DEUXIME SUPPLÉMENT.
APPLICATION DU CALCUL DES PROBABILITÉS AUX OPÉRATIONS GÉODÉSIQUES

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One determines the length of a great arc, on the surface of the Earth, by a chain of triangles which are supported on a base measured with exactitude. But, whatever precision that one brings into the measure of the angles, their inevitable errors can, by accumulating, deviate sensibly from the truth the value of the arc that one has concluded from a great number of triangles. One knows therefore only imperfectly this value, if one is not able to assign the probability that its error is contained within some given limits. The desire to extend the application of the Calculus of Probabilities to natural Philosophy has made me seek the formulas proper to this object.

This application consists in drawing from the observations the most probable results and to determine the probability of the errors of which they are always susceptible. When, these results being known very nearly, one wishes to correct them from a great number of observations, the problem is reduced to determine the probability of one or many linear functions of the partial errors of the observations, the law of probability of these errors being supposed known. I have given, in Book II of my *Théorie analytique des Probabilités*, a method and some general formulas for this object, and I have applied them, in the first Supplement, to some interesting points of the System of the world. In questions of Astronomy, each observation furnishes, in order to correct the elements, an equation of condition: when these equations are very manifold, my formulas give, at the same time, the most advantageous corrections and the probability that the errors, after these corrections, will be contained within some assigned limits, whatever be moreover the law of probability of the errors of each observation. It is so much the more necessary to render itself independent of this law, as the simplest laws are always infinitely less probable, seeing the infinite number of those which are able to exist in nature. But the unknown law which the observations follow of which one makes use introduces into the formulas an indeterminate which would permit not at all to reduce them in numbers, if one did not succeed to eliminate it. This is that which I have done, by means of the sum of the squares of the remainders, when one has substituted, into each equation of condition, the most probable corrections. The geodesic questions offering not at all similar equations, it was necessary to seek another means to eliminate the formulas of undetermined probability dependent on the law of probability

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of the errors of each partial observation. The quantity by which the sum of the angles of each observed triangle surpasses two right angles plus the spherical excess has furnished me this means, and I have replaced by the sum of the squares of these quantities the sum of the squares of the remainders of the equations of condition. Thence, one is able to determine numerically the probability that the final result of a long sequence of geodesic operations does not exceed a given quantity. By applying these formulas to the measure of a perpendicular to the meridian, they will make estimate the errors, not only of the total arc, but also of the difference in longitude of its extreme points, concluded from the chain of triangles which unite them and from the azimuths of the first and of the last side of this chain. If one diminishes, as much as it is possible, the number of triangles and if one gives a great precision to the measure of their angles, two advantages that the use of the repetitive circle and of the reflectors procure, this way to have the difference in longitude of the extreme points of the perpendicular will be one of the better of which one is able to make use.

In order to be assured of the exactitude of a great arc which is supported on a base measured toward one of its extremities, one measures a second base toward the other extremity, and one concluded from one of these bases the length of the other. If the length thus calculated deviates very little from observation, there is everywhere to believe that the chain of triangles is quite nearly exact, thus as the value of the great arc which results from it. One corrects next this value, by modifying the angles of the triangles, in a manner that the bases calculated accord themselves with the measured bases, this which is able to be made in an infinity of ways. Those that one has until the present employed are based on some vague and uncertain considerations. The methods exposed in Book II lead to some very simple formulas, in order to have directly the correction of the total arc which results from the measures of many bases. These measures have not only the advantage to correct the arc, but further to increase that which I have named the *weight* of a result, that is to say to render the probability of its errors more rapidly decreasing, so that the same errors become less probable with the multiplication of the bases. I expose here the laws of probability of the errors that the addition of new bases give birth to. The measure of a second base serves similarly to correct the difference in longitude of the extreme points of a perpendicular to the meridian and to increase the weight of the value of this difference.

After one brought, in the observations and in the calculations, the exactitude that one requires now, one considered the sides of the geodesic triangles as rectilinear, and one supposed the sum of their angles equal to two right angles. Legendre has remarked first that the two errors that one commits thus compensate themselves mutually, that is to say that by subtracting from each angle of a triangle the third of the spherical excess, one is able to neglect the curvature of its sides and to regard them as rectilinear. But the excess of the three observed angles over two right angles is composed of the spherical excess and the sum of the errors of the measure of each of the angles. The analysis of the probabilities shows that one must yet subtract from each angle the third of this sum, in order to have the law of probability of the errors of the results most rapidly decreasing. Thus, by the equal apportionment of the error of the sum observed of the three angles of the triangle considered as rectilinear, one corrects at the same time the spherical excess and the errors of the observations. The weights of the angles thus corrected increase, so that the same errors arrive, by this correction, less probable.

There is therefore advantage to observe the three angles of each triangle, and to correct them as one has just said it. Simple good sense makes a presentiment of this advantage; but the Calculus of probabilities is able alone to estimate and to show that, by this correction, it becomes the greatest that it is possible.

The formulas of which I just spoke are related to some future observations: thus, when one applies them to some past observations, one sets aside all the givens that the comparison of these observations are able to furnish out of the errors, data of which one is able to make use when one knows the law of probability of the errors of the partial observations. If this law is expressed by a constant less than unity, of which the exponent is the square of the error, then my formulas agree to the past observations as to the future observations, and they satisfy to all the data of these observations, as I have shown in n° 25 of Book II. In the case where the angles are measured by means of a repeating circle, each simple angle is the mean result of a great number of measures of the same angle contained in the total arc observed; the error of the angle is therefore the mean of the errors of all these measures; and, by n° 18 of Book II, the probability of this error is expressed by a constant, of which the exponent is equal to the square of the error. The employment of the repeating circle reunites therefore to the benefit of giving a precise measure of the angles the one to establish a law of probability of the errors which satisfies all the data of the observations.

In order to apply with success the formulas of probability to the geodesic observations, it is necessary to return faithfully all those that one would admit if they were isolated, and to reject none of them by the sole consideration that it extended a little from the others. Each angle must be uniquely determined by its measures, without regard to the two other angles of the triangle in which it belongs; otherwise, the error of the sum of the three angles would not be the simple result of the observations, as the formulas of probability supposes it. This remark seems to me important, in order to disentangle the truth in the middle from the slight uncertainties that the observations present.

1. Let us conceive, on a sphere, an arc of great circle A, A', A'', ... and suppose that one has formed about the chain of triangles ACC', $CC'C''$, $C'C''C'''$, $C''C'''C^{iv}$, ..., of which the sides CC', $C'C''$, $C''C'''$, ... cut this arc at A', A'', A''', ... I do not give at all the figure, because it is easy to trace it after these indications. Let A be the angle CAA', $A^{(1)}$ the angle $C'A'A''$, $A^{(2)}$ the angle $C''A''A'''$, ... Let further C be the angle ACC', $C^{(1)}$ the angle $CC'C''$, $C^{(2)}$ the angle $C'C''C'''$, ... One will have

$$A + A^{(1)} + C - \alpha = \pi + t,$$

α being the error of the observed angle C, t being the excess of the angles of the spherical triangle ACA' over π which expresses two right angles or the semi-circumference of which the radius is unity. One will have similarly

$$A^{(1)} + A^{(2)} + C^{(1)} - \alpha^{(1)} = \pi + t^{(1)},$$

$\alpha^{(1)}$ being the error of the observed angle $CC'C''$, and $t^{(1)}$ being the excess of the angles of the spherical triangle $A'C'A''$ over two right angles. One will form similarly the equations

$$A^{(2)} + A^{(3)} + C^{(2)} - \alpha^{(2)} = \pi + t^{(2)},$$
$$A^{(3)} + A^{(4)} + C^{(3)} - \alpha^{(3)} = \pi + t^{(3)},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots;$$

whence one draws easily

$$\begin{aligned}
A^{(2i)} = A &+C -C^{(1)} +C^{(2)} -C^{(3)} +\cdots +C^{(2i-2)} -C^{(2i-1)}\\
&-\alpha +\alpha^{(1)} -\alpha^{(2)} +\alpha^{(3)} -\cdots -\alpha^{(2i-2)} +\alpha^{(2i-1)}\\
&-t +t^{(1)} -t^{(2)} +t^{(3)} -\cdots -t^{(2i-2)} +t^{(2i-1)},\\
A^{(2i-1)} = \pi &-A -C +C^{(1)} -C^{(2)} +C^{(3)} -\cdots -C^{(2i-2)}\\
&+\alpha -\alpha^{(1)} +\alpha^{(2)} -\alpha^{(3)} +\cdots +\alpha^{(2i-2)}\\
&+t -t^{(1)} +t^{(2)} -t^{(3)} +\cdots +t^{(2i-2)};
\end{aligned}$$

by supposing therefore A well known, the error of the angle $A^{(n)}$ is

$$\alpha^{(n-1)} - \alpha^{(n-2)} + \alpha^{(n-3)} - \cdots \pm \alpha$$

the superior sign having place if n is odd, and the inferior sign having place if n is even. The values of $t,\ t^{(1)},\ \dots$ are quite small and are able to be determined with precision.

The concern is now to have the probability that this error will be contained within given limits. For this, I will suppose first that the probability of any error α is proportional to $c^{-h\alpha^2}$, c being the number of which the hyperbolic logarithm is unity. This supposition, the most natural and the most simple of all, results from the use of the repeating circle in the measure of the angles of the triangles. In fact, we name $\phi(q)$ the probability of an error q in the measure of a simple angle, this probability being supposed the same for the positive and for the negative errors. We suppose further that s is the number of simple angles contained in all the series that one has made in order to determine this angle. The probability that the error of the mean result or of the angle concluded from this series will be $\pm\frac{r}{\sqrt{s}}$, by n° 18 of Book II, proportional to

$$c^{-\frac{kr^2}{2k''}}$$

k being equal to $\int dq\,\phi(q)$, the integral being taken from q null to q equal to its greatest value, that one is always able to suppose infinite; by making $\phi(q)$ discontinuous and null beyond the limit of q, k'' is equal to $\int q^2 dq\,\phi(q)$. By supposing therefore

$$r = \alpha\sqrt{s}, \quad h = \frac{ks}{2k''},$$

$c^{-h\alpha^2}$ will be the probability of the error α. One will see, at the end of this article, that the following results always hold, whatever be the probability of α.

Let β and γ be the errors of the two angles $AC'C$ and CAC' of the first triangle ACC'; the probability of the three errors α, β and γ will be proportional to

$$c^{-h\alpha^2 - h\beta^2 - h\gamma^2};$$

but the observation of these angles give the sum $\alpha+\beta+\gamma$ of the three errors; because the sum of the three angles must be equal to two right angles plus the surface of the triangle ACC', if one names T the excess of the three angles observed on this quantity, one will have

$$\alpha+\beta+\gamma=T;$$

the preceding exponential becomes thus

$$c^{-2h(\beta+\frac{1}{2}\alpha-\frac{1}{2}T)^2-\frac{3h}{2}(\alpha-\frac{1}{3}T)^2-\frac{h}{3}T^2},$$

β being susceptible to all the values from $-\infty$ to ∞; it is necessary to multiply this exponential by $d\beta$ and take the integral within these limits, this which gives an integral which has for factor

$$c^{-\frac{3h}{2}(\alpha-\frac{1}{3}T)^2-\frac{h}{3}T^2};$$

the probability of α is therefore proportional to this factor. The value of α most probable is evidently that which renders null the quantity $\alpha-\frac{1}{3}T$; it is necessary therefore to correct the three angles of each triangle by the third of the excess T of their sum observed over two right angles plus the spherical excess. This is that which one does commonly.

We name $\overline{\alpha}$ and $\overline{\beta}$ the quantities $\alpha-\frac{1}{3}T$ and $\beta-\frac{1}{3}T$; the probability of $\overline{\alpha}$ will be proportional therefore to

$$c^{-\frac{3}{2}h\alpha^2}.$$

If one diminishes the angle C by $\frac{1}{3}T$, that is to say if one employs the corrected angles of each triangle, by naming $\overline{C}$, $\overline{C}^{(1)},\ldots$ that which the angles C, $C^{(1)},\ldots$ become, by these corrections, one will have

$$\begin{aligned}A^{(2i)} =&A+\overline{C}-\overline{C}^{(1)}+\overline{C}^{(2)}-\cdots-\overline{\alpha}+\overline{\alpha}^{(1)}-\overline{\alpha}^{(2)}+\cdots-t+t^{(1)}-\cdots\\ A^{(2i-1)} =&\pi-A-\overline{C}+\overline{C}^{(1)}-\cdots+\overline{\alpha}-\overline{\alpha}^{(1)}+\cdots+t-t^{(1)}+\cdots\end{aligned}$$

The probability that the quantity

$$\overline{\alpha}^{(n-1)}-\overline{\alpha}^{(n-2)}-\cdots\pm\overline{\alpha}$$

or the error of the angle $A^{(n)}$ will be contained in the limits $\pm r\sqrt{n}$, will be, by n° 18 cited,

$$\frac{2\sqrt{\frac{3}{2}h}}{\sqrt{\pi}}\int dr\,c^{-\frac{3}{2}hr^2}.$$

One is able to observe here the advantage that the observation of the three angles of each triangle produces, by the correction of these angles. Without this correction, the error of the angle $A^{(n)}$ would be

$$\alpha^{(n-1)}-\alpha^{(n-2)}-\cdots\pm\alpha,$$

and the probability that this error is contained in the limits $\pm r\sqrt{n}$ would be

$$\frac{2\sqrt{h}}{\sqrt{\pi}}\int dr\,c^{-hr^2}.$$

a probability less than the preceding in which the weight of the result is $\frac{3}{2}h$, instead that it is here h.

We determine now the value of h. Among the data of the observations, the quantities of which the sums of the angles of each triangle surpass two right angles plus the spherical excess appear to be the most proper to make known this value. By that which precedes, the probability of the simultaneous existence of $\overline{\alpha}$ and of T is proportional to

$$c^{-\frac{h}{3}T^2-\frac{3h}{2}\overline{\alpha}^2}.$$

By multiplying this exponential by $d\overline{\alpha}$, and taking the integral from $\overline{\alpha}=-\infty$ to $\overline{\alpha}=\infty$, the integral will have for factor $c^{-\frac{h}{3}T^2}$, and this factor will be proportional to the probability of T; this probability will be therefore

$$\frac{dT\,c^{-\frac{h}{3}T^2}}{\int dT\,c^{-\frac{h}{3}T^2}},$$

the integral of the denominator being taken from $T=-\infty$ to $T=\infty$. It will be thus proportional to

$$\frac{\sqrt{\frac{1}{3}h}}{\sqrt{\pi}}\,c^{-\frac{h}{3}T^2}.$$

Here the observed event is that the sum of the angles of the first triangle, of the second, of the third, etc. surpass two right angles plus the spherical excess, respectively, by the quantities T, $T^{(1)}$, $\dots$, $T^{(n-1)}$, n being the number of triangles; the probability of this event will be therefore proportional to

$$\left(\frac{\frac{1}{3}h}{\pi}\right)^{\frac{n}{2}}c^{-\frac{h}{3}\theta^2},$$

by making

$$\theta^2=T^2+T^{(1)2}+\cdots+T^{(n-1)2}.$$

Now, if one considers the diverse values of h as causes of the observed event, the probability of h will be, by the principle of the probability of the causes drawn from observed events, equal to

$$\frac{h^{\frac{n}{2}}\,dh\,c^{-\frac{h}{3}\theta^2}}{\int h^{\frac{n}{2}}\,dh\,c^{-\frac{h}{3}\theta^2}},$$

the integral of the denominator being taken for all the values of h, that is to say from $h=0$ to $h=\infty$. The value of h that it is necessary to choose is evidently the integral of the products of the values of h multiplied by their probabilities; this value is therefore

$$\frac{\int h^{\frac{n+2}{2}}\,dh\,c^{-\frac{h}{3}\theta^2}}{\int h^{\frac{n}{2}}\,dh\,c^{-\frac{h}{3}\theta^2}},$$

the integrals being taken from $h=0$ to $h=\infty$. The integral of the numerator is equal to

$$\frac{3(n+2)}{2\theta^2}\int h^{\frac{n}{2}}\,dh\,c^{-\frac{h}{3}\theta^2}.$$

The preceding fraction becomes thus $\frac{3(n+2)}{2\theta^2}$; this is therefore the value of h that it is necessary to adopt. If one supposes n a great number, this value become very nearly $\frac{3n}{2\theta^2}$. This quantity is the value of h which renders the observed event most probable, the probability of this event, *a priori*, being proportional to $h^{\frac{n}{2}} c^{-\frac{h}{3}\theta^2}$. By taking for h the quantity $\frac{3n}{2\theta^2}$, the probability that the error of the angle $A^{(n)}$ will be contained within the limits $\pm r\sqrt{s}$ is

$$\frac{3\sqrt{n}}{\theta\sqrt{\pi}} \int dr\, c^{-\frac{9}{4}\frac{nr^2}{\theta^2}};$$

the probability that it will be contained in the limits $\pm\frac{2}{3}\theta r'$ is therefore

$$\frac{2\int dr'\, c^{-r^2}}{\sqrt{\pi}},$$

the integral being taken from r' null.

2. Let us suppose the arc $AA'A''\ldots$ perpendicular to the meridian of the point A. Let ϕ be the angle formed by this meridian and by the one of the extreme point $A^{(n)}$, and V the smallest of the angles that this last meridian makes with the arc $AA'\ldots$; one will have

$$\sin\phi = \frac{\cos V}{\sin l},$$

l being the latitude of point A. By designating therefore by $\delta\phi$ and δV the errors of the angles ϕ and V, one will have

$$\delta\phi = -\frac{\delta V \sin V}{\sin l \cos\phi}.$$

If one has measured with a great exactitude the angle that the last side of the chain of triangles forms at $A^{(n)}$ with the meridian of this point, it is easy to see that

$$\delta V = \pm\delta A^{(n)},$$

$\delta A^{(n)}$ being the error of $A^{(n)}$; the preceding integral in r' is therefore the probability that the error $\delta\phi$ of the longitude ϕ concluded from the azimuths observed at A and $A^{(n)}$ will be contained in the limits $\pm\frac{2}{3}\theta r' \frac{\sin V}{\sin l \cos\phi}$.

There results from the analysis exposed in Chapter V of Book III of the *Mécanique céleste* that, if there exists an eccentricity in the terrestrial parallels, it has no sensible influence on the value of ϕ concluded in this manner, provided that the measured arc is not very considerable. In measuring therefore, with a great precision, the angles of the diverse triangles and the amplitudes of the extreme points, one will have quite exactly the difference in longitude of these points, and one will be able, by the preceding formula, to estimate the probability of the small errors to fear respecting this difference.

Let us determine presently the probability that the error of the measure of the line $AA'A''\ldots$ will be contained within some given limits. For this, we suppose that in the triangles CAC', $C'CC''$, $\ldots$ one had corrected the angles as one does ordinarily, that is to say by subtracting from each the third of the quantity of which the sum of the three observed angles surpasses two right angles plus the spherical excess. If one lowers the vertices C, C', C''.... of the perpendiculars CI, $C'I'$, $C''I''$, ... onto the line $AA'A''\dots$; one will have, very nearly,

$$AI = AC\cos IAC.$$

One will have next, quite nearly,

$$II' = CC'\cos A^{(1)}$$

and, generally,

$$I^{(i)}I^{(i+1)} = C^{(i)}C^{(i+1)}\cos A^{(i+1)}.$$

By supposing therefore that δ is the characteristic of the errors, one will have

$$\frac{\delta.I^{(i)}I^{(i+1)}}{I^{(i)}I^{(i+1)}} = \frac{\delta.C^{(i)}C^{(i+1)}}{C^{(i)}C^{(i+1)}} - \delta A^{(i+1)}\tan A^{(i+1)}.$$

One has, by that which precedes,

$$\delta A^{(i+1)} = \overline{\alpha}^{(i)} - \overline{\alpha}^{(i-1)} + \overline{\alpha}^{(i-2)} - \cdots \pm \overline{\alpha};$$

next, one has, in the $(i+1)^{\text{st}}$ triangle,

$$C^{(i)}C^{(i+1)} = \frac{C^{(i)}C^{(i-1)}\sin C^{(i+1)}C^{(i-1)}C^{(i)}}{\sin C^{(i-1)}C^{(i+1)}C^{(i)}},$$

this which gives

$$\frac{\delta.C^{(i)}C^{(i+1)}}{C^{(i)}C^{(i+1)}} = \frac{\delta.C^{(i)}C^{(i-1)}}{C^{(i)}C^{(i-1)}} + \delta C^{(i+1)}C^{(i-1)}C^{(i)}\cot C^{(i+1)}C^{(i-1)}C^{(i)} \\ - \delta C^{(i-1)}C^{(i+1)}C^{(i)}\cot C^{(i-1)}C^{(i+1)}C^{(i)};$$

but $\overline{\alpha}^{(i)}$ is, by that which precedes, the error of the angle $C^{(i)}$ or $C^{(i-1)}C^{(i)}C^{(i+1)}$, corrected by subtracting from it the third of the excess of the sum of the three observed angles of the triangle over two right angles. Let $\overline{\beta}^{(i)}$ be the error of the angle $C^{(i-1)}C^{(i+1)}C^{(i)}$, thus corrected; $-(\overline{\alpha}^{(i)} + \overline{\beta}^{(i)})$ will be the error of the third angle $C^{(i+1)}C^{(i-1)}C^{(i)}$. One will have therefore

$$\frac{\delta.C^{(i)}C^{(i+1)}}{C^{(i)}C^{(i+1)}} = \frac{\delta.C^{(i)}C^{(i-1)}}{C^{(i)}C^{(i-1)}} + (\overline{\alpha}^{(i)} + \overline{\beta}^{(i)})\cot C^{(i+1)}C^{(i-1)}C^{(i)} \\ - \overline{\beta}^{(i)}\cot C^{(i-1)}C^{(i+1)}C^{(i)};$$

this which gives, by observing that, in the first triangle, the side $C^{(i-1)}C$ is AC that I supposed measured very exactly.

$$\frac{\delta.C^{(i)}C^{(i+1)}}{C^{(i)}C^{(i+1)}} = -\text{S}[(\overline{\alpha}^{(i)} + \overline{\beta}^{(i)})\cot C^{(i+1)}C^{(i-1)}C^{(i)} + \overline{\beta}^{(i)}\cot C^{(i-1)}C^{(i+1)}C^{(i)}],$$

the sign S serving to express the sum of all the quantities that it contains from $i = 0$ to i inclusively. One will have therefore thus the value of $\delta.I^{(i)}I^{(i+1)}$. By reuniting all these values, one will have, for the entire error of their sum or of the measured line, an expression of this form

$$(o) \qquad p\overline{\alpha} + q\overline{\beta} + p^{(1)}\overline{\alpha}^{(1)} + q^{(1)}\overline{\beta}^{(1)} + \cdots$$

The probability of the simultaneous values of $\overline{\alpha}$ and of $\overline{\beta}$ is, by that which precedes, proportional to

$$c^{-2h\left(\overline{\beta}+\frac{1}{2}\overline{\alpha}\right)^2-\frac{3}{2}h\alpha^2}.$$

By making

$$\overline{\beta} + \frac{1}{2}\overline{\alpha} = \frac{1}{2}\underline{\alpha}\sqrt{3},$$

the preceding exponential becomes

$$c^{-\frac{3}{2}h\underline{\alpha}^2-\frac{3}{2}h\overline{\alpha}^2};$$

thus the laws of probability of the values of $\underline{\alpha}$ and of $\overline{\alpha}$ are the same. The function (o) takes then this form

$$(o') \qquad r\underline{\alpha} + r^{(1)}\overline{\alpha} + r^{(2)}\underline{\alpha}^{(1)} + r^{(3)}\overline{\alpha}^{(1)} + \cdots$$

The probability that the error of this function, and consequently of the function (o), is contained in the limits $\pm s$, by n° 20 of Book II,

$$\frac{2\int dt\, c^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from t null to t equal to

$$s\sqrt{\frac{\frac{3}{2}h}{r^2 + r^{(1)2} + r^{(2)2} + \cdots}}$$

One has evidently

$$p\overline{\alpha} + q\overline{\beta} = \left(p - \frac{1}{2}q\right)\overline{\alpha} + \frac{1}{2}q\underline{\alpha}\sqrt{3};$$

this which gives, by equating it to $r\underline{\alpha} + r^{(1)}\overline{\alpha}$,

$$r = \frac{1}{2}q\sqrt{3}, \quad r^{(1)} = p - \frac{1}{2}q;$$

the value of t will be therefore, by substituting for h its value $\frac{3n}{2\theta^2}$,

$$\frac{3s}{2\theta}\sqrt{\frac{n}{p^2 - pq + q^2 + p^{(1)2} - p^{(1)}q^{(1)} + q^{(1)2} + \cdots}}.$$

The length of the measured arc makes known that of the osculating radius of the surface at the point A of departure. Let $1 + u$ be the radius drawn from the center of gravity of the Earth at its surface, u being a function of the longitude and of the latitude, the semi-axis of the Earth being taken for unity; if one names R the osculating radius of this point, in the sense AA', one will have, by the Chapter cited from Book III of the *Mécanique céleste*,

$$R = 1 + u - \left(\frac{du}{dl}\right)\tan l + \frac{\left(\frac{d\,du}{d\phi^2}\right)}{\cos^2\, l};$$

and if the name ϵ the length of the measured arc $AA^{(1)}$, one will have, quite nearly,

$$R = \frac{\epsilon}{\phi\cos l}(1 - \frac{1}{3}\epsilon^2\tan^2\, l);$$

this which gives, quite nearly,

$$\delta R = \frac{\delta\epsilon}{\phi\cos l} - \frac{\epsilon\,\delta\phi}{\phi^2\cos l};$$

but one has, by that which precedes,

$$\begin{aligned}\delta\epsilon =& p\overline{\alpha} + q\overline{\beta} + \cdots\ ,\\ \delta\phi =& \frac{\mp\delta A^{(n)}}{\sin l} = \frac{\pm(\overline{\alpha} - \overline{\alpha}^{(1)} + \overline{\alpha}^{(2)} - \cdots)}{\sin l},\end{aligned}$$

the inferior sign having place if n is even, and the superior sign if n is odd. By making therefore

$$\begin{aligned}\overline{p} =& \frac{p}{\phi\cos l} \mp \frac{\epsilon}{\phi^2\sin l\cos l}, & \overline{q} =& \frac{q}{\phi\cos l},\\ \overline{p}^{(1)} =& \frac{p^{(1)}}{\phi\cos l} \mp \frac{\epsilon}{\phi^2\sin l\cos l}, & \overline{q}^{(1)} =& \frac{q^{(1)}}{\phi\cos l},\\ & \cdots\cdots\cdots\cdots\cdots\cdots\ , & & \cdots\cdots\ ,\end{aligned}$$

the probability that the error δR will be contained in the limits $\pm s$ will be

$$\frac{2\int dt\, c^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from t null to

$$t = \frac{3s}{2\theta}\sqrt{\frac{n}{\overline{p}^2 - \overline{pq} + \overline{q}^2 + \overline{p}^{(1)2} - \overline{p}^{(1)}\overline{q}^{(1)} + \cdots}}.$$

The difference in latitude of the extreme points of the perpendicular depend, by the Chapter cited from the *Mécanique céleste*, on the eccentricity of the terrestrial parallels, which introduce into its expression the quantity

$$(u) \qquad -\phi\left[\left(\frac{du}{d\phi}\right)\tan l+\left(\frac{d\,du}{d\phi\,dl}\right)\right];$$

the part of this expression which is independent of this eccentricity is proportional to ϕ^2; thus the small error of which ϕ is susceptible has no sensible influence at all on the difference in latitude. By observing therefore with a great care this difference, the eccentricity of the terrestrial parallels must be manifest, as little as it is sensible.

If the geodesic line has been traced in the sense of the meridian, the azimuth, at the extremity of the measured arc, will make known the eccentricity of the terrestrial parallels, and it is remarkable that this azimuth is the function (u), by changing ϕ into the difference in latitude of the extreme points of the measured arc and by multiplying it by the sine of the latitude divided by the square of the cosine of the latitude at the origin of the arc.

The arc measured in the sense of the meridian will make known the osculating radius of the Earth in this sense, and, by the preceding formulas, one will have the probability of the errors of which its value is susceptible.

One will obtain more precision in all the results by fixing toward the middle of the measured arc the origin of the angles; because then the superior powers of these angles, that one neglects, becomes much smaller.

3. We suppose that, in order to verify the operations, one measures, toward the extremity $A^{(n)}$ of the arc $AA'A''\ldots$, a second base. The expression of the error of this base, concluded from the chain of the triangles and from the base measured at the point A, will be, by that which precedes, of the form

$$(p) \qquad l\overline{\alpha}+m\overline{\beta}+l^{(1)}\overline{\alpha}^{(1)}+m^{(1)}\overline{\beta}^{(1)}+\cdots;$$

let λ be this error which will be known by the direct measure of the second base. If in the function (p) one makes, as previously

$$\beta+\frac{1}{2}\overline{\alpha}=\frac{1}{2}\underline{\alpha}\sqrt{3},$$

it takes this form

$$f\underline{\alpha}+f^{(1)}\overline{\alpha}+f^{(2)}\underline{\alpha}^{(1)}+f^{(3)}\overline{\alpha}^{(1)}+\cdots$$

By designating by s the value of the function (o) or of its equivalent (o') and observing that the probabilities of $\underline{\alpha}$ and of $\overline{\alpha}$ follow the same law and are proportionals to $c^{-\frac{3}{2}h\underline{\alpha}^2}$ and $c^{-\frac{3}{2}h\overline{\alpha}^2}$, the probability of the preceding function will be proportional to

$$c^{-\frac{3}{2}h(\underline{\alpha}^2+\overline{\alpha}^2+\underline{\alpha}^{(1)2}+\overline{\alpha}^{(1)2}+\cdots)}.$$

By supposing the function equal to λ, this exponential becomes

$$c^{-\frac{3}{2}h\left[\left(\underline{\alpha}-\frac{f\lambda}{F}\right)^2+\left(\overline{\alpha}-\frac{f^{(1)}\lambda}{F}\right)^2+\cdots+\frac{\lambda^2}{F}\right]},$$

F expressing the sum of the squares $f^2+f^{(1)2}+f^{(2)2}+\cdots$ The most probable values of $\underline{\alpha}, \overline{\alpha}, \underline{\alpha}^{(1)}, \ldots$ are evidently those which render the exponent of this exponential a minimum, this which gives

$$\underline{\alpha} = \frac{f\lambda}{F}, \quad \overline{\alpha} = \frac{f^{(1)}\lambda}{F}, \quad \underline{\alpha}^{(1)} = \frac{f^{(2)}\lambda}{F}, \quad , \ldots$$

If one observes next that one has, by that which precedes,

$$f = \frac{1}{2}m\sqrt{3}, \quad f^{(1)} = l - \frac{1}{2}m,$$
$$\overline{\beta} = \frac{1}{2}\underline{\alpha}\sqrt{3} - \frac{1}{2}\overline{\alpha},$$

one will have

$$\underline{\alpha} = \frac{(l-\frac{1}{2}m)\lambda}{F}, \qquad \overline{\beta} = \frac{(m-\frac{1}{2}l)\lambda}{F},$$
$$\overline{\alpha}^{(1)} = (l^{(1)} - \frac{1}{2}m^{(1)})\frac{\lambda}{F}, \qquad \overline{\beta}^{(1)} = \frac{(m^{(1)}-\frac{1}{2}l^{(1)})\lambda}{F},$$
$$\cdots\cdots\cdots\cdots, \qquad \cdots\cdots\cdots\cdots,$$

and F will become

$$l^2 - ml + m^2 + l^{(1)2} - m^{(1)}l^{(1)} + m^{(1)2} + \cdots$$

If one substitutes these values into the function (o), one will have the correction resulting from the measure of a second base, by affecting it with a contrary sign. But one is able to arrive to this result directly, by nº 21 of Book II, according to which one sees that, s being the value of the function (o), its probability is proportional to

$$c^{-\frac{\frac{3}{2}h\left(s-\lambda\frac{\mathrm{S}r^{(i)}f^{(i)}}{\mathrm{S}f^{(i)2}}\right)}{\mathrm{S}r^{(i)2}-\frac{(\mathrm{S}r^{(i)}f^{(i)})^2}{\mathrm{S}f^{(i)2}}}},$$

the sign S extending to all the values of i, from $i = 0$ inclusively. The most probable value of s is that which renders null the exponent of c, this which gives

$$s = \lambda\frac{\mathrm{S}r^{(i)}f^{(i)}}{\mathrm{S}f^{(i)2}};$$

it is necessary therefore to subtract from the measured arc $AA^{(1)}\ldots A^{(n)}$ this value of s; and, if one names u the error of the arc thus corrected, the probability of u will be proportional to

$$c^{-\frac{\frac{3}{2}hu^2}{\mathrm{S}r^{(i)2}-\frac{(\mathrm{S}r^{(i)}f^{(i)})^2}{\mathrm{S}f^{(i)2}}}}.$$

One sees by this expression that the weight of the result is increased by virtue of the measure of the second base; because, before this measure, the coefficient of $-s^2$ was, by the preceding section,

$$\frac{\frac{3}{2}h}{\mathrm{S}r^{(i)2}},$$

and, by this measure, the coefficient of $-u^2$ becomes

$$\frac{\frac{3}{2}h}{\mathrm{S}r^{(i)2}-\frac{(\mathrm{S}r^{(i)}f^{(i)})^2}{\mathrm{S}f^{(i)2}}}.$$

The same error becomes therefore less probable by this measure and by the preceding correction of this arc.

One is able to observe here that the preceding values of r, $r^{(1)}$, f and $f^{(1)}$ give

$$\begin{aligned}
r^2 + r^{(1)} &= p^2 - pq + q^2,\\
f^2 + f^{(1)2} &= l^2 - ml + m^2,\\
rf + r^{(1)}f^{(1)} &= l(p - \frac{1}{2}q) + m(q - \frac{1}{2}p).
\end{aligned}$$

One will be able therefore to form easily $\mathrm{S}r^{(i)2}$ and $\mathrm{S}r^{(i)}f^{(i)}$ by means of the coefficients of $\overline{\alpha}$, $\overline{\beta}$, $\overline{\alpha}^{(1)}, \ldots$ in the functions of (o) and (p).

If one had measured some other bases, one would have, by the analysis of nº 21 of Book II, the corrections which it would be necessary to make to the measured arc, and the law of its errors.

The measure of a new base is able to serve to correct, not only the measured arc, but also the difference in longitude of its extreme points or the angle $A^{(n)}$. It will suffice to substitute into the function (o) that here

$$\pm(\overline{\alpha} - \overline{\alpha}^{(1)} + \overline{\alpha}^{(2)} - \cdots)$$

which expresses the error of $A^{(n)}$, the superior sign having place if n is odd, and the inferior if n is even. Then one has

$$p = \pm 1, \quad q = 0, \quad p^{(1)} = \mp 1, \quad q^{(1)} = 0, \quad \ldots;$$

thence it is easy to conclude that, in order to correct the angle $A^{(n)}$, it is necessary to add to it the quantity

$$\frac{\mp\lambda(l - l^{(1)} + l^{(2)} - \cdots - \frac{1}{2}m + \frac{1}{2}m^{(1)} - \cdots)}{l^2 - ml + m^2 + l^{(1)2} - m^{(1)}l^{(1)} + m^{(1)2} + \cdots}.$$

The probability that the error of $A^{(n)}$ thus corrected is within the limits $\pm u$ will be

$$\frac{2\int dt\, c^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from t null to

$$t = \frac{u\sqrt{\frac{3}{2}h}}{\sqrt{n - \frac{(l - l^{(1)} + l^{(2)} - \cdots - \frac{1}{2}m + \frac{1}{2}m^{(1)} - \cdots)^2}{l^2 - ml + m^2 + l^{(1)2} - \cdots}}}$$

4. We are arrived to the preceding results starting from the law of probability of the error α proportional to $c^{-h\alpha^2}$, and we have proved that this law of probability is able to be admitted by regarding it of the angles measured with the repeating circle. We are going to show here that these results hold generally, whatever be the law of probability of error α. Let $\phi(\alpha)$ be this law: We will suppose it such that the same positive and negative errors are equally probable. We will suppose, moreover, that $\phi(\alpha)$ extends from $\alpha = -\infty$ to $\alpha = +\infty$: this supposition is always permitted; because, if the probability becomes null beyond certain limits, the function $\phi(\alpha)$ is then discontinued and null beyond these limits. We seek now the probability of the values of the function (o) of n° 1. This function has been calculated by correcting the angles of each triangle by a third of the observed sum of their errors. We suppose generally that, in the first triangle, one corrects the error α by $(i+\frac{1}{3})T$, the error β by $(i_1+\frac{1}{3})T$, and consequently the third error by $(\frac{1}{3} - i - i_1)T$, by designating by $\underline{\alpha}$ and $\underline{\beta}$ the errors α and β thus corrected, one will have

$$\alpha = \underline{\alpha} + (i + \frac{1}{3})T, \quad \beta = \underline{\beta} + (i_1 + \frac{1}{3})T.$$

By designating similarly by $\underline{\alpha}^{(1)}$ and $\underline{\beta}^{(1)}$ the errors $\alpha^{(1)}$ and $\beta^{(1)}$ respectively corrected from $(i^{(1)} + \frac{1}{3})T^{(1)}$, $(i_1^{(1)} + \frac{1}{3})T^{(1)}$, one will have

$$\alpha^{(1)} = \underline{\alpha}^{(1)} + (i^{(1)} + \frac{1}{3})T^{(1)}, \quad \beta^{(1)} = \underline{\beta}^{(1)} + (i_1^{(1)} + \frac{1}{3})T^{(1)},$$

and thus consecutively. The function (o) is, by n° 1, equal to

$$p\overline{\alpha} + q\overline{\beta} + p^{(1)}\overline{\alpha}^{(1)} + q^{(1)}\overline{\beta}^{(1)} + \cdots;$$

next, one has

$$\alpha = \overline{\alpha} + \frac{1}{3}T = \underline{\alpha} + (i + \frac{1}{3})T;$$

this which gives

$$\overline{\alpha} = \underline{\alpha} + iT;$$

one has similarly

$$\overline{\beta} = \underline{\beta} + i_1 T, \quad \overline{\alpha}^{(1)} = \underline{\alpha}^{(1)} + iT, \quad \ldots$$

The function (o) becomes thus

$$p\underline{\alpha} + q\underline{\beta} + p^{(1)}\underline{\alpha}^{(1)} + q^{(1)}\underline{\beta}^{(1)} + \cdots + \mathrm{S}(pi + qi_1)T,$$

$\mathrm{S}(pi + qi_1)T$ designating the sum

$$(pi + qi_1)T + (p^{(1)}i^{(1)} + q^{(1)}i_1^{(1)})T^{(1)} + \cdots$$

The correction of the function (o) relative to the values of $i,\ i_1,\ i^{(1)},\ \ldots$ is therefore

$$-\mathrm{S}(pi + qi_1)T,$$

and then this function thus corrected becomes

$$(\epsilon) \qquad p\underline{\alpha} + q\underline{\beta} + p^{(1)}\underline{\alpha}^{(1)} + q^{(1)}\underline{\beta}^{(1)} + p^{(2)}\underline{\alpha}^{(2)} + \cdots ,$$

In order to have the probability of the values of this last function, we will observe that the probability of the simultaneous existence of the values of α, β and T is

$$\frac{d\alpha\, d\beta\, dT \phi(\alpha)\phi(\beta)\phi(T - \alpha - \beta)}{\iiint d\alpha\, d\beta\, dT \phi(\alpha)\phi(\beta)\phi(T - \alpha - \beta)},$$

the integrals of the denominator being taken within their infinite positive and negative limits. Let us designate by k the integral $\int d\alpha\phi(\alpha)$, taken within these limits; it is easy to see that this denominator will be equal to k^3. The preceding fraction becomes thus

$$\frac{d\alpha\, d\beta\, dT}{k^3}\phi(\alpha)\phi(\beta)\phi(T - \alpha - \beta);$$

the probability of the simultaneous existence of the values of $\underline{\alpha}, \underline{\beta}$ and T will be therefore

$$\frac{\underline{d\alpha}\,\underline{d\beta}\, dT}{k^3}\phi[\underline{\alpha} + (i + \frac{1}{3}T)]\phi[\underline{\beta} + (i_1 + \frac{1}{3}T)]\phi[(\frac{1}{3} - i - i_1)T - \underline{\alpha} - \underline{\beta}]$$

T being supposed to be able to be varied from $-\infty$ to $+\infty$, one will have the probability of the simultaneous values of $\underline{\alpha}$ and $\underline{\beta}$ by integrating the preceding function with respect to T, within the infinite limits. We name $\frac{\underline{d\alpha}\,\underline{d\beta}}{k^3}\psi(\underline{\alpha},\ \underline{\beta})$ this integral. One sees, by n° 20 of Book II, that by designating by s the value of the function (ϵ), the probability of s will be proportional to

$$(\mathrm{H}) \qquad \int dw\, c^{-sw\sqrt{-1}} \left\{ \begin{array}{l} \iint \underline{d\alpha}\, \underline{d\beta}\, \psi(\underline{\alpha}, \underline{\beta}) \cos(p\underline{\alpha} + q\underline{\beta})w \\ \times \iint \underline{d\alpha}^{(1)}\, \underline{d\beta}^{(1)}\, \psi(\underline{\alpha}^{(1)}, \underline{\beta}^{(1)}) \cos(p^{(1)}\underline{\alpha}^{(1)} + q^{(1)}\underline{\beta}^{(1)})w \\ \times \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array} \right\},$$

the integral relative to w being taken from $w = -\pi$ to $w = \pi$ and the integrals relative to $\underline{\alpha}$ and $\underline{\beta}$ being taken within their infinite limits. We develop into a series, ordered with respect to the powers of w, the function contained in the parenthesis. The logarithm of $\iint \underline{d\alpha}\, \underline{d\beta}\, \psi(\underline{\alpha}, \underline{\beta}) \cos(p\underline{\alpha} + q\underline{\beta})w$ is equal to

$$\log \iint \underline{d\alpha}\, \underline{d\beta}\, \psi(\underline{\alpha}, \underline{\beta}) - \frac{\frac{w^2}{2} \iint \underline{d\alpha}\, \underline{d\beta}\, \psi(\underline{\alpha}, \underline{\beta})(p\underline{\alpha} + q\underline{\beta})^2}{\iint \underline{d\alpha}\, \underline{d\beta}\, \psi(\underline{\alpha}, \underline{\beta})} - \cdots$$

Now one has

$$\begin{aligned} &\iint \underline{d\alpha}\, \underline{d\beta}\, \psi(\underline{\alpha}, \underline{\beta}) \\ &\quad = \iiint \underline{d\alpha}\, \underline{d\beta}\, dT \phi[\underline{\alpha} + (i + \frac{1}{3}T)]\phi[\underline{\beta} + (i_1 + \frac{1}{3}T)]\phi[(\frac{1}{3} - i - i_1)T - \underline{\alpha} - \underline{\beta}]. \end{aligned}$$

The integrals being taken within their infinite limits, it is easy to see, by the known theory of multiple integrals, that the second member of this equation is equal to

$$\iiint d\alpha\, d\beta\, dT' \phi(\alpha)\phi(\beta)\phi(T'),$$

T' being equal to $T - \alpha - \beta$; it is therefore equal to k^3.

One has next

$$(u) \qquad \left\{ \begin{aligned} &\iint \underline{d\alpha}\, \underline{d\beta}\, \psi(\underline{\alpha}, \underline{\beta})(p\underline{\alpha} + q\underline{\beta})^2 \\ &\quad = \iiint d\alpha\, d\beta\, dT' \phi(\alpha)\phi(\beta)\phi(T')(p\underline{\alpha} + q\underline{\beta})^2, \end{aligned} \right.$$

by substituting for $\underline{\alpha}$ and $\underline{\beta}$ their values in α, β, and T' in the quantity $(p\underline{\alpha}+q\underline{\beta})^2$. Now it follows from that which precedes that one has

$$\begin{aligned} \underline{\alpha} &= (\frac{2}{3} - i)\alpha - (i + \frac{1}{3})\beta - (i + \frac{1}{3})T', \\ \underline{\beta} &= (\frac{2}{3} - i)\beta - (i + \frac{1}{3})\alpha - (i_1 + \frac{1}{3})T'. \end{aligned}$$

By substituting these values into the quantity $(p\underline{\alpha} + q\underline{\beta})^2$, one will be able, in its development, to neglect the terms dependent on the products $\alpha\beta$, $\alpha T'$, $\beta T'$, because the triple integral

$$(u) \qquad \iiint d\alpha\, d\beta\, dT' \phi(\alpha)\phi(\beta)\phi(T')(p\underline{\alpha} + q\underline{\beta})^2$$

being taken within its infinite limits, and the function $\phi(\alpha)$ being supposed the same for the values $+\alpha$ and $-\alpha$, it is clear that the elements of this integral depending on $+\alpha\beta$ will be destroyed by the negative elements depending on $-\alpha\beta$. If one observes next that by designating $\int \alpha^2\, d\alpha\, \phi(\alpha)$ by k'', one has

$$\iiint \alpha^2 d\alpha\, d\beta\, dT' \phi(\alpha)\phi(\beta)\phi(T') = k^2 k'',$$

the function (u) will become

$$k^2 k''[\frac{2}{3}(p^2 - pq + q^2) + 3(pi + qi_1)^2];$$

the logarithm of

$$\iint \underline{d\alpha}\, \underline{d\beta}\, \psi(\underline{\alpha}, \underline{\beta}) \cos(p\underline{\alpha} + q\underline{\beta})w$$

being thus

$$\log k^3 - \frac{k''}{2k} w^2 [\frac{2}{3}(p^2 - pq + q^2) + 3(pi + qi_1)^2] - \cdots$$

By passing again from logarithms to the numbers and neglecting, consistently with the analysis of n$^{\text{o}}$ 20 of Book II, the powers of w superior to the square, the integral (H) will take this form

$$k^{3n} \int dw\, c^{-sw\sqrt{-1} - \frac{k'' w^2}{2k}[\frac{2}{3}\mathrm{S}(p^2 - pq + q^2) + 3\mathrm{S}(pi + qi_1)^2]},$$

$\mathrm{S}(p^2 - pq + q^2)$ representing the sum of the quantities

$$p^2 - pq + q^2 + p^{(1)2} - p^{(1)} q^{(1)} + \cdots;$$

$\mathrm{S}(pi + qi_1)^2$ representing the sum of the quantities

$$(pi + qi_1)^2 + (p^{(1)} i^{(1)} + q^{(1)} i^{(1)})^2 + \cdots,$$

and n being the number of triangles. Let us give to the preceding integral this form

$$k^{3n} \int dw\, c^{-Q\left(w + \frac{s\sqrt{-1}}{2Q}\right)^2 - \frac{s^2}{4Q}},$$

Q being equal to

$$\frac{k''}{2k}[\frac{2}{3}\mathrm{S}(p^2 - pq + q^2) + 3\mathrm{S}(pi + qi_1)^2]$$

The integral must be taken from $w = -\pi$ to $w = \pi$, and one has seen, in the section cited from Book II, that it is able to be extended from $w = -\infty$ to $w = \infty$; then the preceding integral, or the probability of s, becomes proportional to $\mathrm{c}^{-\frac{s^2}{4Q}}$ or to

$$c^{-\frac{3ks^2}{4k''[\mathrm{S}(p^2 - pq + q^2) + \frac{9}{2}\mathrm{S}(pi + qi_1)^2]}}$$

It is necessary now to determine the value of $\frac{k}{k''}$. For this we will make, as above, use of the observed values of T, $T^{(1)}$, $T^{(2)}$, ... When these values are in great number, the sum of their squares divided by their number will be, quite nearly, by that which we have established in Book II, the mean value of T^2; by making therefore

$$\theta^2 = T^2 + T^{(1)2} + T^{(2)2} + \cdots,$$

$\frac{\theta^2}{n}$ will be this mean value. Now one has this value by multiplying each possible value of T^2 by its probability and by taking the sum of all these products; the expression of the mean value of T^2 will be therefore

$$\frac{\iiint d\alpha\, d\beta\, dT . T^2 \phi(\alpha)\phi(\beta)\phi(T-\alpha-\beta)}{\iiint d\alpha\, d\beta\, dT \phi(\alpha)\phi(\beta)\phi(T-\alpha-\beta)},$$

the integrals being taken within their infinite limits. Let, as above,

$$T' = T - \alpha - \beta;$$

the preceding fraction will become

$$\frac{\iiint (T'-\alpha-\beta)^2 \; d\alpha\, d\beta\, dT' \phi(\alpha)\phi(\beta)\phi(T')}{\iiint d\alpha\, d\beta\, dT' \phi(\alpha)\phi(\beta)\phi(T')},$$

all these integrals being taken again within their infinite limits. It is easy to see, by the preceding analysis, that the numerator of this fraction is equal to $3k^2k''$, and that its denominator is equal to k^3; the fraction becomes thus $\frac{3k''}{k}$; by equating it to $\frac{\theta^2}{n}$, one will have

$$\frac{k''}{k} = \frac{\theta^2}{3n};$$

the probability of s is therefore proportional to

$$c^{-\frac{9ns^2}{4\theta^2[\text{S}(p^2-pq+q^2)+\frac{9}{2}\text{S}(pi+qi_1)^2]}}$$

It is clear that the values of i and of i_1, which render this probability the most rapidly decreasing are those which give $pi + qi_1 = 0$; and then the preceding correction of the measured arc becomes null. The case of i and i_1 nulls give therefore the law of probability of the geodesic errors, the most rapidly decreasing, a law which must be evidently adopted.

Thence, it is easy to conclude that the probability that the value of s will be contained within the limits $\pm s$ is equal to

$$\frac{2\int dt\, c^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from t null to

$$t = \frac{3s}{2\theta}\sqrt{\frac{n}{\text{S}(p^2-pq+q^2)}},$$

this which is conformed to that which we have deduced in n$^{\text{o}}$ 1 from the particular law of probability of the errors α proportional to $c^{-h\alpha^2}$.

Let us express, as in n$^{\text{o}}$ 2, the error of a new base concluded from the first by the function

$$l\overline{\alpha} + m\overline{\beta} + l^{(1)}\overline{\alpha}^{(1)} + m^{(1)}\overline{\beta}^{(1)} + \cdots$$

By making, as previously,

$$\underline{\alpha} = \overline{\alpha} - iT, \quad \underline{\beta} = \overline{\beta} - i_1 T, \quad \underline{\alpha} = \overline{\alpha}^{(1)} - i^{(1)}T^{(1)}, \quad \cdots,$$

the correction of this function, relative to the values of i, i_1, $i^{(1)}$, ... will be $-\mathrm{S}(li + mi_1)T$, and the error of the new base thus corrected will be

$$(\lambda) \qquad l\underline{\alpha} + m\underline{\beta} + l^{(1)}\underline{\alpha}^{(1)} + m^{(1)}\underline{\beta}^{(1)} + \cdots$$

Let s' be the value of this function; the probability of the simultaneous existence of the values of s and s' of the functions (ϵ) and (λ) will be, by nº 21 of Book II, proportional to

$$\iint dw\, dw' c^{-sw\sqrt{-1}-s'w'\sqrt{-1}-Qw^2-2Q_1ww'-Q_2w'^2},$$

the integrals being taken from w to w' equal to $-\infty$ to w to w' equal to $+\infty$. One sees next, by the analysis of the section cited, that one has

$$Qw^2 + 2Q_1ww' + Q_2w'^2$$
$$= \frac{\frac{1}{2}\mathrm{S}\iiint\, d\alpha\, d\beta\, dT'\phi(\alpha)\phi(\beta)\phi(T')[(p\underline{\alpha} + q\underline{\beta})w + (l\underline{\alpha} + m\underline{\beta})w']^2}{\iiint\, d\alpha\, d\beta\, dT'\phi(\alpha)\phi(\beta)\phi(T')},$$

the integrals relative to α, β and T' being taken within their infinite limits; this which gives, by substituting for $\underline{\alpha}$ and $\underline{\beta}$ their previous values,

$$Q = \frac{1}{3}\frac{k''}{k}[\mathrm{S}(p^2 - pq + q^2) + \frac{9}{2}\mathrm{S}(pi + qi_1)^2],$$
$$Q_1 = \frac{1}{3}\frac{k''}{k}\left\{\mathrm{S}\left[\left(p - \frac{q}{2}\right)l + \left(q - \frac{p}{2}\right)m\right] + \frac{9}{2}\mathrm{S}((pi + qi_1)(li + mi_1)\right\},$$
$$Q_2 = \frac{1}{3}\frac{k''}{k}[\mathrm{S}(l^2 - ml + m^2) + \frac{9}{2}\mathrm{S}(li + mi_1)^2];$$

whence one concludes, by the analysis of the section cited, that the probability of the simultaneous existence of the values of s and of s' is proportional to

$$c^{-\frac{(Q_2s^2 - 2Q_1ss' + Qs'^2)}{4(QQ_2 - Q_1^2)}}$$

or

$$c^{-\frac{Q_2(s - s'\frac{Q_1}{Q_2})^2}{4(QQ_2 - Q_1^2)} - \frac{s'^2}{4Q_2}}$$

The measure of the second base determines the value of s'; and, by naming λ as above, the probability of s will be proportional to

$$c^{-\frac{Q_2(s - \frac{\lambda Q_1}{Q_2})^2}{4(QQ_2 - Q_1^2)}}$$

The most probable value of s is that which renders null the exponent of c; this which gives

$$s = \lambda\frac{Q_1}{Q_2};$$

by making therefore

$$s = \lambda \frac{Q_1}{Q_2} + u,$$

u will be the error of the arc measured and diminished by $\frac{\lambda Q_1}{Q_2}$; and the probability of this error will be proportional to

$$c^{-\frac{Q_2 u^2}{4(QQ_2 - Q_1^2)}}$$

The values of $i,\ i_1,\ i^{(1)}, \ldots$ must be determined by the condition that the coefficient of u^2, in this exponential, is a maximum; we see therefore what are the values of these quantities of these quantities which render the fraction

$$\frac{Q_2}{QQ_2 - Q_1^2}$$

a maximum. If one names Q' that which the expression of Q becomes when one diminishes the finite integral $\mathrm{S}(pi + qi_1)^2$ by the element $(pi + qi_1)^2$, one will have

$$Q' = Q - \frac{3}{2} \frac{k''}{k} (pi + qi_1)^2.$$

If one names similarly Q_1' that which the expression of Q_1 becomes when one diminishes the finite integral $\mathrm{S}(pi + qi_1)(li + mi_1)$ by the element $(pi + qi_1)(li + mi_1)$, one will have

$$Q_1' = Q_1 - \frac{3}{2} \frac{k''}{k} (pi + qi_1)(li + mi_1).$$

Finally, if one names Q_2' that which Q_2 becomes when one diminishes the finite integral $\mathrm{S}(li + mi_1)^2$ by the element $(li + mi_1)^2$, one will have

$$Q_2' = Q_2 - \frac{3}{2} \frac{k''}{k} (li + mi_1)^2.$$

The fraction

$$\frac{Q_2'}{Q' Q_2' - Q_1'^{\,2}}$$

surpasses the fraction

$$\frac{Q_2}{QQ_2 - Q_1^2};$$

because, by substituting into the first, instead of Q' , Q_1' and Q_2', their values, and reducing to the same denominator its excess over the second, the numerator of this excess becomes

$$\frac{3}{2} \frac{k''}{k} [Q_2(pi + qi_1) - Q_1(li + mi_1)]^2.$$

We name further Q'' that which Q' becomes when we subtract $\frac{3}{2} \frac{k''}{k} (p^{(1)} i^{(1)} + q^{(1)} i_1^{(1)})^2$ from it; and, consequently, that which the expression of Q becomes when one diminishes the integral $\mathrm{S}(pi+qi_1)^2$ by the two elements $(pi+qi_1)^2+(p^{(1)}i^{(1)}+q^{(1)}i_1^{(1)})^2$. We name similarly Q_1'' that which Q_1' becomes when one subtracts from it

$$\frac{3}{2}\frac{k''}{k}(p^{(1)}i^{(1)}+q^{(1)}i_1^{(1)})(l^{(1)}i^{(1)}+m^{(1)}i_1^{(1)});$$

finally, we name Q_2'' that which Q_2' becomes when one subtracts from it

$$\frac{3}{2}\frac{k''}{k}(l^{(1)}i^{(1)}+m^{(1)}i_1^{(1)})^2;$$

one will see, by the same process, that the fraction

$$\frac{Q_2''}{Q''Q_2''-Q_1''^2}$$

surpasses the fraction

$$\frac{Q_2'}{Q'Q_2'-Q_1'^2};$$

and, consequently, the fraction

$$\frac{Q_2}{QQ_2-Q_1^2}.$$

By continuing thus, one sees that this last fraction becomes to its maximum when the finite integrals $\mathrm{S}(pi+qi_1)^2$, $\mathrm{S}(pi+qi_1)(li+mi_1)^2$ and $\mathrm{S}(li+mi_1)^2$ are null in the expressions of Q, Q_1 and Q_2, that which reverts to supposing null the values of i, i_1, $i^{(1)}$, $\ldots$; this supposition gives therefore the law of probability of the most rapidly decreasing values of Q, and then one has

$$\begin{aligned} Q &= \frac{\theta^2}{9n}\mathrm{S}(p^2-pq+q^2),\\ Q_1 &= \frac{\theta^2}{9n}\mathrm{S}\left[\left(p-\frac{q}{2}\right)l+\left(q-\frac{p}{2}\right)m\right],\\ Q_2 &= \frac{\theta^2}{9n}\mathrm{S}(l^2-ml+m^2). \end{aligned}$$

The weight of the error u becomes thus

$$\frac{-\frac{9n}{4\theta^2}}{\mathrm{S}(p^2-pq+q^2)-\frac{\left[\mathrm{S}\left(p-\frac{q}{2}\right)l+\left(q-\frac{p}{2}\right)m\right]^2}{\mathrm{S}(l^2-ml+m^2)}}$$

It is easy to see that this result coincides with the analogous result of nº 3.

On the probability of the results deduced, by any processes whatsoever, of a great number of observations.

The true march of the natural sciences consists in showing, through the path of induction, from the phenomena to the laws and from the laws to the forces. One comes down next from these forces to the complete explication of the phenomena as far as into their smallest details. The attentive inspection of a great assembly of observations and their comparisons multiplied make presentiment the laws that it conceals. The analytic expression of these laws depend on constant coefficients that one name *elements*. One determines, by the theory of probabilities, the most probable values of these elements, and if, by substituting them into the analytic expressions, these expressions satisfy all the observations, within the limits of the possible errors, one will be sure that these laws are those of nature, or at least they are very little different from them. One sees thence how much the application of the Calculus of Probabilities is useful to natural Philosophy, and how much it is essential to have some methods in order to draw from observations the most advantageous results. These results are evidently those with which one same error is less probable than with each other result. Thus the condition that it is necessary to fulfill in the choice of a result is that the law of probability of its errors is most rapidly decreasing. Before the application of the Calculus of Probabilities to this object, each calculator subjected the results of the observations to the conditions which to him appeared to be most natural. Now if one has certain formulas in order to obtain the most advantageous result, it is no longer necessary to have uncertainty in this regard, at least when one makes use of the factors. One is able, not only to determine this result, but further to assign the probability of the errors of the results obtained by some other processes and to compare these processes to the most advantageous method. The excessive length of the calculations that this method requires, when one employs a very great number of observations, does not permit then to make use of it. But, by grouping conveniently the equations of condition and by applying this method to the equations which result from each of these groups, one is able at the same time to simplify considerably the calculations and to conserve a part of the advantages which are attached to them, as one will see it in the following. Whatever be the process of which one makes use, it is very useful to have a means to determine the probability of the results to which one arrives, especially when there is a question of the important elements. One will have easily this probability by the following method.

1. We consider first a quite simple case, the one of the angles measured by means of a repeating circle. We suppose that at the end of each partial observation one reads the corresponding division of the circle; one will have, by departing from the point of departure, a sequence of terms of which the first will be the angle itself, the second will be the double of this angle, the third will be the triple of it, and thus consecutively. We designate by A_1, A_2, $\ldots$, A_n these different terms, and by a_1, $a_2, \ldots$, a_n the n

partial angles successively measured. One will have

$$\begin{aligned} A_1 &= a_1, \\ A_2 &= a_2 + a_1, \\ A_3 &= a_3 + a_2 + a_1, \\ \cdots & \quad \cdots\cdots\cdots ; \end{aligned}$$

and, if one names y the simple true angle, one will have this sequence of equations

$$(a) \qquad \begin{cases} y - a_1 + x_1 = 0, \\ y - a_2 + x_2 = 0, \\ y - a_3 + x_3 = 0, \\ \cdots\cdots\cdots\cdots\cdots ; \\ y - a_n + x_n = 0, \end{cases}$$

$x_1,\ x_2,\ x_3, \ldots$ being the errors of the angles $a_1,\ a_2,\ a_3, \ldots$ One will have, by n° 20 of Book II, the most advantageous result by multiplying by unity each of the preceding equations and by adding them, this which gives

$$y = \frac{a_1 + a_2 + \cdots + a_n}{n} + \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

By supposing $x_1,\ x_2,\ \ldots$ null, one will have the result of the most advantageous method, and the error of this result will be $\frac{x_1+x_2+\cdots+x_n}{n}$. By designating by u this error, one sees, by the section cited, that the probability of u is proportional to $c^{-\frac{knu^2}{2k''}}$, k being equal to $\int dx\, \phi(x)$ and k'' being equal to $\int x^2\, dx\, \phi(x)$, $\phi(x)$ being the law of probability of the errors x of the partial observations, this law being supposed the same for the positive and negative errors and being able to be extended to infinity; c is always the number of which the hyperbolic logarithm is unity.

Svangerg, in his excellent Work on the degree of Lapland, exposes, in order to determine y, a new process founded on the following considerations. Each term of the sequence $A_1,\ A_2, \ldots$ is able to give its value, which is able to be equally determined by the difference $A_{s'} - A_s$ of any two terms whatsoever of this sequence, s' being greater than s. This difference, divided by $s' - s$, gives a value of y so much more exact as this divisor is greater. By multiplying it therefore by this divisor, one will render it preponderant by reason of its exactitude. If one makes next a sum of these products and if one divides it by the number of simple angles that it contains, one will have a value of y which, concluded from all the combinations of the quantities $A_1,\ A_2, \ldots$ by giving to each of these combinations the influence that it must have, seems to have to approach to the truth the nearest that it is possible. This would be just, in fact, if all these values of y were independent. But their mutual dependence makes that the same simple angles are employed many times and in a different manner for each of them, this which must change the respective probabilities of the values of y and, consequently, the probability of the mean value. This is a new example of the illusions to which one is exposed in these delicate researches.

The process of which there is question reverts to forming the sum of the differences $A_{s'} - A_s$, s' being greater than s and having with this condition to be extended from $s' = 1$ to $s' = n$; s must be extended from $s = 0$ to $s = n - 1$, and one must make $A_0 = 0$. By dividing next this sum by the number of simple angles that it contains, one has the value of y. It is easy to see that this value is

$$\gamma = \frac{n\mathrm{S}A_n - 2\mathrm{SS}A_{n-1}}{\frac{n(n+1)(n+2)}{1.2.3}},$$

$\mathrm{S}A_n$ expressing the sum of the quantities $A_1,\ A_2,\ \ldots,\ A_n$; $\mathrm{SS}A_{n-1}$ is the sum of the quantities

$$\begin{aligned}
&A_1,\\
&A_1 + A_2,\\
&A_1 + A_2 + A_3,\\
&\ldots\ldots\ldots\ldots,\\
&A_1 + A_2 + \cdots + A_{n-1};
\end{aligned}$$

the angle a_1 is contained $n - i + 1$ times in $\mathrm{S}A_n$, it is contained $\frac{(n-1)(n-i+1)}{1.2}$ times in the function $\mathrm{SS}A_{n-1}$; it is therefore contained $\frac{i(n-i+1)}{\frac{n(n+1)(n+2)}{1.2.3}}$ times in the preceding expression of y. Thence it follows that this process reverts to multiplying the equations (a) respectively by the factors

$$\frac{n}{\frac{n(n+1)(n+2)}{6}},\quad \frac{2(n-1)}{\frac{n(n+1)(n+2)}{6}},\quad \frac{3(n-2)}{\frac{n(n+1)(n+2)}{6}},\quad ;$$

and then one finds, by nº 20 from Book II, that the probability of the error u in the preceding expression of y is proportional to

$$c^{-\frac{k}{k''}\frac{u^2}{\mathrm{S}M_i^2}},$$

M_i being here equal to $\frac{i(n-i+1)}{\frac{n(n+1)(n+2)}{6}}$; the integral $\mathrm{S}M_i^2$ having to comprehend all the values of M_i^2 from $i = 1$ to $i = n$ inclusively. One has thus

$$\mathrm{S}M_i^2 = \frac{6}{5}\frac{n^2 + 2n + 2}{n(n+1)(n+2)}.$$

n being supposed very great, this value of $\mathrm{S}M_i^2$ is reduced very nearly to $\frac{6}{5n}$; the probability of the error u is therefore proportional to

$$c^{-\frac{5}{6}\frac{k}{k''}nu^2}.$$

One has just seen that, in the most advantageous method, the probability of a similar error of the result is proportional to

$$c^{-\frac{knu^2}{2k''}}.$$

Thus, in order that the same errors become equally probable, the observations must be, in the process of Svanberg, more numerous than in the ordinary process, following the ratio of six to five.

One would be able to believe that, the result obtained by the process of Svanberg being a new datum from the observations, its combination with the result of the ordinary method must give a more exact result, and of which the law of probability of the errors is more rapidly decreasing. But the analysis proves that this is not. We consider, in fact, the system of equations

$$(b) \qquad \begin{cases} p_1 y - a_1 + x_1 = 0, \\ p_2 y - a_2 + x_2 = 0, \\ \cdots\cdots\cdots\cdots ; \\ p_n y - a_n + x_n = 0, \end{cases}$$

$x_1, x_2, \ldots$ being, as above, the errors of the observations. The most advantageous method prescribes to multiply these equations, respectively, by p_1, p_2, ... and to add them, this which gives

$$y = \frac{\mathrm{S}p_i a_i}{\mathrm{S}p_i^2} - \frac{\mathrm{S}p_i x_i}{\mathrm{S}p_i^2},$$

the sign S comprehending, as above, all the values that it precedes, from $i = 1$ to $i = n$ inclusively. The first term of this expression will be the value of y given by the most advantageous method, and its error will be $\frac{\mathrm{S}p_i x_i}{\mathrm{S}p_i^2}$; in designating it by u, its probability will be, by n° 20 of Book II, proportional to

$$c^{-\frac{k}{2k''} u^2 \mathrm{S}p_i^2}.$$

If one multiplies the equations (b) respectively by m_1, m_2, m_3, ..., their sum will give

$$y = \frac{\mathrm{S}m_i a_i}{\mathrm{S}m_i p_i} - \frac{\mathrm{S}m_i x_i}{\mathrm{S}m_i p_i}.$$

The first term of this expression will be the value of y relative to the system of factors m_1, m_2, ..., and $\frac{\mathrm{S}m_i x_i}{\mathrm{S}m_i p_i}$ will be the error of this value, an error that we will designate by u'. If one makes

$$l = \mathrm{S}p_i x_i, \quad l' = \mathrm{S}m_i x_i,$$

the probability of the simultaneous existence of l and of l' will be, by n° 21 of Book II, proportional to

$$c^{-\frac{k}{2k''E}(l^2 \mathrm{S}m_i^2 - 2ll' \mathrm{S}m_i p_i + l'^2 \mathrm{S}p_i^2)},$$

E being equal to $\mathrm{S}m_i^2 \mathrm{S}p_i^2 - (\mathrm{S}m_i p_i)^2$. Now one has

$$l = u\mathrm{S}p_i^2, \quad l' = u' \mathrm{S}m_i p_i;$$

the simultaneous existence of u and of u' is therefore proportional to

$$c^{-\frac{k}{2k''} \frac{\mathrm{S}p_i^2}{E} [u^2 E + (u'-u)^2 (\mathrm{S}m_i p_i)^2]}.$$

Let e be the difference of the preceding values from y; one has

$$e = \frac{\mathrm{S}p_i a_i}{\mathrm{S}p_i^2} - \frac{\mathrm{S}m_i a_i}{\mathrm{S}m_i p_i};$$

the equality of these values, corrected respectively of their errors u and u', give

$$e = u - u';$$

the preceding exponential becomes thus

$$c^{-\frac{k}{2k''}\mathrm{S}p_i^2\left[u^2+e^2\frac{(\mathrm{S}m_i p_i)^2}{E}\right]}.$$

e is a quantity given by the observations; the value of u which renders this exponential a maximum is evidently $u = 0$; thus the consideration of the result given by the system of factors $m_1,\ m_2,\ \dots$ add no correction to the result of the most advantageous method and changes not at all the law of probability of its error u, which remains always proportional to

$$c^{-\frac{k}{2k''}u^2\mathrm{S}p_i^2}.$$

If the very great number of equations of condition do not permit to apply this method to them, there will be always advantage to apply it to some equations resulting from groups of these equations. We suppose that one has r groups, each formed from s equations, so that $n = rs$; one will have the following r equations

$$\text{(V)}\qquad \begin{cases} P_1 y - A_1 + X_1 = 0, \\ P_2 y - A_2 + X_2 = 0, \\ \dots\dots\dots\dots\dots, \\ P_r y - A_r + X_r = 0, \end{cases}$$

and one has

$$\begin{aligned} P_1 &= p_1 + p_2 + \cdots + p_s, \\ A_1 &= a_1 + a_2 + \cdots + a_s, \\ X_1 &= x_1 + x_2 + \cdots + x_s, \\ \\ P_2 &= p_{s+1} + p_{s+2} + \cdots + p_{2s}, \\ &\dots\dots\dots\dots\dots \end{aligned}$$

By applying to the equations (V) the process of the most advantageous method, one has

$$y = \frac{\mathrm{S}P_t A_t}{\mathrm{S}P_t^2} - \frac{\mathrm{S}P_t X_t}{\mathrm{S}P_t^2};$$

the sign S embraces all the quantities which it precedes, from $t = 1$ to $t = r$ inclusively. $\frac{\mathrm{S}P_t X_t}{\mathrm{S}P_t^2}$ is the error of the value $\frac{\mathrm{S}P_t A_t}{\mathrm{S}P_t^2}$ taken for y; by designating this error by u, its probability will be, by n° 20 of Book II, proportional to

$$c^{-\frac{k}{2k''}\frac{u^2}{\mathrm{S}m_i^2}};$$

$m_1,\ m_2,\ \ldots$ being the coefficients of $x_1,\ x_2,\ \ldots$ in the expression of u; and the integral $\mathrm{S}m_i^2$ being extended from $i=0$ to $i=n$ inclusively. Now it is easy to see that one has

$$\begin{array}{llll} m_1 = \dfrac{P_1}{\mathrm{S}P_t^2}, & m_2 = \dfrac{P_1}{\mathrm{S}P_t^2}, & \cdots, & m_s = \dfrac{P_1}{\mathrm{S}P_t^2}, \\ m_{s+1} = \dfrac{P_2}{\mathrm{S}P_t^2}, & \ldots\ldots\ldots\ldots, & \cdots, & m_{2s} = \dfrac{P_2}{\mathrm{S}P_t^2}, \\ m_{2s+1} = \dfrac{P_3}{\mathrm{S}P_t^2}, & \ldots\ldots\ldots\ldots, & \cdots, & \ldots\ldots; \end{array}$$

thence it is easy to conclude that one has

$$\mathrm{S}m_i^2 = \frac{s}{\mathrm{S}P_t^2} = \frac{n}{r\mathrm{S}P_t^2};$$

the probability of u is therefore proportional to

$$c^{-\frac{k}{2k''}\frac{r}{n}u^2\mathrm{S}P_t^2}.$$

If one reunited all the equations of a single group, the probability of u would be proportional to

$$c^{-\frac{k}{2k''}\frac{u^2}{n}(\mathrm{S}p_i)^2};$$

because then r would become unity, P_1 would become $\mathrm{S}p_i$, $P_2,\ P_3,\ \ldots$ would be nulls. The weight of the result or the coefficient of $-u^2$ would be therefore, in the first case,

$$\frac{k}{2k''}\frac{r}{n}\mathrm{S}P_t^2,$$

and, in the second case, it would be

$$\frac{k}{2k''n}(\mathrm{S}p_i)^2$$

Now the first of these quantities surpasses the second; in fact,

$$(\mathrm{S}p_i)^2 = (P_1 + P_2 + \cdots + P_r)^2.$$

If, in the development of this last square, one substitutes, instead of the product $2P_1P_2$, its value $P_2^1 + P_2^2 - (P_1 - P_2)^2$, and thus of the other products, one sees that this square is equal to $r\mathrm{S}P_t^2$, less a positive quantity; there is therefore advantage to partition the equations of condition into many groups to which one applies the most advantageous method.

One sees further that there is advantage to augment the number of groups; because, if one supposes r even and equal to $2r'$, the weight of the result relative to the number r' of groups will be proportional to

$$r'[(P_1 + P_2)^2 + (P_3 + P_4)^2 + \cdots + (P_{2r'+1} + P_{2r'})^2];$$

and the weight of the result relative to $2r'$ groups will be proportional to

$$2r'(P_1^2 + P_2^2 + \cdots + P_{2r'}^2).$$

This last quantity surpasses the preceding, as one sees it by observing that

$$2(P_1^2 + P_2^2) > (P_1 + P_2)^2.$$

If the equations of condition contain many unknown elements, y, y', ... there will be always advantage to partition them into groups in order to apply to the equations resulting from these groups the most advantageous method. The more one will multiply these groups, the more one will augment the weight of the results.

But, from whatever manner that one has obtained these results, one will be able always to determine, by the following theorem, the probability of their errors. If one has, by any process whatsoever, drawn from the equations of condition the equation $y - a = 0$, it is clear that one has multiplied the equations of condition, respectively, by some factors M_1, M_2, M_3, ... such that the unknowns have disappeared, with the exception of y which has unity for factor. The error u of the result $y = a$ is evidently $M_1x_1 + M_2x_2 + \cdots$; the probability of this error will be therefore, by n° 20 of Book II, proportional to

$$c^{-\frac{k}{2k''}\frac{u^2}{\mathrm{S}M_i^2}},$$

the sign S being extended to all the values of i from $i = 1$ to $i = n$, n being the number of observations. All is reduced therefore to determine, in the process that one has followed, the factors $M_1, M_2, \ldots$

If, for example, the equations of condition contain two unknowns y and y' and if, in order to form the final two equations, one adds together all these equations: 1° by changing the signs of the equations in which y has the sign $-$; 2° by changing the signs of the equations in which y' has the sign $-$, one will obtain, by this process of which one has often made use, two equations that we will represent by the following:

$$Py + Ry' - A = 0,$$
$$P_1y + R_1y' - A_1 = 0.$$

In multiplying the first of these equations by

$$\frac{R_1}{PR_1 - P_1R}$$

and the second by

$$\frac{-R}{PR_1 - P_1R},$$

one will have, by adding them,

$$\gamma - \frac{AR_1 - A_1R}{PR_1 - P_1R} = 0.$$

In the equations of condition, x_i has been multiplied by ± 1; the sign $-$ having place if, in order to form the final equations, one has changed the signs of the i^{th} equation. Thence it is easy to conclude that, if one designates by s the number of equations of condition in which the coefficients of y and of y' have the same sign, one will have

$$\mathrm{S}M_i^2 = \frac{s(R_1 - R)^2 + (n - s)(R_1 + R)^2}{(PR_1 - P_1R)^2}.$$

One will simplify the calculation by preparing the equations of condition in a manner that in each the coefficient of y has the sign $+$. One will form next a first final equation by adding the s equations in which the coefficient of y' has the sign $+$. One will form a second final equation by adding the $n-s$ equations in which the coefficient of y' has the sign $-$. Let

$$fy + gy' - h = 0,$$
$$f_1y + g_1y' - h_1 = 0$$

be these two equations. By multiplying the first by $\frac{g_1}{fg_1+f_1g}$ and the second by $\frac{g}{fg_1+f_1g}$, one will have

$$y - \frac{hg_1 + h_1g}{fg_1 + f_1g} = 0,$$

and it is easy to see that

$$\mathrm{S}M_i^2 = \frac{sg_1^2 + (n-s)g^2}{(fg_1 + f_1g)^2}.$$

These values of y and of $\mathrm{S}M_i^2$ coincide with the preceding, as it is easy to see it by observing that one has

$$P = f + f_1, \qquad R = g - g_1, \qquad A = h + h_1,$$
$$P_1 = f - f_1, \qquad R_1 = g + g_1, \qquad A_1 = h - h_1.$$

The equations of condition being represented generally by the following

$$0 = x_i - a_i + p_iy + q_iy',$$

if one multiplies them respectively by $m_1,\ m_2,\ \ldots$ and if one adds them, one will have the final equation

$$0 = \mathrm{S}m_ix_i - \mathrm{S}m_ia_i + y\mathrm{S}m_ip_i + y'\mathrm{S}m_iq_i;$$

if one multiplies next the same equations, respectively by $n_1,\ n_2,\ \ldots,$ one will have, by adding them, the final equation

$$0 = \mathrm{S}n_ix_i - \mathrm{S}n_ia_i + y\mathrm{S}n_ip_i + y'\mathrm{S}n_iq_i.$$

By multiplying the first of these equations by $\frac{\mathrm{S}n_iq_i}{I}$ and the second by $-\frac{\mathrm{S}m_iq_i}{I}$, I being equal to

$$\mathrm{S}m_ip_i\mathrm{S}n_iq_i - \mathrm{S}n_ip_i\mathrm{S}m_iq_i,$$

one will have

$$0 = y - \frac{\mathrm{S}m_ia_i\mathrm{S}n_iq_i - \mathrm{S}n_ia_i\mathrm{S}m_iq_i}{I} + \frac{\mathrm{S}m_ix_i\mathrm{S}n_iq_i - \mathrm{S}n_ix_i\mathrm{S}m_iq_i}{I}.$$

This last term is the error of the value that one obtains for y, by supposing nulls $x_1,\ x_2,\ \ldots$: one has therefore then

$$M_i = \frac{m_i\mathrm{S}n_iq_i - n_i\mathrm{S}m_iq_i}{I};$$

whence it is easy to conclude

$$c^{-\frac{k}{2k''}\frac{u^2}{\mathrm{S}M_i^2}} = c^{-\frac{k}{2k''}u^2\frac{I^2}{H}},$$

by making

$$H = \mathrm{S}m_i^2(\mathrm{S}n_iq_i)^2 - 2\mathrm{S}m_in_i\mathrm{S}m_iq_i\mathrm{S}n_iq_i + \mathrm{S}n_i^2(\mathrm{S}m_iq_i)^2,$$

a result which coincides with the one of n° 21 of Book II, in which we have proved that the maximum of the coefficient of $-u^2$ in this exponential takes place when one supposes generally $m_i = p_i$, $n_i = q_i$; this supposition gives therefore the most advantageous result or the one of which the weight is a maximum.

One will determine the value of $\frac{k}{2k''}$ by means of the squares of the remainders which take place when one substitutes into the equations of condition the values determined for y and y'. By designating by ϵ_i this remainder in the i^{th} equation of condition

$$0 = x_i - a_i + p_iy + q_iy',$$

and designating by u and u' the errors of these values, one will have

$$0 = x_i + \epsilon_i - p_iu - q_iu';$$

this which gives

$$\mathrm{S}\epsilon_i^2 = \mathrm{S}x_i^2 - 2u\mathrm{S}p_ix_i - 2u'\mathrm{S}q_ix_i + u^2\mathrm{S}p_i^2 + 2uu'\mathrm{S}p_iq_i + u'^2\mathrm{S}q_i^2.$$

One has, by n° 19 of Book II,

$$\mathrm{S}x_i^2 = \frac{k''}{k}n;$$

next, the values u and u' cease to be probable, when they surpass the quantities of order $\frac{1}{\sqrt{n}}$. The values of $\mathrm{S}p_ix_i$ and $\mathrm{S}q_ix_i$ cease to be probable when they surpass some quantities of order $\sqrt{n}$; the values of $-2u\mathrm{S}p_ix_i$ and $-2u'\mathrm{S}q_ix_i$ cease therefore to be probable when they cease to be of a finite order, n being supposed infinitely great. $\mathrm{S}p_i^2$, $\mathrm{S}p_iq_i$ and $\mathrm{S}q_i^2$ being of order n, the values of $u^2\mathrm{S}p_i^2$, $2uu'\mathrm{S}p_iq_i$, $u'^2\mathrm{S}q_i^2$ cease to be probable when they cease to be finite quantities. One is able therefore to neglect all these quantities and to suppose, what ever be the process of which one makes use,

$$\mathrm{S}\epsilon_i^2 = \frac{k''}{k}n,$$

this which gives

$$\frac{k}{2k''} = \frac{n}{2\mathrm{S}\epsilon_i^2}.$$

2. The preceding methods are reduced to multiplying each equation of condition by a factor and to adding all these products in order to form a final equation. But one is able to employ some other considerations in order to obtain the result sought: for example, one is able to choose that of the equations of condition which must most approach to

the truth. The process that I have given in n° 40 of Book III of the *Mécanique céleste* is of this kind. By supposing the equations (b) of the previous section prepared in a manner that p_1, p_2, $p_3, \dots$ are positive and that the values $\frac{a_1}{p_1}$, $\frac{a_2}{p_2}$, $\dots$ of y, data by these equations under the supposition of x_1, x_2, $\dots$ nulls, form a decreasing sequence, the process of which there is question consists in choosing the equation of r^{th} condition, such that one has

$$p_1 + p_2 + \cdots + p_{r-1} < p_r + p_{r+1} + \cdots + p_n,$$
$$p_1 + p_2 + \cdots + p_r > p_{r+1} + p_{r+1} + \cdots + p_n,$$

and in supposing

$$y = \frac{a_r}{p_r}.$$

This value of y renders a minimum a sum of all the deviations from the other values, taken positively; because by naming x_1, x_2, $\dots$ these deviations, x_1, x_2, $\dots$, x_{r-1} will be positive and x_{r+1}, x_{r+2}, $\dots$, x_n will be negative. If one increases the preceding value of y by the infinitely small quantity δy, the sum of the positive deviations x_1, x_2, $\dots$, x_{r-1} will diminish by the quantity

$$\delta y(p_1 + p_2 + \cdots + p_{r-1});$$

but the sum of the negative deviations, taken with the sign $+$, will increase by the quantity

$$\delta y(p_r + p_{r+1} + \cdots + p_n);$$

the deviation x_r will become $p_r \delta y$. The sum of the deviations, taken all positively, will be therefore increased by the quantity

$$\delta y(p_{r+1} + p_{r+2} + \cdots + p_n - p_1 - p_2 - \cdots - p_{r-1});$$

by the conditions to which the choice of the r^{th} equation is subject, this quantity is positive. One will see, in the same manner, that if one diminishes $\frac{a_r}{p_r}$ by δy, the sum of the deviations taken positively will be increased by the positive quantity

$$\delta y(p_1 + p_2 + \cdots + p_r - p_{r+1} - p_{r+2} - \cdots - p_n).$$

Thus, in the two cases of an increase and of a diminution of the value $\frac{a_r}{p_r}$ by y, the sum of the deviations, taken positively, is increased. This consideration seems to give a great advantage to the preceding value of y, which, when there is a question to choose a middle among the results of an odd number of observations, becomes the result equidistant from the extremes. But the Calculus of probabilities is able alone to estimate this advantage: I am going therefore to apply it to this delicate question.

The sole data of which we will make use are that the equation of condition

$$0 = x_r - a_r + p_r y$$

gives, setting aside the errors, a value of y smaller than the $r - 1$ anterior equations and greater than the $n - r$ posterior equations; and that one has

$$p_1 + p_2 + \cdots + p_{r-1} < p_r + p_{r+1} + \cdots + p_n,$$
$$p_1 + p_2 + \cdots + p_r > p_{r+1} + p_{r+1} + \cdots + p_n,$$

One has

$$y = \frac{a_1}{p_1} - \frac{x_1}{p_1} = \frac{a_r}{p_r} - \frac{x_r}{p_r};$$

this which gives

$$\frac{x_1}{p_1} = \frac{a_1}{p_1} - \frac{a_r}{p_r} + \frac{x_r}{p_r}.$$

Thus, $\frac{a_1}{p_1}$ surpassing $\frac{a_r}{p_r}$, $\frac{x_1}{p_1}$ surpasses $\frac{x_r}{p_r}$. It is of it the same of $\frac{x_2}{p_2}$, $\frac{x_3}{p_3}$, ... to $\frac{x_{r-1}}{p_{r-1}}$. One will see in the same manner that $\frac{x_{r+1}}{p_{r+1}}$, $\frac{x_{r+2}}{p_{r+2}}$, ..., $\frac{x_n}{p_n}$ are less than $\frac{x_r}{p_r}$. Thus, the sole conditions to which we subject the errors and the equations of condition are the following:

$$(c) \qquad \begin{cases} s > r, & s < r, \\ \dfrac{x_s}{p_s} < \dfrac{x_r}{p_r}, & \dfrac{x_s}{p_s} > \dfrac{x_r}{p_r}; \end{cases}$$

$$p_1 + p_2 + \cdots + p_{r-1} < p_r + p_{r+1} + \cdots + p_n,$$
$$p_1 + p_2 + \cdots + p_r > p_{r+1} + p_{r+1} + \cdots + p_n,$$

It is uniquely according to these data from the observations that we are going to determine the probability of the error x_r. We will have besides no regard to the order that the first $r-1$ equations of condition and the $n-r$ last observe among them, nor to the values of the quantities $a_1,\ a_2,\ \ldots,\ a_n$.

We represent, as above, by $\phi(x)$ the law of probability of the error x of the observations and, in order to express that this probability is the same for the positive and negative errors, we suppose $\phi(x)$ a function of x^2.

Now, if one supposes x_r positive, the probability that x_1 will surpass $p_1 \frac{x_r}{p_r}$ will be

$$\frac{1}{2} - \frac{\frac{1}{2}\int dx\,\phi(x)}{k},$$

the integral $\int dx\,\phi(x)$ being taken from $x = 0$ to $x = p_1 \frac{x_r}{p_r}$ and k being, as above, this integral taken from x null to x infinity. The probability that the quantities $\frac{x_1}{p_1}$, $\frac{x_2}{p_2}$, ..., $\frac{x_{r-1}}{p_{r-1}}$ will be all greater than $\frac{x_r}{p_r}$ is therefore proportional to the product of the $r-1$ factors

$$1 - \frac{\int dx\,\phi(x)}{k}, \quad 1 - \frac{\int dx\,\phi(x)}{k}, \quad \ldots;$$

the integral of the first factor being taken from $x = 0$ to $x = p_1 \frac{x_r}{p_r}$; the integral of the second factor being taken from $x = 0$ to $x = p_2 \frac{x_r}{p_r}$; and thus consecutively.

Similarly, all the quantities $\frac{x_{r+1}}{p_{r+1}}$, $\frac{x_{r+2}}{p_{r+2}}$, ..., $\frac{x_n}{p_n}$ being supposed smaller than $\frac{x_r}{p_r}$, one sees, by the same reasoning, that the probability of this supposition is proportional to the product of the $n-r$ factors

$$1 + \frac{\int dx\,\phi(x)}{k}, \quad 1 + \frac{\int dx\,\phi(x)}{k}, \quad \ldots;$$

the integral of the first factor being taken from $x = 0$ to $x = p_{r+1} \frac{x_r}{p_r}$, that of the second factor being taken from $x = 0$ to $x = p_{r+2} \frac{x_r}{p_r}$, and thus consecutively. The probability of the error x_r is $\phi(x_r)$; thus the probability that the error of the r^{th} observation will be x_r and that the value of y given by the r^{th} equation will be smaller than the values given by the preceding equations, and will surpass the values given by the following equations, this probability, I say, will be proportional to the product of the $n-1$ preceding factors and of $\phi(x_r)$.

x being supposed very small, one has, to the quantities near of order x^3,

$$\int dx\,\phi(x) = x\phi(0) + \frac{1}{2}x^2\phi'(0),$$

$\phi'(0)$ being that which $\frac{d\phi(x)}{dx}$ becomes when x is null. In the present question, $\phi(x)$ being a function of x^2, one has $\phi'(0)=0$, and then one has

$$\int dx\,\phi(x) = x\phi(0).$$

The preceding factors will become thus, by making $\frac{x_r}{p_r}=\zeta$,

$$\begin{aligned}
&1 - p_1\zeta\frac{\phi(0)}{k},\\
&1 - p_2\zeta\frac{\phi(0)}{k},\\
&\cdots\cdots\cdots\cdots\cdots,\\
&1 - p_{r-1}\zeta\frac{\phi(0)}{k},\\
&1 + p_{r+1}\zeta\frac{\phi(0)}{k},\\
&\cdots\cdots\cdots\cdots\cdots,\\
&1 + p_n\zeta\frac{\phi(0)}{k}.
\end{aligned}$$

If one designates by $\phi''(0)$ the value of $\frac{d^2\phi(x)}{dx^2}$ when x is null, $\phi(x_r)$ becomes

$$\phi(0) + \frac{1}{2}p_r^2\zeta^2\phi''(0).$$

The sum of the hyperbolic logarithms of all these factors is, to the quantities near of order ζ^3, by dividing the factor $\phi(x_r)$ by $\phi(0)$,

$$\begin{aligned}
&-\zeta\frac{\phi(0)}{k}(p_1+p_2+\cdots+p_{r-1}-p_{r+1}-p_{r+2}-\cdots-p_n)\\
&\quad-\frac{\zeta^2}{2}\left[\frac{\phi(0)}{k}\right]^2(p_1^2+p_2^2+\cdots+p_r^2+p_{r+1}^2+\cdots+p_n^2)\\
&\quad+\frac{1}{2}p_r^2\zeta^2\left\{\frac{\phi''(0)}{k}+\left[\frac{\phi(0)}{k}\right]^2\right\}.
\end{aligned}$$

The probability of ζ is therefore proportional to the base c of the hyperbolic logarithms, elevated to a power of which the exponent is the preceding function. One must observe that by virtue of the conditions to which the choice of the r^{th} equation is subject, the quantity

$$p_1 + p_2 + \cdots + p_{r-1} - p_{r+1} - p_{r+2} - \cdots - p_n$$

is, setting aside the sign, a quantity less than p_r, and that thus, by supposing ζ of order $\frac{1}{\sqrt{n}}$, the number n of the observations being quite great, the term depending on the first power of ζ, in the preceding function, is of order $\frac{1}{\sqrt{n}}$; one is able therefore to neglect it, thus as the last term of this function. By designating therefore by $\mathrm{S}p_i^2$ the entire sum

$$p_1^2 + p_2^2 + \cdots + p_n^2,$$

the probability of ζ will be proportional to

$$c^{-\frac{\zeta^2}{2}\left[\frac{\phi(0)}{k}\right]^2 \mathrm{S}p_i^2},$$

ζ or $\frac{x_r}{p_r}$ being the error of the value $\frac{a_r}{p_r}$ given for y by the r^{th} equation. The value given by the most advantageous method is, by the preceding section,

$$y = \frac{\mathrm{S}p_i a_i}{\mathrm{S}p_i^2},$$

and the probability of an error ζ in this result is proportional to

$$c^{-\frac{k}{2k''}\zeta^2 \mathrm{S}p_i^2},$$

k'' being always the integral $\int x^2\, dx \phi(x)$, taken from x null to x infinity. The result of the method that we just examined, and that we will name method of situation, will be preferable to the one of the most advantageous method, if the coefficient of $-\zeta^2$, which is relative to it, surpasses the coefficient relative to the most advantageous method, because then the law of probability of the errors will be more rapidly decreasing there. Thus, the method of situation must be preferred if one has

$$\left[\frac{\phi(0)}{k}\right]^2 > \frac{k}{k''};$$

in the contrary case, the most advantageous method is preferable. If one has, for example,

$$\phi(x) = c^{-hx^2},$$

k becomes $\frac{\sqrt{\pi}}{2\sqrt{h}}$ and k'' becomes $\frac{\sqrt{\pi}}{4h\sqrt{h}}$; this which gives $\frac{k}{k''} = 2h$. The quantity $\left[\frac{\phi(0)}{k}\right]^2$ becomes $\frac{4h}{\pi}$; or one has $2h > \frac{4h}{\pi}$; the most advantageous method must therefore then be preferred.

By combining the results of these two methods, one is able to obtain a result of which the law of probability of the errors is more rapidly decreasing. We name always ζ the error of the result of the method of situation, and we designate by ζ' the error of the result of the most advantageous method. The first of these results is, as one has seen, $\frac{a_r}{p_r}$, and the second is $\frac{\mathrm{S}p_i a_i}{\mathrm{S}p_i^2}$. If one designates $\mathrm{S}p_i x_i$ by l, $\frac{l}{\mathrm{S}p_i^2}$ will be the error of this last result; thus one will have $l = \zeta'\mathrm{S}p_i^2$. The probability of the simultaneous existence of l and of ζ is, by nº 21 of Book II, proportional to

$$\int dw\, c^{-lw\sqrt{-1}}\phi(p_r\zeta)c^{p_r\zeta w\sqrt{-1}}\int dx\phi(x)c^{p_1xw\sqrt{-1}}\int dx\phi(x)c^{p_2xw\sqrt{-1}}\cdots,$$

the integral relative to w being taken from $w = -\pi$ to $w = \pi$. The integral relative to x, in the factor $\int dx\phi(x)c^{p_1xw\sqrt{-1}}$, must be taken, by that which precedes, from $x = p_1\zeta$ to $x = \infty$. In developing this factor according to the powers of x, it becomes

$$\int dx\phi(x) + p_1w\sqrt{-1}\int xdx\phi(x) - p_1^2\frac{w^2}{2}\int x^2dx\phi(x) + \cdots$$

By taking the integral within the preceding limits, one has, to the quantities near of order ζ^3,

$$\int dx\phi(x) = k - p_1\zeta\phi(0).$$

By neglecting similarly the quantities of the orders ζ^2w, ζ^3w^2, ..., one has

$$p_1w\sqrt{-1}\int xdx\phi(x) = k'p_1w\sqrt{-1}, \quad \frac{-p_1^2}{2}w^2\int x^2dx\phi(x) = -\frac{k''}{2}p_1^2w^2,$$

k' being the integral $\int xdx\phi(x)$ taken from $x = 0$ to x infinity. The factor of which there is question becomes therefore, by neglecting w^3, conformably to the analysis of the section cited from Book II,

$$k - p_1\zeta\phi(0) + k'p_1w\sqrt{-1} - \frac{k''}{2}p_1^2w^2.$$

Its hyperbolic logarithm is

$$p_1\zeta\frac{\phi(0)}{k} + \frac{k'}{k}p_1w\sqrt{-1} - \frac{k''}{2k}p_1^2w^2 - \frac{p_1^2}{2}\left[\zeta\frac{\phi(0)}{k} - \frac{k'}{k}w\sqrt{-1}\right]^2 + \log k.$$

By changing p_1 successively into $p_2, p_3, \ldots, p_{r-1}$, one will have the logarithms of the factors following, to the factor relative to p_{r-1}.

In the factor $\int dx\phi(x)c^{p_{r+1}xw\sqrt{-1}}$, the integral must be taken from $x = -\infty$ to $x = p_{r+1}\zeta$; then $\int xdx\phi(x)$ becomes $-k'$, the logarithm of this factor is

$$\begin{aligned} p_{r+1}\zeta\frac{\phi(0)}{k} - \frac{k'}{k}p_{r+1}w\sqrt{-1} - \frac{k''}{2k}p_{r+1}^2w^2 \\ - \frac{p_{r+1}^2}{2}\left[\zeta\frac{\phi(0)}{k} - \frac{k'}{k}w\sqrt{-1}\right]^2 + \log k. \end{aligned}$$

One will have the logarithms of the factors following by changing p_{r+1} successively into $p_{r+2}, p_{r+3}, \ldots, p_n$. The factor $\phi(p_r\zeta)c^{p_r\zeta w\sqrt{-1}}$ is equal to

$$\left[\phi(0) + \frac{p_r^2\zeta^2}{2}\right]\phi''(0)c^{p_r\zeta w\sqrt{-1}},$$

and its logarithm is

$$\frac{p_r^2}{2}\zeta^2\frac{\phi''(0)}{\phi(0)} + p_r\zeta w\sqrt{-1} + \log\phi(0).$$

Now, if one reassembles all these logarithms, if one considers next the conditions (c) to which the r^{th} equation is subject, finally if one passes again from the logarithms to the numbers, one finds, by neglecting that which it is permissible to neglect, that the probability of the simultaneous existence of l and of ζ is proportional to

$$\int d\phi\, c^{-lw\sqrt{-1}-\left\{\left[\zeta\frac{\phi(0)}{k}-\frac{k'}{k}w\sqrt{-1}\right]^2+\frac{k''}{k}w^2\right\}^2\frac{\mathrm{S}p_i^2}{2}}$$

By making therefore

$$F = \left(\frac{k''}{k}-\frac{k'^2}{k^2}\right)^2\frac{\mathrm{S}p_i^2}{2},$$

the probability of the simultaneous existence of ζ and of ζ' will be proportional to

$$c^{-\frac{\zeta^2}{2}\left[\frac{\phi(0)}{k}\right]^2\mathrm{S}p_i^2-\frac{\left[\zeta'-\zeta\frac{k'}{k}\frac{\phi(0)}{k}\right]^2}{4F}(\mathrm{S}p_i^2)^2}\int dw\, c^{-F\left\{w+\frac{\left[\zeta'-\zeta\frac{k'}{k}\frac{\phi(0)}{k}\right]\sqrt{-1}\,\mathrm{S}p_i^2}{2F}\right\}}.$$

By the analysis of n° 21 of Book II, the integral relative to w is able to be taken from $w=-\infty$ to $w=\infty$, and then the preceding probability becomes proportional to

$$c^{-\frac{\zeta^2}{2}\mathrm{S}p_i^2\left[\frac{\phi(0)}{k}\right]^2-\frac{\left[\zeta'-\zeta\frac{k'}{k}\frac{\phi(0)}{k}\right]^2}{2\left(\frac{k''}{k}-\frac{k'^2}{k^2}\right)}\mathrm{S}p_i^2},$$

an expression that one is able to set yet under this form

$$c^{-\frac{k}{2k''}\zeta'^2\mathrm{S}p_i^2-\frac{k''}{k}\frac{\left[\zeta\frac{\phi(0)}{k}-\zeta'\frac{k'}{k}\right]^2}{2\left(\frac{k''}{k}-\frac{k'^2}{k^2}\right)}\mathrm{S}p_i^2}.$$

If one names e the excess of the value of y given by the most advantageous method over that which the method of *situation* gives, one will have $\zeta=\zeta'-e$. We suppose

$$\zeta' = u + \frac{e\frac{\phi(0)}{k}\left[\frac{\phi(0)}{k}-\frac{k'}{k''}\right]}{\frac{k}{k''}-\frac{k'^2}{k''^2}+\left[\frac{\phi(0)}{k}-\frac{k'}{k''}\right]^2};$$

the probability of u will be proportional to

$$c^{-\frac{u^2}{2}\mathrm{S}p_i^2\left\{\frac{k}{k''}+\frac{\frac{k''}{k}\left[\frac{\phi(0)}{k}-\frac{k'}{k''}\right]^2}{\frac{k''}{k}-\frac{k'2}{k2}}\right\}};$$

the result of the most advantageous method must therefore be diminished by the quantity

$$\frac{e\frac{\phi(0)}{k}\left[\frac{\phi(0)}{k}-\frac{k'}{k''}\right]}{\frac{k}{k''}-\frac{k'^2}{k''^2}+\left[\frac{\phi(0)}{k}-\frac{k'}{k''}\right]^2};$$

and the probability of the error u, in this result thus corrected, will be proportional to the preceding exponential. The weight of the new result will be augmented, if $\frac{\phi(0)}{k} - \frac{k'}{k''}$ is not null; there is therefore advantage to correct thus the result of the most advantageous method. Ignorance where one is of the law of probability of the errors of the observations renders this correction impractical; but it is remarkable that, in the case where this probability is proportional to c^{-hx^2}, that is to say where one has $\phi(x) = c^{-hx^2}$, the quantity $\frac{\phi(0)}{k} - \frac{k'}{k''}$ is null. Then the result of the most advantageous method receives no correction of the result of the method of situation, and the law of probability of errors remains the same.