BOOK II
CHAPTER XI
DE LA PROBABILITÉ DES TÉMOIGNAGES.

Pierre Simon Laplace*

Théorie Analytique des Probabilités **7** §§44–50, pp. 455–470

ON THE PROBABILITY OF TESTIMONIES

One has extracted a ball from an urn which contains the number n of them; a witness of this drawing, of whom the veracity and the probability that he is not mistaken at all are supposed known, announces the exit of the n^o^ i; one demands the probability of this exit. N° 44.

One has extracted a ball from an urn which contains $n - 1$ black balls and one white ball. A witness to the drawing announces that the extracted ball is white; one demands the probability of this exit. If the number n is very great, this which renders extraordinary the exit of the white ball, the probability of the error or of the falsehood of the witness becomes quite near to certitude, this which shows how the extraordinary facts weaken the belief due to the testimonies. N° 45.

Urn A contains n white balls, urn B contains the same number of black balls; one has extracted a ball from one of these urns and one has put it into the other urn from which one has next extracted a ball. A witness of the first drawing has seen a white ball exit. A witness of the second drawing announces that he has seen similarly a white ball extracted. One demands the probability of this double exit. In order that this double exit take place, it is necessary that a white ball extracted from urn A in the first drawing, put next into urn B, has been extracted from it in the second drawing, this which is a quite extraordinary event, when the number n of black balls with which one has mixed it is very considerable. The probability of this event becomes then very small; whence it follows that the probability of the fact, resulting from the collection of many testimonies, decreases in measure as this fact becomes more extraordinary. N° 46.

Two witnesses attest to the exit of the n° i from an urn which contains the number n of them, and of which one has extracted only one ticket. One demands the probability of this exit.

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One of the witnesses attests to the exit of the n^o^ i and the other attests to the exit of the n^o^ i'; to determine the probability of the exit of the n^o^ i. N° 47.

One or many traditional successions of r witnesses transmit the exit of the n^o^ i from an urn which contains the number n of them; to determine the probability of this exit. N° 48.

One knows the respective veracities of two witnesses, of whom at least one, and perhaps two, attest to the exit of the n° i from one urn which contains the number n of them; to determine the probability of this exit. N° 49.

The judgments of the tribunals are able to be assimilated to the witnesses. *To determine the probability of the goodness of these judgments*. N° 50.

44. I will first consider a single witness. The probability of his testimony is composed of his veracity, of the possibility of his error and of the possibility of the fact in itself. In order to fix the ideas, we imagine that one has extracted a ticket from an urn which contains the number n of them, and that a witness of the drawing announces that the n^o i is exited. The observed event is here the witness announcing the exit of the n^o i. Let p be the veracity of the witness, or the probability that he will not at all seek to deceive; let further r be the probability that he is not deceived at all. This put:

One is able to form the following four hypotheses. Either the witness does not deceive at all and is not deceived at all; or he deceives not at all and is deceived; or he deceives and is not deceived at all; finally, or he deceives and is deceived at the same time. We will see what is, *a priori*, under each of these hypotheses, the probability that the witness will announce the exit of the n^o i.

If the witness does not deceive at all and is not deceived at all, the n^o i will be exited: but the probability of this exit is a priori $\frac{1}{n}$; by multiplying it by the probability pr of the hypothesis, one will have $\frac{pr}{n}$ for the entire probability of the observed event under this first hypothesis.

If the witness does not deceive at all and is deceived, the n^o i must not be exited at all, in order that he announces its exit; the probability of that is $\frac{n-1}{n}$. But the error of the witness must carry over one of the non-exited tickets. We suppose that it is able to carry equally over all: the probability that it will carry over the n^o i will be $\frac{1}{n-1}$; the probability that the witness not deceiving at all and being deceived will announce the n^o i is therefore $\frac{n-1}{n}\frac{1}{n-1}$ or $\frac{1}{n}$. By multiplying it by the probability $p(1-r)$ of the hypothesis itself, one will have $\frac{p(1-r)}{n}$ for the probability of the observed event under this second hypothesis.

If the witness deceives and is not deceived at all, the n^o i will not be exited at all, and the probability of that is $\frac{n-1}{n}$; but the witness must choose, among the $n-1$ tickets not exited, the n^o i. If one supposes that his choice is able equally to carry over each of them, $\frac{1}{n-1}$ will be the probability that his choice will be fixed on the n^o i; $\frac{n-1}{n}\frac{1}{n-1}$ or $\frac{1}{n}$ is therefore the probability that the witness will announce the n^o i. By multiplying it by the probability $(1-p)r$ of the hypothesis, one will have $\frac{(1-p)r}{n}$ for the entire probability of the observed event under this third hypothesis.

Finally, if the witness deceives and is deceived, the probability that he will not believe the n^o i exited will be $\frac{n-1}{n}$, and the probability that he will choose it among the $n-1$ tickets that he will not believe exited will be $\frac{1}{n-1}$; $\frac{n-1}{n}\frac{1}{n-1}$ or $\frac{1}{n}$ will be therefore the probability that he will announce the exit of n^o i. By multiplying it by the probability $(1-p)(1-r)$ of the hypothesis, one will have $\frac{(1-p)(1-r)}{n}$ for the probability of the observed event under this fourth hypothesis.

This hypothesis contains one case in which the n^o i is exited, namely the case in which, the n^o i being exited, the witness believes it not exited, and he chooses it among the $n-1$ tickets that he believes not exited. The probability of that is the product of $\frac{1}{n}$ by $\frac{1}{n-1}$. By multiplying this product by the probability $(1-p)(1-r)$ of the hypothesis, one will have $\frac{(1-p)(1-r)}{n(n-1)}$ for the probability of the case of which there is concern.

One is able to arrive to the same results in this manner. Let a, b, c, d, i, ... be the n tickets. Since the witness is deceived, he must not believe exited at all the exited ticket, and since he deceives, he must not announce at all as exited the ticket that he believes exited. We put therefore, in the first place the exited ticket, in the second the ticket that the witness believes exited, and in the third the ticket that he announces. Among all the possible combinations of the tickets three by three, without excluding those where they are repeated, there are compatibles with the present hypothesis only those where the ticket which occupies the second place occupies neither the first, nor the third; such are the combinations aba, abc, ... Now it is easy to see that the number of combinations which satisfy the two preceding conditions is $n(n-1)^2$; because the combination ab is able to be combined with the $n-1$ tickets other than b, and the number of combinations ab, ba, ac is $n(n-1)$. However the combinations in which the n° i is announced without being exited are of the form abi, bai, aci, ..., and the number of these combinations is $(n-1)(n-2)$; thus the probability that one of these combinations will take place is $\frac{n-2}{n(n-1)}$. The combinations in which the n° i being exited, it is announced, are of the form iai, ibi, ... and the number of these combinations is clearly $n-1$; the probability that one of these combinations will take place is therefore $\frac{1}{n(n-1)}$. It is necessary to multiply all these combinations by the probability $(1-p)(1-r)$ of the hypothesis, and then one will have the preceding results.

Now, in order to have the probability of the exit of the n° i , one must make a sum of all the preceding probabilities, relative to this exit, and to divide it by the sum of all these probabilities, this which gives, for this probability,

$$\frac{\frac{pr}{n}+\frac{(1-p)(1-r)}{n(n-1)}}{\frac{pr}{n}+\frac{p(1-r)}{n}+\frac{(1-p)r}{n}+\frac{(1-p)(1-r)}{n}} \quad \text{or} \quad pr+\frac{(1-p)(1-r)}{n-1}.$$

If r is equal to unity, or if the witness is not deceived at all, the probability of the exit of the n° i will be p, that is to say the probability of the veracity of the witness.

If n is a very great number, this probability will be very nearly pr or the probability of the veracity of the witness, multiplied by the probability that he is not deceived at all.

We have supposed that the error of the witness, when he is deceived, is able equally to fall on all the tickets non-exited; but this supposition ceases to hold, if some of them have more resemblance than the others with the exited ticket, because the mistake in this regard is easier. We have further supposed that the witness, when he deceives, has no motive in order to choose one ticket rather than another, this which is able to not take place. But it will be very difficult to make enter into a formula all these particular considerations.

45. Suppose now that the urn contains $n-1$ black balls and one white ball, and that by having extracted one ball, a witness of the drawing announces the exit of a white ball. We determine the probability of this exit. We will form the same hypotheses as we have just made. In the first, the probability of the exit of the white ball is, as above, $\frac{pr}{n}$. Under the second hypothesis, the witness is being deceived without deceiving, a black ball must be exited, and the probability of that is $\frac{n-1}{n}$, and as the witness, supposed truthful, must announce the exit of a white ball, by that alone that he is mistaken, the probability of this announcement will be therefore $\frac{n-1}{n}$, a probability that it is

necessary to multiply by the probability $p(1-r)$ of the hypothesis, this which gives $\frac{p(1-r)(n-1)}{n}$ for the probability of the event observed under this hypothesis. Under the third hypothesis, the witness being supposed deceiving and not at all being deceived, a black ball must be exited, and the probability of that is $\frac{n-1}{n}$. By multiplying it by the probability $(1-p)r$ of this hypothesis, one will have $\frac{(1-p)r(n-1)}{n}$ for the probability of the observed event, under this hypothesis. Finally, under the fourth hypothesis, the witness, deceiving and being deceived, is able to announce the exit of the white ball only as long as it will be exited. The probability of this exit is $\frac{1}{n}$. By multiplying it by the probability $(1-p)(1-r)$ of the hypothesis, one will have $\frac{(1-p)(1-r)}{n}$ for the probability of the observed event, under this hypothesis.

Presently, if one reunites among the preceding probabilities those in which the white ball is exited, one will have the probability of this exit, by dividing their sum by the sum of all the probabilities, this which gives

$$\frac{pr+(1-p)(1-r)}{pr+(1-p)(1-r)+[p(1-r)+(1-p)r](n-1)}$$

for the probability of the exit of the white ball; consequently

$$\frac{[p(1-r)+(1-p)r](n-1)}{pr+(1-p)(1-r)+[p(1-r)+(1-p)r](n-1)}$$

is the probability that the fact attested by the witness of the drawing has not taken place.

One is able to observe here that, if one names q the probability that the witness announces the truth, one will have

$$q = pr + (1-p)(1-r);$$

because it is clear that he spoke true, in the case of which there is concern, either that he deceives not at all and is not deceived at all, or that he deceives and is deceived. This expression of q gives

$$1-q = p(1-r) + (1-p)r.$$

In fact, the probability $1-q$ that he does not enunciate the truth is the probability that he deceives not at all and is deceived, plus the probability that he deceives and is not deceived at all. The preceding expression of the probability that the attested fact is false becomes thus

$$\frac{(1-q)(n-1)}{q+(1-q)(n-1)}.$$

If the number $n-1$ of black balls is very great, this probability becomes to very nearly equal to unity or to certitude, if the error or the mistake of the witness is in the least probable. Then the fact that he attests becomes extraordinary. Thus one sees how the extraordinary facts weaken the belief due to the witnesses, the mistake or the error becoming so much more possible as the attested fact is the more extraordinary in itself.

46. We consider presently two urns A and B, of which the first contains a great number n of white balls, and the second the same number of black balls. One draws

from one of these urns a ball that one replaces into the second urn; next one draws a ball from this second urn. A witness of the first drawing attests that one white ball is exited; a witness of the second drawing attests similarly that he has extracted a white ball. Each of these testimonies, considered isolated, offers nothing of the unlikely. But the consequence which results from their collection is that the same ball, exited from the first drawing, has reappeared in the second, this which is a phenomenon so much more extraordinary as n is a greater number. We will see how the value of these testimonies is weakened from it.

Let us name q the probability that the first witness enunciates the truth. One sees, by the preceding section, that in the present case this probability is composed of the probability that the witness deceives not at all and is not deceived at all, added to the probability that he deceives and is deceived at the same time; because the witness, in these two cases, enunciates the truth. Let q' be the same probability relative to the second witness. One is able to form these four hypotheses: either the first and the second witness say the truth; or the first says the truth, the second does not say it; or the second witness says the truth, the first does not say the truth at all; or finally neither of the two say the truth. We determine *a priori*, under each of these hypotheses, the probability of the observed event.

This event is the announcement by the exit of one white ball at each drawing. The probability that one white ball is exited at the first drawing is $\frac{1}{2}$, since the ball extracted is able to be equally exited from urn A or from urn B. In the case where it has been extracted from urn A and put into urn B, $n+1$ balls are contained in this last urn, and the probability to extract from it a white ball is $\frac{1}{n+1}$; the product of $\frac{1}{2}$ by $\frac{1}{n+1}$ is therefore the probability *a priori* of the extraction of one white ball in the two consecutive drawings. In multiplying it by the probability qq' that the two witnesses say the truth, one will have

$$\frac{qq'}{2(n+1)}$$

for the probability of the observed event, under the first hypothesis.

Under the second hypothesis, the ball has been extracted from urn A and put into urn B: the probability of this extraction is $\frac{1}{2}$. Moreover, since the second witness does not say the truth, a black ball has been extracted from urn B, and the probability of this extraction is $\frac{n}{n+1}$. By multiplying therefore $\frac{1}{2}$ by $\frac{n}{n+1}$, and the product by the probability $q(1+q')$ that the first witness says the truth while the second does not say it, one will have

$$\frac{q(1-q')n}{2(n+1)}$$

for the probability of the observed event under the second hypothesis.

Under the third hypothesis, a black ball has been extracted from urn B and put into urn A: the probability of this extraction is $\frac{1}{2}$. Moreover, a white ball has been further extracted from urn A, and the probability of this extraction is $\frac{n}{n+1}$; by multiplying therefore $\frac{1}{2}$ by $\frac{n}{n+1}$, and the product by the probability $(1-q)q'$ that the second witness says the truth, while the first does not say it, one will have

$$\frac{(1-q)q'n}{2(n+1)},$$

for the probability relative to the third hypothesis.

Finally, under the fourth hypothesis, a black ball has first been extracted from urn B, and the probability of this extraction is $\frac{1}{2}$. Next this black ball, put into urn A, has been extracted from it in the second drawing, and the probability of this extraction is $\frac{1}{n+1}$; by multiplying therefore the product of these two probabilities by the probability $(1-q)(1-q')$ that none of the witnesses says the truth, one will have

$$\frac{(1-q)(1-q')}{2(n+1)}$$

for the probability relative to the fourth hypothesis.

Now the probability of the fact which results from the collection of the two testimonies, namely, that a white ball extracted in the first drawing has reappeared in the second drawing, is clearly equal to the probability relative to the first hypothesis divided by the sum of the probabilities relative to the four hypotheses; this probability is therefore

$$\frac{qq'}{qq' + (1-q)(1-q') + [q(1-q') + q'(1-q)]n}.$$

The phenomenon of the reappearance of a white ball in the second drawing becomes so much more extraordinary as the number n of balls of each urn is more considerable, and then the preceding probability becomes very small. One sees therefore that the probability of the fact resulting from the collection of the witnesses is extremely weak, when it is extraordinary.

47. Consider simultaneous witnesses: we suppose two witnesses in accord on a fact, and we determine its probability. In order to fix the ideas, we suppose that the fact is the extraction of the n° i from an urn which contains the number n of them, such that the observed event is the accord of two witnesses of the drawing to enunciate the exit of the n° i. We name p and p' their respective veracities, and we suppose, in order to simplify, that they are not deceived at all. This put, one is able to form only these two hypotheses: the witnesses say the truth; the witnesses deceive.

Under the first hypothesis, the n° i is exited, and the probability of this event is $\frac{1}{n}$. By multiplying it by the product of the veracities p and p' of the witnesses, one will have $\frac{pp'}{n}$ for the probability of the observed event, under this hypothesis.

In the second, the n° i is not exited, and the probability of this event is $\frac{n-1}{n}$; but the two witnesses are agreed to choose the n° i among the $n-1$ non-exited tickets. Now the number of different combinations which are able to result from their choice is $(n-1)^2$, and in this number they must choose that where the n° i is combined with itself; the probability of this choice is therefore $\frac{1}{(n-1)^2}$. By multiplying it by the preceding probability $\frac{n-1}{n}$, and by the products of the probabilities $1-p$ and $1-p'$ that the witnesses deceive, one will have $\frac{(1-p)(1-p')}{n(n-1)}$ for the probability of the observed event, under the second hypothesis.

Now, one will have the probability of the exit of the n° i by dividing the probability relative to the first hypothesis by the sum of the probabilities relative to the two hypotheses; one will have therefore, for this probability,

$$(o) \qquad \frac{pp'}{pp' + \frac{(1-p)(1-p')}{n-1}}.$$

If $n = 2$, then the exit of the n$^{\text{o}}$ i is as probable as its non-exit, and the probability of its exit, resulting from the accord of the testimonies, is

$$\frac{pp'}{pp' + (1-p)(1-p')}.$$

This is generally the probability of a fact attested by two witnesses, when the existence of the fact is as probable as its nonexistence. If the two witnesses are equally truthful, this which gives $p' = p$, this probability becomes

$$\frac{p^2}{p^2 + (1-p)^2}.$$

In general, if the number r of the equally truthful witnesses affirm the existence of a fact of this kind, its probability resulting from the testimonies will be

$$\frac{p^r}{p^r + (1-p)^r}.$$

But this formula is applicable only in the case where the existence of the fact and its nonexistence are in themselves equally probable.

If the number n of the tickets of the urn is very great, formula (o) becomes to very nearly unity, and consequently the exit of n$^{\text{o}}$ i is extremely probable. That holds to this that it is very little credible that the witnesses, wishing to deceive, are agreed to enunciate the same ticket, when the urn contains a great number of them. Simple good sense indicates this result from the calculus; but one sees at the same time that the probability of the exit of the n$^{\text{o}}$ i is much diminished, if the two witnesses, seeking to deceive, have been able to hear one another.

Let us suppose now that the first witness affirms the exit of the n$^{\text{o}}$ i, and that the second witness affirms the exit of the n$^{\text{o}}$ i'. One is able to form then the following three hypotheses: the first witness says the truth and the second deceives; in this case the n$^{\text{o}}$ i is exited, and the probability of this event is $\frac{1}{n}$; moreover, the second witness, who deceives, must choose among the other non-exited tickets the n$^{\text{o}}$ i', and the probability of this choice is $\frac{1}{n-1}$. The product of these two probabilities by the product of the probabilities p and $1-p'$, that the first witness not deceive and that the second deceive, will be the probability of the observed event or of the enunciation of the exit of the n$^{\text{os}}$ i and i', under this hypothesis, a probability which is thus $\frac{p(1-p')}{n(n-1)}$.

Under the second hypothesis, the first witness deceives and the second does not deceive. Then the n$^{\text{o}}$ i' is exited, and the probability of this event is $\frac{1}{n}$. Moreover, the first witness chooses the n$^{\text{o}}$ i out of the $n-1$ non-exited tickets, and the probability of this choice is $\frac{1}{n-1}$. By multiplying the product of these two probabilities by the product of the probabilities $1-p$ and p', that the first witness deceives and that the second does not deceive, one will have $\frac{(1-p)p'}{n(n-1)}$.

Finally, under the third hypothesis, the two witnesses deceive at the same time. Then none of the two tickets i and i' is exited. The probability of this event is $\frac{n-2}{n}$. Moreover, the first witness must choose the n° i, and the second must choose the n° i', among the $n-1$ non-exited tickets, and the probability of this composite event is $\frac{1}{(n-1)^2}$. By multiplying the product of these two probabilities by the product of the probabilities $1-p$ and $1-p'$ that the first and the second witness deceive, one will have $\frac{(n-2)(1-p)(1-p')}{n(n-1)^2}$ for the probability of the observed event, under this hypothesis.

Now one will have the probability of the exit of the n° i, by dividing the probability relative to the first hypothesis by the sum of the probabilities relative to the three hypotheses; the probability of this exit is therefore

$$\frac{p(1-p')}{1-pp'-\frac{(1-p)(1-p')}{n-1}}.$$

If $n=2$, that is to say if the existence of each fact attested by the two witnesses is *a priori* as probable as its nonexistence, then the preceding probability becomes $\frac{1}{2}$, when $p=p'$, this which is clear besides, the two testimonies destroying themselves reciprocally. In general, if a fact of this kind is attested by r witnesses and denied by r' witnesses, all equally truthful, it is easy to see that its probability will be

$$\frac{p^{r-r'}}{p^{r-r'}+(1-p)^{r-r'}},$$

that is to say the same as if the fact were attested by $r-r'$ witnesses.

48. We will consider presently a traditional chain of r witnesses, and we suppose that the transmitted fact is the exit of the n° i from an urn which contains n tickets. We designate by y_r its probability. The addition of a new witness will change this probability into y_{r+1}, a probability which will be formed: 1° from the product of y_r by the veracity of the new witness, a veracity that we will designate by p_{r+1}; 2° from the product of the probability $1-p_{r+1}$ that this new witness deceives, by the probability $1-y_r$ that the preceding witness has not said the truth, and by the probability $\frac{1}{n-1}$ that the new witness will choose the exited ticket, in the number of the $n-1$ other tickets than the one which has been indicated to him by the preceding witness; therefore one will have

$$y_{r+1}=p_{r+1}y_r+\frac{1}{n-1}(1-p_{r+1})(1-y_r),$$

an equation of which the integral is

$$y_r=\frac{1}{n}+C\frac{(np_1-1)(np_2-1)\cdots(np_r-1)}{(n-1)^r},$$

C being an arbitrary constant. In order to determine it, one will observe that the probability of the fact, after the first testimony, is, by that which precedes, equal to p_1; one has therefore $y_1=p_1$, this which gives $C=\frac{n-1}{n}$; hence

$$y_r=\frac{1}{n}+\frac{n-1}{n}\,\frac{(np_1-1)(np_2-1)\cdots(np_r-1)}{(n-1)^r}.$$

If n is infinite, one has

$$y_r = p_1 p_2 \dots p_r.$$

If $n = 2$, that is to say if the existence of the fact is as probable as its nonexistence, one has

$$y_r = \frac{1}{2} + \frac{1}{2}(2p_1 - 1)(2p_2 - 1)\cdots(2p_r - 1).$$

In general, in measure as the traditional chain is prolonged, y_r approaches indefinitely to its limit $\frac{1}{n}$, a limit which is the probability, *a priori*, of the exit of the n° i. The term $\frac{n-1}{n}\,\frac{np_1-1}{n-1}\cdots$ of the expression of y_r is therefore that which the chain of witnesses adds to this probability. One sees thus how the probability is weakened in measure as the tradition is prolonged. In truth, the monuments, printings and other causes are able to diminish this inevitable effect of the times; but they are never able to entirely destroy it.

If one has two traditional chains, each of r witnesses, if one supposes the witnesses of these chains equally truthful and if the last witness of the one of the chains accords with the last of the other to affirm the exit of the n° i, one will have the probability of this exit, by substituting y_r for p and p' in formula (o) of the preceding section, which becomes thence

$$\frac{y_r^2}{y_r^2 + \frac{(1-y_r)^2}{n-1}}.$$

49. We consider two witnesses of which p and p' are the respective veracities. One knows that both or at least one of the them, without being contradicted by the other who, in this case, has not pronounced at all, affirm that the n° i is exited from one urn which contains the number n of them. By supposing always that one has extracted only a single ticket, one demands the probability of the exit of the n° i.

Let r and r' be the respective probabilities that the witnesses pronounce. One is able to make here only the following four hypotheses: 1° the two witnesses pronounce and say the truth; 2° the two witnesses pronounce and deceive; 3° one of the witnesses pronounces and says the truth, and the other witness does not pronounce; 4° one of the witnesses pronounces and deceives, and the other does not pronounce.

Under the first hypothesis, the n° i is exited, and the probability of this event is $\frac{1}{n}$. It is necessary to multiply it by the product of the probabilities r and r' that the two witnesses have pronounced, and by the product of the probabilities p and p' that they say the truth; one will have thus

$$\frac{pp'.rr'}{n}$$

for the probability of the observed event, under this hypothesis.

In the second, the n° i is not exited, and the probability of this event is $\frac{n-1}{n}$. But, if the two witnesses deceive without hearing one another, the probability that they will agree to enunciate the same n° i is $\frac{1}{(n-1)^2}$. It is necessary to multiply the product of these probabilities by the probability rr' that the two witnesses pronounce at the same time, and by the probability $(1-p)(1-p')$ that they both deceive. One will have thus

$$\frac{(1-p)(1-p')rr'}{n(n-1)}$$

for the probability of the observed event, under the second hypothesis.

Under the third, the n° i is exited, and the probability of this event is $\frac{1}{n}$. It is necessary to multiply by the probability $pr(1-r')+p'r'(1-r)$ that one of the witnesses pronounces by saying the truth, while the other witness does not pronounce it at all. One will have thus

$$\frac{pr(1-r')+p'r'(1-r)}{n}$$

for the probability of the observed event under this hypothesis.

Finally, under the fourth, the n° i is not exited, and the probability of this event is $\frac{n-1}{n}$; but the witness who deceives must choose it in the $n-1$ non-exited tickets, and the probability of this choice is $\frac{1}{n-1}$. It is necessary to multiply the product of these probabilities by the probability $(1-p)r(1-r')+(1-p')r'(1-r)$ that one of the witnesses pronouncing deceives, while the other witness does not pronounce it at all. One has thus

$$\frac{(1-p)r(1-r')+(1-p')r'(1-r)}{n}$$

for the probability corresponding to the fourth hypothesis.

Now one will have the probability of the exit of the n° i, by dividing the sum of the probabilities relative to the first and to the third hypothesis by the sum of the probabilities relative to all the hypotheses, this which gives, for this probability,

$$\frac{pp'rr'+pr(1-r')+p'r'(1-r)}{pp'.rr'+r(1-r')+r'(1-r)+\frac{(1-p)(1-p')rr'}{n-1}}.$$

These examples indicate sufficiently the method to subject to calculation of the probability of the testimonies.

50. One is able to assimilate the judgment of a tribunal which pronounces between two contradictory opinions to the result of the testimonies of many witnesses of the extraction of a ticket from one urn which contains only two tickets. In expressing by p the probability that the judge pronounces the truth, the probability of the goodness of a judgment rendered by unanimity will be, by that which precedes,

$$\frac{p^r}{p^r+(1-p)^r},$$

r being the number of judges. One is able to determine p by the observation of the ratio of the judgments rendered by unanimity by the tribunal to the total number of judgments. When this number is very great, by designating it by n, and by i the number of judgments rendered by unanimity, one will have to very nearly

$$p^r+(1-p)^r=\frac{i}{n};$$

the resolution of this equation will give the veracity p of the judges. This equation is reduced to a degree less than half, by making $p=1+\sqrt{u}$. It becomes thus

$$(1+\sqrt{u})^r+(1-\sqrt{u})^r=\frac{i}{n},$$

an equation which, developed, is of degree $\frac{r}{2}$ or $\frac{r-1}{2}$, according as r is even or odd.

The probability of the goodness of a new judgment rendered by unanimity will be

$$1 - \frac{n}{i}(1-p)^r.$$

If one supposes the tribunal formed of three judges, one will have

$$p = \frac{1}{2} \pm \sqrt{\frac{4i-n}{12n}}.$$

We will adopt the $+$ sign; because it is natural to suppose to each judge a greater probability for the truth than for error. If the half of the judgments rendered by the tribunal have been rendered by unanimity, then $\frac{i}{n} = \frac{1}{2}$, and one finds $p = 0,789$. The probability of a new judgment rendered by unanimity will be $0,981$. If this judgment is rendered only by plurality, its probability will be p or $0,789$.

In general, one sees that the probability $1 - \frac{n}{i}(1-p)^r$ of the goodness of a new judgment rendered by unanimity is so much greater as r is a greater number and as the values of p and of $\frac{i}{n}$ are greater, this which depends on the wisdom of the judges. There is therefore a great advantage to form the tribunals of appeal, composed of a great number of judges chosen among the most enlightened persons.