal the stake of Pierre. Therefore we must have

$$y = \frac{N+1}{2} + \frac{x+y}{2^{N+1}}$$

or

$$y = \frac{(N+1)2^N + x}{2^{N+1} - 1}.$$

Using the condition that $y < 2^{N+1} - x$, we have

$$x < \frac{2^{N+2} - N - 3}{2}.$$

Similarly, using the condition that $y \geq 2^N - x$, we have

$$x \geq \frac{2^{N+1} - N - 2}{2}$$

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Given x, the resources of Pierre, we can determine N since

$$\frac{2^{N+1}-N-2}{2} \leq x < \frac{2^{N+2}-N-3}{2}$$

and consequently given now both N and x, y is determined.

Fontaine's example: If $x = 1,000,000$ livres, then $x = 333,333\frac{1}{3}$ écus. With $N = 18$, we see that the inequality on x is satisfied

$$262,134 \leq 333,333\frac{1}{3} < 524,277.5$$

It follows that $y = \frac{(18+1)2^{18}+333,333\frac{1}{3}}{2^{18+1}-1} = 10.13580221$ écus or 30.40740663 livres.

On the other hand, using the same kind of reasoning as above, we find that

$$2y - 3 < N \leq 2y - 2$$

and

$$x = (2^{N+1} - 1)y - (N+1)2^N$$

Contrary to what Fontaine says, if y is a given integer, then $N = 2y - 2$ and we must have

$$x = 4^{y-1} - y$$

Consequently, if Paul is willing to wager 5, then Pierre must have 251 in assets and the game must terminate at no more than the 9th trial.

Given N we have the following table providing ranges of x and y.

N	$\frac{2^{N+1}-N-2}{2} \leq x < \frac{2^{N+2}-N-3}{2}$	y
1	$0.5 \leq x < 2.0$	$1.5 \leq y < 2.0$
2	$2.0 \leq x < 5.5$	$2.0 \leq y < 2.5$
3	$5.5 \leq x < 13.0$	$2.5 \leq y < 3.0$
4	$13.0 \leq x < 28.5$	$3.0 \leq y < 3.5$
5	$28.5 \leq x < 60.0$	$3.5 \leq y < 4.0$
6	$60.0 \leq x < 123.5$	$4.0 \leq y < 4.5$
7	$123.5 \leq x < 251.0$	$4.5 \leq y < 5.0$
8	$251.0 \leq x < 506.5$	$5.0 \leq y < 5.5$
9	$506.5 \leq x < 1018.0$	$5.5 \leq y < 6.0$
10	$1018.0 \leq x < 2041.5$	$6.0 \leq y < 6.5$