CHAPTER III

DES LOIS DE LA PROBABILITÉ QUI RÉSULTENT DE LA MULTIPLICATION INDÉFINIE DES ÉVÉNEMENTS

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Théorie Analytique des Probabilités §§16–17, pp. 280–308

ON THE LAWS OF PROBABILITY WHICH RESULT FROM THE INDEFINITE MULTIPLICATION OF EVENTS

p being the probability of the arrival of a simple event at each trial and $1 - p$ that of its non-arrival, to determine the probability that, out of a very great number of trials n, the number of times that the event will take place will be comprehended within some given limits. Solution of the problem. The number of times the most probable is np. Expression of the probability that this number of times will be comprehended within the limits $np \pm l$. The limits $\pm l$ remaining the same, this probability increases with the number of trials n: the probability remaining the same, the ratio of the interval $2l$ of the limits to the number n is tightened when n increases, and, in the case of n infinite, this ratio becomes null and the probability is changed into certitude. The solution of the preceding problem serves further to determine the probability that the value of p, supposed unknown, is comprehended within some given limits, when, out of a very great number of trials n, one knows the number of events i corresponding to p which are arrived: p is very nearly $\frac{i}{n}$, and generally when, in a trial, there must happen any one of many simple events, the respective probabilities of these events are very nearly proportionals to the number of times that they will arrive in a very great number of trials n. P being the probability of the arrival of an event composed of two simple events, of which p and $1 - p$ are the respective probabilities and $1 - P$ being the probability of the non-arrival of this composite event, if out of a very great number n of arrivals and of non-arrivals of the same event, one knows the number of these arrivals i, one has the probability that the value of P will be comprehended within some given limits, and, as P is a known function of p, one concludes from it the probability that the value of p will be comprehended within some given limits. N° 16.

An urn A containing a very great number n of white and black balls, at each drawing, one extracts one from it that one replaces with a black ball; one demands the probability that, after r drawings, the number of white balls will be x.

The solution of the problem depends on a linear equation in partial finite differences of the first order, with variable coefficients. Reduction of this equation to an equation in the infinitely small partial differences. Integration of this last equation. Application of the solution to the case where the urn is originally filled in this manner: one projects a right prism of which the base, being a regular polygon of $p+q$ sides, is rather narrow in order that the prism never falls on it; on the $p+q$ lateral faces, p are white and q are black and one puts, into urn A, at each projection, a ball of the color of the face on which the prism falls again.

Two urns A and B each contain a very great number n of white and black balls, the number of whites being equal to the one of the blacks in the totality $2n$ of balls; one draws at the same time a ball from each urn, and one places again into one urn the ball extracted from the other. By repeating this operation any number r times, one demands the probability that there will be x white balls in urn A.

The problem depends on a linear equation in the partial finite differences of the second order, with variable coefficients. Reduction of this equation to an equation in the infinitely small partial differences of the second order. Integration of this last equation by means of a definite integral. Development of this integral into series. Determination of the constants of the series by means of its initial value. Analytic theorems relative to this object. Application of the solution in the case where urn A is originally filled, as in the preceding problem. Mean value of the white balls in each urn, after r drawings. General expression of this value, in the case where one has a number e of urns disposed circularly and each containing a great number n of balls, some white and the others black, each drawing consisting in extracting at the same time one ball from each urn and placing it again into the following, by departing from one of them, in a determined sense. N° 17.

16. In measure as the events are multiplied, their respective probabilities are developed more and more; their mean results and the profits or the losses which depend on them converge toward some limits which they bring together with the probabilities always increasing. The determination of these increases and of these limits is one of the most interesting and most delicate parts of the analysis of chances.

We will consider first the manner in which the possibilities of two simple events, of which one alone must arrive at each trial, is developed when one multiplies the number of trials. It is clear that the event of which the facility is greatest must probably arrive more often in a given number of trials, and one is carried naturally to think that by repeating the trials a great number of times, each of these events will arrive proportionally to its facility, that one will be able thus to discover by experience. We will demonstrate analytically this important theorem.

One has seen in n° 6 that, if p and $1-p$ are the respective probabilities of two events a and b, the probability that in $x+x'$ trials the event a will arrive x times and the event b, x' times, is equal to

$$\frac{1.2.3\ldots(x+x')}{1.2.3\ldots x.1.2.3\ldots x'}p^x(1-p)^{x'};$$

it is the $(x'+1)^{\text{st}}$ term of the binomial $[p+(1-p)]^{x+x'}$. We will consider the greatest of these terms that we will designate by k. The anterior term will be $\frac{kp}{1-p}\frac{x'}{x+1}$, and the following term will be $k\frac{1-p}{p}\frac{x}{x'+1}$. In order that k be the greatest term, it is necessary that one has

$$\frac{x}{x'+1} < \frac{p}{1-p} < \frac{x+1}{x'};$$

it is easy to conclude from it that, if one makes $x + x' = n$, one will have

$$(n+1)p - 1 < x < (n+1)p;$$

thus x is the greatest whole number contained within $(n+1)p$; by making therefore

$$x = (n+1)p - s,$$

this which gives

$$p = \frac{x+s}{n+1}, \quad 1-p = \frac{x'+1-s}{n+1}, \quad \frac{p}{1-p} = \frac{x+s}{x'+1-s},$$

s will be less than unity. If x and x' are very great numbers, one will have, very nearly,

$$\frac{p}{1-p} = \frac{x}{x'},$$

that is to say that the exponents of p and of $1-p$ in the greatest term of the binomial are quite nearly in the ratio of these quantities; so that, in all the combinations which are able to take place in a very great number n of trials, the most probable is that in which each event is repeated proportionally to its probability.

The l^{th} term, after the greatest, is

$$\frac{1.2.3\ldots n}{1.2.3\ldots(x-l).1.2.3\ldots(x'+l)} p^{x-l}(1-p)^{x'+l}.$$

One has, by n° 33 of Book I,

$$1.2.3\ldots n = n^{n+\frac{1}{2}} c^{-n} \sqrt{2\pi}\left(1 + \frac{1}{12n} + \cdots\right),$$

this which gives

$$\frac{1}{1.2.3\ldots(x-l)} = (x-l)^{l-x-\frac{1}{2}}\frac{c^{x-l}}{\sqrt{2\pi}}\left[1-\frac{1}{12(x-l)}-\cdots\right],$$
$$\frac{1}{1.2.3\ldots(x'+l)} = (x'+l)^{-x'-l-\frac{1}{2}}\frac{c^{x'+l}}{\sqrt{2\pi}}\left[1-\frac{1}{12(x'+l)}-\cdots\right].$$

We develop the term $(x-l)^{l-x-\frac{1}{2}}$. Its hyperbolic logarithm is

$$\left(l-x-\frac{1}{2}\right)\left[\log x+\log\left(1-\frac{l}{x}\right)\right];$$

now one has

$$\log\left(1-\frac{l}{x}\right) = -\frac{l}{x}-\frac{l^2}{2x^2}-\frac{l^3}{3x^3}-\frac{l^4}{4x^4}-\cdots;$$

we will neglect the quantities of order $\frac{1}{n}$, and we will suppose that l^2 does not surpass at all the order n; then one will be able to neglect the terms of order $\frac{l^4}{x^3}$, because x and x' are of order n. One will have thus

$$\left(l-x-\frac{1}{2}\right)\left[\log x+\log\left(1-\frac{l}{x}\right)\right]$$
$$= \left(l-x-\frac{1}{2}\right)\log x + l + \frac{l}{2x} - \frac{l^2}{2x} - \frac{l^3}{6x^2},$$

this which gives, by passing again from the logarithms to the numbers,

$$(x-l)^{l-x-\frac{1}{2}} = c^{l-\frac{l^2}{2x}}x^{l-x-\frac{1}{2}}\left(1+\frac{l}{2x}-\frac{l^3}{6x^2}\right);$$

one will have similarly

$$(x'+l)^{-l-x'-\frac{1}{2}} = c^{-l-\frac{l^2}{2x'}}x'^{-l-x'-\frac{1}{2}}\left(1+\frac{l}{2x'}-\frac{l^3}{6x'^2}\right).$$

One has next, by that which precedes, $p=\frac{x+s}{n+1}$, s being less than unity; by making therefore $p=\frac{x-z}{n}$, z will be contained within the limits $\frac{x}{n+1}$ and $-\frac{n-x}{n+1}$, and consequently it will be, setting aside the sign, below unity. The value of p gives $1-p=\frac{x'+z}{n}$; one will have, by the preceding analysis,

$$p^{x-l}(1-p)^{x'+l} = \frac{x^{x-l}x'^{x'+l}}{n^n}\left(1+\frac{nzl}{xx'}\right);$$

thence one draws

$$\frac{1.2.3\dots n}{1.2.3\dots(x-l).1.2.3\dots(x'+l)}p^{x-l}(1-p)^{x'+l}$$
$$= \frac{\sqrt{n}c^{-\frac{nl^2}{2xx'}}}{\sqrt{\pi}\sqrt{2xx'}}\left[1+\frac{nzl}{xx'}+\frac{l(x'-x)}{2xx'}-\frac{l^3}{6x^2}+\frac{l^3}{6x'^2}\right].$$

One will have the anterior to the greatest term and which is extended from it at the distance l, by making l negative in this equation; by reuniting next these two terms, their sum will be

$$\frac{2\sqrt{n}}{\sqrt{\pi}\sqrt{2xx'}}c^{-\frac{nl^2}{2xx'}}.$$

The finite integral

$$\sum\frac{2\sqrt{n}}{\sqrt{\pi}\sqrt{2xx'}}c^{-\frac{nl^2}{2xx'}},$$

taken from $l=0$ inclusively, will express therefore the sum of all the terms of the binomial $[p+(1-p)]^n$, contained between the two terms, of which the one has p^{x+l} for factor, and the other has p^{x-l} for factor, and which are thus equidistant from the greatest term; but it is necessary to subtract from this sum the greatest term which is evidently contained twice.

Now, in order to have this finite integral, we will observe that one has, by n° 10 of Book I, y being function of l,

$$\sum y = \frac{1}{c^{\frac{dy}{dl}}-1} = \left(\frac{dy}{dl}\right)^{-1} - \frac{1}{2}\left(\frac{dy}{dl}\right)^{0} + \frac{1}{12}\frac{dy}{dl} + \cdots,$$

whence one draws, by the preceding number,

$$\sum y = \int y\,dl - \frac{1}{2}y + \frac{1}{12}\frac{dy}{dl} + \cdots + \text{ const.};$$

y being here equal to $\frac{2\sqrt{n}}{\sqrt{\pi}\sqrt{2xx'}}c^{-\frac{nl^2}{2xx'}}$, the successive differentials of y acquire for factor $\frac{nl}{2xx'}$, and its powers. Thus, l not being supposed to be able to be more than order $\sqrt{n}$, this factor is of order $\frac{1}{\sqrt{n}}$, and consequently its differentials, divided by the respective powers of dl, decrease more and more; by neglecting therefore, as one has done previously, the terms of order $\frac{1}{n}$, one will have, by starting with l the two finite and infinitely small integrals, and designating by Y the greatest term of the binomial,

$$\sum y = \int ydl - \frac{1}{2}y + \frac{1}{2}Y.$$

The sum of all the terms of the binomial $[p + (1 - p)]^n$ contained between the two terms equidistant from the greatest term of the number l being equal to $\sum y - \frac{1}{2}Y$, it will be

$$\int ydl - \frac{1}{2}y,$$

and if one adds there the sum of these extreme terms, one will have, for the sum of all these terms,

$$\int ydl + \frac{1}{2}y.$$

If one makes

$$t = \frac{l\sqrt{n}}{\sqrt{2xx'}},$$

this sum becomes

$$(o) \qquad \frac{2}{\sqrt{\pi}}\int dt\, c^{-t^2} + \frac{\sqrt{n}}{\sqrt{\pi}\sqrt{2xx'}}c^{-t^2}.$$

The terms that one has neglected being of the order $\frac{1}{n}$, this expression is so much more exact as n is greater; it is rigorous when n is infinity. It would be easy, by the preceding analysis, to have regard to the terms of order $\frac{1}{n}$ and of the superior orders.

One has, by that which precedes, $x = np + z$, z being a number smaller than unity; one has therefore

$$\frac{x+l}{n} - p = \frac{l+z}{n} = \frac{t\sqrt{2xx'}}{n\sqrt{n}} + \frac{z}{n};$$

thus formula (*o*) expresses the probability that the difference between the ratio of the number of times that the event a must arrive to the total number of trials, and the facility p of this event, is contained within the limits

$$(l) \qquad \pm\frac{t\sqrt{2xx'}}{n\sqrt{n}} + \frac{z}{n}.$$

$\sqrt{2xx'}$ being equal to

$$n\sqrt{2p(1-p) + \frac{2z}{n}(1-2p) - \frac{2z^2}{n^2}},$$

one sees that the interval contained between the preceding limits is of order $\frac{1}{\sqrt{n}}$.

If the limit of t, that we will designate by T, is supposed invariable, the probability determined by the function (*o*) remains very nearly the same; but the interval comprehended between the limits (*l*) diminishes without ceasing in measure as the trials repeat themselves, and it becomes null, when their number is infinite.

This interval being supposed invariable, when the events are multiplied, T increases without ceasing, and quite nearly as the square root of the number of trials. But, when T is considerable, formula (*o*) becomes, by n° 27 of Book I,

$$1 - \frac{c^{-T^2}}{2T\sqrt{\pi}}\frac{1}{1+\frac{q}{1+\frac{2q}{1+\frac{3q}{1+\ddots}}}} + \frac{c^{-T^2}}{\sqrt{2\pi n\left[p(1-p) + \frac{z}{n}(1-2p) - \frac{z^2}{n^2}\right]}},$$

q being equal to $\frac{1}{2T^2}$. When one makes T increase, c^{-T^2} diminishes with extreme rapidity, and the preceding probability approaches rapidly to unity, to which it becomes equal, when the number of trials is infinite.

There are here two sorts of approximations: the one of them is relative to the limits taken on both sides of the facility of the event a; the other approximation is related to the probability that the ratio of the arrivals of this event to the total number of trials will be contained within these limits. The indefinite repetition of the trials increases more and more this probability, the limits remaining the same; it narrows more and more the interval of these limits, the probability remaining the same. Into infinity, this interval becomes null, and the probability is changed into certitude.

The preceding analysis reunites to the advantage to demonstrate this theorem the one to assign the probability that, in a great number n of trials, the ratio of the arrivals of each event will be comprehended within some given limits. We suppose, for example, that the facilities of the births of boys and of girls are in the ratio of 18 to 17, and that there are born in one year 14000 infants; one demands the probability that the number of boys will not surpass 7363, and will not be less than 7037.

In this case, one has

$$p = \frac{18}{35}, \quad x = 7200, \quad x' = 6800, \quad n = 14000, \quad l = 163;$$

formula (*o*) gives quite nearly 0.994303 for the sought probability.

If one knows the number of times that out of n trials the event a is arrived, formula (*o*) will give the probability that its facility p, supposed unknown, will be comprehended within the given limits. In effect, if one names i this number of times, one will have, by that which precedes, the probability that the difference $\frac{i}{n} - p$ will be comprehended within the limits $\pm \frac{T\sqrt{2xx'}}{n\sqrt{n}} + \frac{z}{n}$; consequently, one will have the probability that p will be comprehended within the limits

$$\frac{i}{n} \mp \frac{T\sqrt{2xx'}}{n\sqrt{n}} - \frac{z}{n}.$$

The function $\frac{T\sqrt{2xx'}}{n\sqrt{n}}$ being of the order $\frac{1}{\sqrt{n}}$, one is able, by neglecting the quantities of order $\frac{1}{n}$, to substitute there i instead of x and $n-i$ instead of x'; the preceding limits become thus, by neglecting the terms of order $\frac{1}{n}$,

$$\frac{i}{n} \mp \frac{T\sqrt{2i(n-i)}}{n\sqrt{n}},$$

and the probability that the facility of the event a is comprehended within these limits is equal to

$$(o') \qquad \frac{2}{\sqrt{\pi}}\int dt\, c^{-t^2} + \frac{\sqrt{n}c^{-T^2}}{\sqrt{\pi}\sqrt{2i(n-i)}}.$$

One sees thus that, in measure as the events are multiplied, the interval of the limits is narrowed more and more, and the probability that the value of p falls within these limits approaches more and more unity or certitude. It is thus that the events, in being developed, make known their respective probabilities.

One arrives directly to these results, by considering p as a variable which is able to be extended from zero to unity, and by determining, after the observed events, the probability of its diverse values, as one will see it when we will treat the probability of causes deduced from observed events.

If one has three or a greater number of events a, b, c, $\dots$, of which one alone must arrive at each trial, one will have, by that which precedes, the probability that, in a very great number n of trials, the ratio of the number x of times that one of these events, a for example, will arrive, to the number n, will be comprehended within the limits $p \pm \alpha$, α being a very small fraction, and one sees that, in the extreme case of the number n infinite, the interval 2α of these limits is able to be supposed null, and the probability is able to be supposed equal to certitude, so that the numbers of arrivals at each event will be proportional to their respective facilities.

Sometimes the events, instead of making known directly the limits of the value of p, give those of a function of this value; then one concludes from it the limits of p, by the resolution of equations. In order to give a quite simple example of it, we will consider two players A and B, of whom the respective skills are p and $1 - p$, and playing together on this condition, that the set is won by the one of the two players who, out of three trials, will have vanquished twice his adversary, the third trial being not played, as useless, when one of the players has vanquished in the first two trials.

The probability of A to win the set is the sum of the first two terms of the binomial $[p + (1 - p)]^3$; it is consequently equal to $p^3 + 3p^2(1 - p)$. Let P be this function; by raising the binomial $P + (1 - P)$ to the power n, one will have, by the preceding analysis, the probability that, out of the number n of sets, the number of sets won by A will be comprehended within the given limits. It suffices for that to change p into P in formula (o).

If one names i the number of sets won by A, formula (o') will give the probability that P will be comprehended within the limits

$$\frac{i}{n} \pm \frac{T\sqrt{2i(n-i)}}{n\sqrt{n}}.$$

Let therefore p' be the real and positive root of the equation

$$p^3 + 3p^2(1-p) = \frac{i}{n};$$

by designating by $p' \mp \delta p$ the limits of p, the corresponding limits of P will be very nearly $3p'^2 - 2p'^3 \mp 6p'(1-p')\delta p$; by equating these limits to the preceding, one will have

$$\delta p = \frac{T\sqrt{2i(n-i)}}{6p'(1-p')n\sqrt{n}};$$

thus formula (o') will give the probability that p will be comprehended within the limits

$$p' \mp \frac{T\sqrt{2i(n-i)}}{6p'(1-p')n\sqrt{n}}.$$

The number n of sets does not determine the number of trials, since one is able to have some sets of two trials, and others of three trials. One will have the probability that the number of sets of two trials will be comprehended within the given limits, by observing that the probability of a set with two trials is $p^2 + (1-p)^2$; we designate this function by P'. By elevating the binomial $P' + (1-P')$ to the power n, formula (o) will give the probability that the number of sets of two trials will be comprehended within the limits $nP' \pm l$; now the number of sets of two trials being $nP' \pm l$, the number of sets with three trials will be $n(1-P') \mp l$; the total number of trials will be therefore $3n - nP' \mp l$; formula (o) will give therefore the probability that the number of trials will be comprehended within the limits

$$2n(1+p-p^2) \mp T\sqrt{2nP'(1-P')}.$$

17. We consider an urn A containing a very great number n of white and black balls, and we suppose that at each drawing one draws a ball from the urn, and that one replaces it with a black ball. One demands the probability that after r drawings the number of white balls will be x.

We name $y_{x,r}$ this probability. After a new drawing, it becomes $y_{x,r+1}$. But, in order that there are x white balls after $r+1$ drawings, it is necessary that there are either $x+1$ white balls after the drawing r and that the following drawing makes a white ball exit, or x white balls after the drawing r and that the following drawing makes a black ball exit. The probability that there will be $x+1$ white balls after r drawings is $y_{x+1,r}$ and the probability that then the following drawing will make a white ball exit is $\frac{x+1}{n}$; the probability of the composite event is therefore $\frac{x+1}{n}y_{x+1,r}$; this is the first part of $y_{x,r+1}$. The probability that there will be x white balls after the drawing r is $y_{x,r}$, and the probability that then there will exit a black ball is $\frac{n-x}{n}$, because the number of black balls in the urn is $n-x$; the probability of the composite event is therefore $\frac{n-x}{n}y_{x,r}$; this is the second part of $y_{x,r+1}$. Thus one has

$$y_{x,r+1} = \frac{x+1}{n}y_{x+1,r} + \frac{n-x}{n}y_{x,r}.$$

If one makes

$$x = nx', \quad r = nr', \quad y_{x,r} = y'_{x,r},$$

this equation becomes

$$y'_{x,r'+\frac{1}{n}} = \left(x' + \frac{1}{n}\right)y'_{x'+\frac{1}{n},r'} + (1-x')y'_{x',r'};$$

n being supposed a very great number, one is able to reduce into convergent series $y_{x,r'+\frac{1}{n}}$ and $y_{x'+\frac{1}{n},r'}$; one will have therefore, by neglecting the squares and the superior powers of $\frac{1}{n}$,

$$\frac{1}{n}\frac{\partial y'_{x',r'}}{\partial r'} = \frac{x'}{n}\frac{\partial y'_{x',r'}}{\partial x'} + \frac{1}{n}y'_{x,r'};$$

the integral of this equation in partial differences is

$$y'_{x,r'} = c^{r'}\phi(x'c^{r'}),$$

$\phi(x'c^{r'})$ being an arbitrary function of $x'c^{r'}$, that it is necessary to determine through the value of $y'_{x,0}$.

We suppose that urn A has been replenished in this manner. One projects a right prism of which the base, being a regular polygon of $p+q$ sides, is rather narrow so that the prism never falls on it. On the $p+q$ lateral faces, p are white and q are black, and one puts into urn A, at each projection, a ball of the color of the face on which the prism falls. After n projections, the number of white balls will be quite nearly, by the preceding section, $\frac{np}{p+q}$, and the probability that it will be $\frac{np}{p+q}+l$ is, by the same section,

$$\frac{p+q}{\sqrt{2npq\pi}}c^{-\frac{(p+q)^2l^2}{2npq}}.$$

If one makes

$$x=\frac{np}{p+q}+l,\qquad \frac{(p+q)^2}{2pq}=i^2,$$

this function becomes

$$\frac{i}{\sqrt{\pi n}}c^{-\frac{i^2}{n}\left(x-\frac{np}{p+q}\right)^2};$$

this is the value of $y_{x,0}$ or of $y'_{x',0}$; but the preceding value of $y'_{x',r'}$ gives

$$y_{x,0}=\phi\left(\frac{x}{n}\right);$$

one has therefore

$$\phi\left(\frac{x}{n}\right)=\frac{i}{\sqrt{n\pi}}c^{-i^2n\left(\frac{x}{n}-\frac{p}{p+q}\right)^2};$$

hence

$$y'_{x',r'}=\frac{ic^{r'}}{\sqrt{n\pi}}c^{-i^2n\left(\frac{xc^{r'}}{n}-\frac{p}{p+q}\right)^2};$$

whence one draws

$$y_{x,r}=\frac{ic^{\frac{r}{n}}}{\sqrt{n\pi}}c^{-\frac{i^2}{n}\left(xc^{\frac{r}{n}}-\frac{np}{p+q}\right)^2}.$$

The most probable value of x is that which renders $xc^{\frac{r}{n}}-\frac{np}{p+q}$ null, and consequently it is equal to

$$\frac{np}{(p+q)c^{\frac{r}{n}}};$$

the probability that the value of x will be contained within the limits

$$\frac{np}{(p+q)c^{\frac{r}{n}}} \pm \frac{\mu\sqrt{n}}{c^{\frac{r}{n}}}$$

is

$$2\int \frac{i\,d\mu}{\sqrt{\pi}} c^{-i^2\mu^2},$$

the integral being taken from $\mu = 0$.

We seek now the mean value of the number of white balls contained within urn A, after r drawings. This value is the sum of all the possible numbers of white balls, multiplied by their respective probabilities; it is therefore equal to

$$\frac{2np}{(p+q)c^{\frac{r}{n}}} \int \frac{i\,d\mu}{\sqrt{\pi}} c^{-i^2\mu^2},$$

the integral being taken from $\mu = 0$ to $\mu = \infty$. This value is thus

$$\frac{np}{(p+q)c^{\frac{r}{n}}};$$

consequently, it is the same as the most probable value of x.

We consider now two urns A and B containing each the number n of balls, and we suppose that, in the total number $2n$ of balls, there are as many white as black. We imagine that one draws at the same time one ball from each urn, and that next one sets into one urn the ball extracted from the other. We suppose that one repeats this operation any number r times, by agitating at each time the urns, in order to well mix the balls; and we seek the probability that after this number r of operations, there will be x white balls in urn A.

Let $z_{x,r}$ be this probability. The number of possible combinations in r operations is n^{2r}; because at each operation the n balls of urn A are able to be combined with each of n balls from urn B, this which produces n^2 combinations; $n^{2r} z_{x,r}$ is therefore the number of combinations in which it is possible to have x white balls in urn A after these operations. Now, it is able to happen that the $(r+1)^{\text{st}}$ operation makes a white ball exit from urn A, and makes a white ball return; the number of cases in which this is able to arrive is the product of $n^{2r}z_{x,r}$, by the number x of white balls of urn A, and by the number $n-x$ of white balls which must be then in urn B, since the total number of white balls of the two urns is n. In all these cases, there remains x white balls in urn A; the product $x(n-x)n^{2r}z_{x,r}$ is therefore one of the parts of $n^{2r+2}z_{x,r+1}$.

It is able to happen that the $(r+1)^{\text{st}}$ operation makes a black ball exit and return into urn A, this which conserves in this urn x white balls. Thus $n-x$ being, after the r^{th} operation, the number of black balls of urn A, and x being the one of black balls of urn B, $(n-x)xn^{2r}x_{x,r}$ is further a part of $n^{2r+2}z_{x,r+1}$.

If there are $x-1$ white balls in urn A after the r^{th} operation and if the operation following makes a black ball to exit from it and makes a white ball return there, there will be x white balls in urn A after the $(r+1)^{\text{st}}$ operation. The number of cases in which that is able to arrive is the product of $n^{2r}x_{x,r}$, by the number $n-x+1$ of the black balls of urn A after the r^{th} drawing, and by the number $n-x+1$ of white balls of urn B, after the same operation; $(n-x+1)^2n^{2r}z_{x-1,r}$ is therefore again a part of $n^{2r+2}z_{x,r+1}$.

Finally, if there are $x+1$ white balls in urn A after the r^{th} operation, and if the operation following makes a white ball exit from it and makes a black ball return there, there will be further, after this last operation, x white balls in the urn. The number of cases in which that is able to arrive is the product of $n^{2r}z_{x+1,r}$, by the number $x+1$ of white balls of urn A, and by the number $x+1$ of black balls of urn B after the r^{th} operation; $(x+1)^2n^{2r}z_{x+1,r}$ is therefore again part of $n^{2r+2}z_{x,r+1}$.

By reuniting all these parts and by equating their sum to $n^{2r+2}z_{x,r+1}$, one will have the equation in partial finite differences

$$z_{x,r+1}=\left(\frac{x+1}{n}\right)^2 z_{x+1,r}+\frac{2x}{n}\left(1-\frac{x}{n}\right)z_{x,r}+\left(1-\frac{x-1}{n}\right)^2 z_{x-1,r}.$$

Although this equation is in differences of the second order with respect to the variable x, however its integral contains only one arbitrary function which depends on the probability of the diverse values of x in the initial state of urn A. In effect, it is clear that, if one knew the values of $z_{x,0}$ corresponding to all the values of x from $x=0$ to $x=n$, the preceding equation will give all the values of $z_{x,1}$, $z_{x,2},\ldots,$ by observing that, the negative values of x being impossible, $z_{x,r}$ is null when x is negative.

If n is a very great number, this equation is transformed into an equation in partial differences, that one obtains thus. One has then, very nearly,

$$
\begin{aligned}
z_{x+1,r} &= z_{x,r} + \frac{\partial z_{x,r}}{\partial x} + \frac{1}{2}\frac{\partial^2 z_{x,r}}{\partial x^2},\\
z_{x-1,r} &= z_{x,r} - \frac{\partial z_{x,r}}{\partial x} + \frac{1}{2}\frac{\partial^2 z_{x,r}}{\partial x^2},\\
z_{x,r+1} &= z_{x,r} + \frac{\partial z_{x,r}}{\partial x}.
\end{aligned}
$$

Let

$$x = \frac{n+\mu\sqrt{n}}{2}, \quad r = nr', \quad z_{x,r} = U;$$

the preceding equation in the partial finite differences will become, by neglecting the terms of order $\frac{1}{n^2}$,

$$\frac{\partial U}{\partial r'} = 2U + 2\mu\frac{\partial U}{\partial \mu} + \frac{\partial^2 U}{\partial \mu^2}.$$

In order to integrate this equation, which, as one is able to assure oneself by the method that I have given for this object, in the *Mémoires de l'Académie des Sciences* of the year 1773, is integrable in finite terms only in the manner of definite integrals, we make

$$U = \int \phi\, dt c^{-\mu t},$$

ϕ being a function of t and of r'. One will have

$$
\begin{aligned}
2\mu\frac{\partial U}{\partial \mu} &= 2c^{-\mu t}t\phi - 2\int c^{-\mu t}(\phi\, dt + t d\phi),\\
\frac{\partial^2 U}{\partial \mu^2} &= \int c^{-\mu t}t^2\phi\, dt;
\end{aligned}
$$

the equation in the partial differentials in U become thus

$$\int c^{-\mu t}\frac{\partial U}{\partial r'}dt = 2c^{-\mu t}t\phi + \int c^{-\mu t}dt\left(t^2\phi - 2t\frac{d\phi}{dt}\right).$$

By equating between them the terms affected of the $\int$ sign, one will have the equation in the partial differentials

$$\frac{\partial\phi}{\phi r'} = t^2\phi - 2t\frac{\partial\phi}{\partial t}.$$

The term outside the $\int$ sign, equated to zero, will give, for the equation in the limits of the integral,

$$0 = t\phi c^{-\mu t}.$$

The integral of the preceding equation in the partial differentials of ϕ is

$$\phi = c^{\frac{1}{4}t^2}\psi\left(\frac{t}{c^{2r'}}\right),$$

$\psi\left(\frac{t}{c^{2r'}}\right)$ being an arbitrary function of $\frac{t}{c^{2r'}}$; one has therefore

$$U = \int dt c^{-\mu t + \frac{1}{4}t^2}\psi\left(\frac{t}{c^{2r'}}\right).$$

Let be

$$t = 2\mu + 2s\sqrt{-1}$$

the expression of U will take this form

(A) $$U = c^{-\mu^2}\int ds\, c^{-s^2}\Gamma\left(\frac{s-\mu\sqrt{-1}}{c^{2r'}}\right).$$

It is easy to see that the preceding equation, in the limits of the integral, requires that the limits of the integral relative to s are taken from $s = -\infty$ to $s = \infty$. By taking the radical $\sqrt{-1}$ with the $-$ sign, one will have for U an expression of this form

$$U = c^{-\mu^2}\int ds\, c^{-s^2}\Pi\left(\frac{s+\mu\sqrt{-1}}{c^{2r'}}\right),$$

the arbitrary function $\Pi(s)$ being able to be different from $\Gamma(s)$. The sum of these two expressions of U will be its complete value. But it is easy to be assured that, the integrals being taken from $s = -\infty$ to $s = \infty$, the addition of this new expression of U adds nothing to the generality of the first, in which it is comprehended.

We develop now the second member of equation (A), according to the powers of $\frac{1}{c^{2r'}}$, and we consider one of the terms of this development, such as

$$\frac{H^{(i)}c^{-\mu^2}}{c^{4ir'}}\int ds\, c^{-s^2}(s-\mu\sqrt{-1})^{2i};$$

this term becoming, after the integrations,

$$\frac{1.3.5\ldots(2i-1)}{2^i}\sqrt{\pi}\frac{H^{(i)}c^{-\mu^2}}{c^{4ir'}} \times\left[1-\frac{i(2\mu)^2}{1.2}+\frac{i(i-1)(2\mu)^4}{1.2.3.4}-\frac{i(i-1)(i-2)(2\mu)^6}{1.2.3.4.5.6}+\cdots\right]$$

We will consider further one term of this development, relative to the odd powers of $\frac{1}{c^{2r'}}$, such as

$$\frac{L^{(i)}\sqrt{-1}c^{-\mu^2}}{c^{(4i+2)r'}}\int ds\, c^{-s^2}(s-\mu\sqrt{-1})^{2i+1},$$

This term becomes, after the integrations,

$$\frac{1.3.5\ldots(2i+1)L^{(i)}\sqrt{\pi}\mu c^{-\mu^2}}{2^i c^{(4i+2)r'}}\left[1-\frac{i(2\mu)^2}{1.2.3}+\frac{i(i-1)(2\mu)^4}{1.2.3.4.5}-\cdots\right].$$

One will have thus the general expression of the probability U, developed into a series ordered according to the powers of $\frac{1}{c^{2r'}}$, a series which becomes very convergent when r' is a considerable number. This expression must be such that $\int U\,dx$ or $\frac{1}{2}\int U\,d\mu\sqrt{n}$ is equal to unity, the integrals being extended to all the values of x and of μ, that is to say from x null to $x=n$, and from $\mu=-\sqrt{n}$ to $\mu=\sqrt{n}$; because it is certain that, one of the values of x having to take place, the sum of the probabilities of all these values must be equal to unity. By taking the integral $\int c^{-\mu^2}d\mu$ within the limits of μ, one has the same result, to very nearly, as by taking it from $\mu=-\infty$ to $\mu=\infty$; the difference is only of the order $\frac{c^{-n}}{\sqrt{n}}$, and seeing the extreme rapidity with which c^{-n} diminishes in measure as n increases, one sees that this difference is insensible when n is a great number. This put, we will consider in the integral $\frac{1}{2}\int U\,d\mu\sqrt{n}$ the term

$$\frac{1.3.5\ldots(2i-1)H^{(i)}\sqrt{n\pi}}{2^i c^{4ir'}}\int d\mu c^{-\mu^2}\left[1-\frac{i(2\mu)^2}{1.2}+\frac{i(i-1)(2\mu)^4}{1.2.3.4}-\cdots\right].$$

By extending the integral from $\mu=-\infty$ to $\mu=\infty$, this term becomes

$$\frac{1.3.5\ldots(2i-1)H^{(i)}\pi\sqrt{n}}{2^i c^{4ir'}}\left[1-i+\frac{i(i-1)}{1.2}+\frac{i(i-1)(i-2)}{1.2.3}+\cdots\right].$$

The factor $1-i+\frac{i(i-1)}{1.2}-\cdots$ is equal to $(1-1)^i$; it is therefore null, except in the case of $i=0$, where it is reduced to unity. It is clear that the terms of the expression of U which contain the odd powers of μ give a null result in the integral $\frac{1}{2}\int U d\mu\sqrt{n}$, extended from $\mu=-\infty$ to $\mu=\infty$; because these terms have for factor $c^{-\mu^2}$, and one has generally within these limits

$$\int \mu^{2i+1} d\mu c^{-\mu^2}=0.$$

There is therefore only the first term of the expression of U, a term that we will represent by $Hc^{-\mu^2}$, which is able to give a result in the integral $\frac{1}{2}\int U d\mu\sqrt{n}$, and this result is $\frac{1}{2}H\sqrt{n\pi}$; one has therefore

$$\frac{1}{2}H\sqrt{n\pi}=1;$$

consequently,

$$H=\frac{2}{\sqrt{n\pi}}.$$

The general expression of U has thus the following form

$$U=\frac{2c^{-\mu^2}}{\sqrt{n\pi}}\left\{\begin{array}{l} 1+\dfrac{Q^{(1)}(1-2\mu^2)}{c^{4r'}}+\dfrac{Q^{(2)}(1-4\mu^2+\frac{4}{3}\mu^4)}{c^{8r'}}+\cdots \\ +\dfrac{L^{(0)}\mu}{c^{2r'}}+\dfrac{L^{(1)}\mu(1-\frac{2}{3}\mu^2)}{c^{6r'}}\dfrac{L^{(2)}\mu(1-\frac{4}{3}\mu^2+\frac{4}{15}\mu^4)}{c^{10r'}}+\cdots \end{array}\right\}$$

$Q^{(1)}$, $Q^{(2)}$, $\ldots$, $L^{(0)}$, $L^{(1)}$, $\ldots$ being some indeterminate constants, which depend on the initial value of U.

We suppose that U becomes X when r is null, X being a given function of μ. One has generally these two theorems,

$$0 = Q^{(i)}\int \mu^{2q} d\mu\, U_i c^{-\mu^2},$$
$$0 = L^{(i)}\int \mu^{2q+1} d\mu\, U_i' c^{-\mu^2},$$

when q is less than i; U_i and U_i' being some functions of μ, by which $\frac{2Q^{(i)}c^{-\mu^2}}{\sqrt{n\pi}c^{4ir'}}$ and $\frac{2L^{(i)}c^{-\mu^2}}{\sqrt{n\pi}c^{(4i+2)r'}}$ are multiplied in the expression of U. In order to demonstrate these theorems, we will observe that, by that which precedes, $\frac{2Q^{(i)}c^{-\mu^2}U_i}{\sqrt{n\pi}}$ is equal to

$$(\sqrt{-1})^2 H^{(i)} c^{-\mu^2}\int ds\, c^{-s^2}(\mu + s\sqrt{-1})^{2i};$$

it is necessary therefore to show that one has

$$0 = \int\int \mu^{2q} ds\, d\mu\, c^{-\mu^2-s^2}(\mu + s\sqrt{-1})^{2i},$$

the integrals being taken from μ and s equal to $-\infty$ to μ and s equal to $+\infty$. By integrating first with respect to μ, this term becomes

$$\frac{2q-1}{2}\int\int \mu^{2q-2} ds\; d\mu\; c^{-\mu^2-s^2}(\mu + s\sqrt{-1})^{2i}$$
$$+\, i\int\int \mu^{2q-1} ds\, d\mu\, c^{-\mu^2-s^2}(\mu + s\sqrt{-1})^{2i-1}.$$

By continuing to integrate thus by parts relative to μ, one arrives finally to some terms of the form

$$k\int\int d\mu\, ds\, c^{-\mu^2-s^2}(\mu + s\sqrt{-1})^{2e},$$

e not being zero, and, by that which precedes, these terms are null.

One will prove in the same manner that one has

$$0 = L^{(i)}\int \mu^{2q+1}\, d\mu\, U_i'\, c^{-\mu^2}.$$

Thence it follows that one has generally

$$0=\int U_i\,U_{i'}d\mu\,c^{-\mu^2},\quad 0=\int U_i\;U_i'\;d\mu\,c^{-\mu^2},$$

i and i' being some different numbers. Because if, for example, i' is greater than i, all the powers of μ in U_i are less than $2i'$; each of the terms of U_i will give therefore, by that which precedes, a result null in the integral $\int U_i\,U_{i'}d\mu\,c^{-\mu^2}$. The same reasoning holds for the integral $\int U_i'\;U_i'\;d\mu c^{-\mu^2}$.

But these integrals are not nulls, when $i=i'$. One will obtain them in this case in this manner. One has, by that which precedes,

$$U_i=\frac{2^i(\sqrt{-1})^{2i}\int ds\,c^{-s^2}(\mu+s\sqrt{-1})^{2i}}{1.3.5\ldots(2i-1)\sqrt{\pi}}.$$

The term which has for factor μ^{2i} in this expression is

$$\frac{2^i(\sqrt{-1})^{2i}\mu^{2i}}{1.3.5\ldots(2i-1)};$$

now one is able to consider only this term in the first factor U_i of the integral $\int U_iU_id\mu\,c^{-\mu^2}$; because the inferior powers of μ, in this factor, become a null result in the integral. One has therefore

$$\int U_iU_id\mu\,c^{-\mu^2}=\frac{2^{2i}}{[1.3.5\ldots(2i-1)]^2\sqrt{\pi}}\int\int\mu^{2i}d\mu\,ds\,c^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i}.$$

One has, by integrating with respect to μ, form $\mu=-\infty$ to $\mu=\infty$,

$$\begin{aligned}\int\int\mu^{2i}d\mu\,ds\,c^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i}&\\ =\frac{2i-1}{2}\int\int\mu^{2i-2}d\mu\,ds\,c^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i}&\\ +\frac{2i}{2}\int\int\mu^{2i-1}d\mu\,ds\,c^{-\mu^2-s^2}(\mu+s\sqrt{-1})^{2i-1}&\end{aligned}$$

The first term of the second member of this equation is null by that which precedes; this member is reduced therefore to its second term. One finds in the same manner that one has

$$\int\int \mu^{2i-1} d\mu\, ds\, c^{-\mu^2-s^2}(\mu + s\sqrt{-1})^{2i-1}$$
$$= \frac{2i-1}{2}\int\int \mu^{2i-2} d\mu\, ds\, c^{-\mu^2-s^2}(\mu + s\sqrt{-1})^{2i-2},$$

and thus consecutively; one has therefore

$$\int\int \mu^{2i} d\mu\, ds\, c^{-\mu^2-s^2}(\mu + s\sqrt{-1})^{2i} = \frac{1.2.3\ldots 2i\pi}{2^{2i}};$$

consequently,

$$\int U_i U_i d\mu\, c^{-\mu^2} = \frac{2.4.6\ldots 2i\sqrt{\pi}}{1.3.5\ldots(2i-1)}.$$

One will find in the same manner

$$\int U_i' U_i' d\mu\, c^{-\mu^2} = \frac{1}{2}\,\frac{2.4.6\ldots 2i\sqrt{\pi}}{1.3.5\ldots(2i+1)}.$$

One has evidently

$$\int U_i U_{i'}' d\mu\, c^{-\mu^2} = 0,$$

in the same case where i and i' are equal, because the product $U_i U_{i'}$ contains only some odd powers of μ.

This put, the general expression of U gives, for its initial value, that we have designated by X,

$$X = \frac{2c^{-\mu^2}}{\sqrt{n\pi}}\left[1 + Q^{(1)}(1-2\mu^2) + \cdots + L^{(0)}\mu + L^{(1)}\mu\left(1 - \frac{3}{2}\mu^2\right) + \cdots\right].$$

If one multiplies this equation by $U_i d\mu$, and if one takes the integrals from $\mu = -\infty$ to $\mu = \infty$, one will have, by virtue of the preceding theorems,

$$\int XU_i d\mu = \frac{2}{\sqrt{n\pi}} Q^{(i)} \int U_i U_i d\mu c^{-\mu^2},$$

whence one draws

$$Q^{(i)} = \frac{1.3.5\ldots(2i-1)\frac{1}{2}\sqrt{n}}{2.4.6\ldots 2i}\int XU_i d\mu;$$

one will find, in the same manner,

$$L^{(i)} = \frac{1.3.5\ldots(2i+1)\sqrt{n}}{2.4.6\ldots 2i}\int XU_i' d\mu.$$

One will have therefore thus the successive values of $Q^{(1)}$, $Q^{(2)}, \ldots,$ $L^{(0)}$, $L^{(1)}$, $\ldots$ by means of definite integrals, when X or the initial value of U will be given.

In the case where X is equal to $\frac{2i}{\sqrt{n\pi}}c^{-i^2\mu^2}$, the general expression of U takes a very simple form. Then the arbitrary function $\Gamma\left(\frac{s-\mu\sqrt{-1}}{c^{2r'}}\right)$ from formula (A) is of the form $kc^{-\beta\left(\frac{s-\mu\sqrt{-1}}{c^{2r'}}\right)^2}$. In order to determine the constants β and k, we will observe that by supposing

$$\beta' = \frac{\beta}{c^{4r'}},$$

one will have

$$U = kc^{-\frac{\mu^2}{1+\beta'}}\int ds\, c^{-(1+\beta')\left(s-\frac{\beta'\mu\sqrt{-1}}{1+\beta'}\right)^2}.$$

By making next

$$\sqrt{1+\beta'}\left(s - \frac{\beta'\mu\sqrt{-1}}{1+\beta'}\right) = s',$$

and observing that the integral relative to s must be taken from $s = -\infty$ to $s = \infty$, the integral relative to s' must be taken within the same limits, one will have

$$U = \frac{k\sqrt{\pi}}{\sqrt{1+\beta'}}c^{-\frac{\mu^2}{1+\beta'}}.$$

By comparing this expression to the initial value of U, which is

$$U = \frac{2i}{\sqrt{n\pi}} c^{-i^2\mu^2},$$

and observing that β is the initial value of β', one will have

$$i^2 = \frac{1}{1+\beta},$$

whence on draws

$$\beta = \frac{1-i^2}{i^2}, \quad \beta' = \frac{1-i^2}{i^2 c^{4r'}}.$$

One must have next

$$\frac{k\sqrt{\pi}}{\sqrt{1+\beta}} = \frac{2i}{\sqrt{n\pi}},$$

this which gives

$$k\sqrt{\pi} = \frac{2}{\sqrt{n\pi}},$$

a value that one obtains next by the condition that $\frac{1}{2}\int U\,d\mu\sqrt{n} = 1$, the integral being taken from $\mu = -\infty$ to $\mu = \infty$; one will have, for the expression of U, whatever be r',

$$U = \frac{2}{\sqrt{n\pi(1+\beta')}} c^{-\frac{\mu^2}{1+\beta'}}.$$

One finds, in effect, that this value of U, substituted into the equation in the partial differentials in U, satisfies it.

β' diminishing without ceasing when r' increases, the value of U varies without ceasing and arrives to its limit, when r' is infinity,

$$U = \frac{2}{\sqrt{n\pi}} c^{-\mu^2}.$$

In order to give an application of these formulas, we imagine, in an urn C, a very great number m of white balls and a parallel number of black balls. These balls having been mixed, we suppose that one draws from the urn n balls, that one puts into urn A. We suppose next that one puts into urn B as many white balls as there are black balls in urn A, and as many black balls as there are white balls in the same urn. It is clear that the number of cases in which there will be x white balls, and consequently $n-x$ black balls in urn A, is equal to the product of the number of combinations of the m white balls of urn C, taken x by x, by the number of combinations of the m black balls of the same urn, taken $n-x$ by $n-x$. This product is, by n° 3, equal to

$$\frac{m(m-1)(m-2)\cdots(m-x+1)}{1.2.3\ldots x}\,\frac{m(m-1)(m-2)\cdots(m-n+x+1)}{1.2.3\ldots(n-x)}$$

or to

$$\frac{(1.2.3\ldots m)^2}{1.2.3\ldots x.1.2.3\ldots(n-x).1.2.3\ldots(m-x).1.2.3\ldots(m-n+x)}.$$

The number of all possible cases is the number of combinations of the $2m$ balls from urn C, taken n by n; this number is

$$\frac{1.2.3\ldots 2m}{1.2.3\ldots n.1.2.3\ldots(2m-n)};$$

by dividing the preceding fraction by that here, one will have, for the probability of x or for the initial value of U,

$$\frac{(1.2.3\ldots m)^2 1.2.3\ldots n.1.2.3\ldots(2m-n)}{1.2.3\ldots x.1.2.3\ldots(m-x).1.2.3\ldots(n-x).1.2.3\ldots(m-n+x).1.2.3\ldots 2m}.$$

Now, if one observes that one has very nearly, when s is a great number,

$$1.2.3\ldots s = s^{s+\frac{1}{2}}c^{-s}\sqrt{2\pi},$$

one will find easily, after all the reductions, by making

$$x = \frac{n+\mu\sqrt{n}}{2},$$

and by neglecting the quantities of order $\frac{1}{n}$ which are not multiplied by μ^2,

$$U = \frac{2}{\sqrt{n\pi}}\sqrt{\frac{m}{2m-n}}c^{-\frac{m\mu^2}{2m-n}};$$

by making therefore

$$i^2 = \frac{m}{2m-n},$$

one will have

$$U = \frac{2i}{\sqrt{n\pi}} c^{-i^2\mu^2}.$$

If the number m is infinite, then $i^2 = \frac{1}{2}$, and the initial value of U is

$$U = \frac{2}{\sqrt{n\pi}} c^{-\frac{1}{2}\mu^2}.$$

Its value, after any number of drawings, is

$$U = \frac{2}{\sqrt{n\pi\left(1 + c^{-\frac{4r}{n}}\right)}} c^{-\frac{\mu^2}{1+c^{-\frac{4r}{n}}}}.$$

The case of m infinite returns to the one in which the urns A and B would be filled, by projecting n times a coin which would bring forth indifferently *heads* or *tails*, and putting into urn A a white ball each time that *heads* would arrive, and a black ball each time that *tails* would arrive, and making the inverse for urn B. Because it is clear that the probability to draw a white ball from urn C is then $\frac{1}{2}$, as that to bring forth *heads* or *tails*.

By taking the integral $\int U\,dx$ or $\frac{1}{2}\int U\,d\mu\sqrt{n}$ from $\mu = -a$ to $\mu = a$, one will have the probability that the number of white balls of urn A will be comprehended within the limits $\pm a\sqrt{n}$.

One is able to generalize the preceding result, by supposing the urn A filled, as at the beginning of this section, by the projection of a prism of $p+q$ lateral faces, of which p are white and q are black. One has seen that then, if one makes

$$i^2 = \frac{(p+q)^2}{2pq},$$

one has, at the origin or when r is null,

$$U = \frac{i}{\sqrt{n\pi}} c^{-\frac{i^2}{n}\left(x - \frac{np}{p+q}\right)^2}.$$

We suppose p and q very little different, so that one has

$$p = \frac{p+q}{2}\left(1 + \frac{a}{\sqrt{n}}\right),$$
$$q = \frac{p+q}{2}\left(1 - \frac{a}{\sqrt{n}}\right),$$

one will have

$$i^2 = \frac{2}{1 - \frac{a^2}{n}}$$

or, very nearly, $i^2 = 2$; therefore

$$U = \frac{2}{\sqrt{2n\pi}} c^{-\frac{2}{n}\left(x - \frac{n}{2} - \frac{a\sqrt{n}}{2}\right)^2}.$$

By making therefore

$$x = \frac{n + \mu\sqrt{n}}{2},$$

one will have

$$U = \frac{2}{\sqrt{2n\pi}} c^{-\frac{1}{2}(\mu - a)^2}.$$

We suppose now that after any number whatsoever of drawings one has

$$U = \frac{2}{\sqrt{n\beta\pi}} c^{-\frac{(\mu-\alpha)^2}{\beta}},$$

β and α being some functions of r'. If one substitutes this value into the equation in the partial differences in U, one will have

$$
\begin{aligned}
&-\frac{d\beta}{dr'}\left[1-\frac{2(\mu-\alpha)^2}{\beta}\right]+4\frac{d\alpha}{dr'}(\mu-\alpha)\\
&\quad=4(\beta-1)\left[1-\frac{2(\mu-\alpha)^2}{\beta}\right]-8\alpha(\mu-\alpha),
\end{aligned}
$$

whence one draws the two following equations:

$$\frac{\frac{d\beta}{dr'}}{\beta-1}=-4,\quad \frac{d\alpha}{dr'}=-2\alpha.$$

By integrating them and observing that at the origin of r', $\alpha=a$ and $\beta=2$, one will have

$$\beta=1+c^{-4r'},\quad \alpha=ac^{-2r'},$$

this which gives

$$U=\frac{2}{\sqrt{n\pi(1+c^{-4r'})}}c^{-\frac{(\mu-ac^{-2r'})^2}{1+c^{-4r'}}}.$$

We seek now the mean value of the number of white balls contained in urn A, after r drawings. This value is the sum of the products of the diverse numbers of white balls, multiplied by their respective probabilities; it is therefore equal to the integral

$$\int\frac{n+\mu\sqrt{n}}{2}U\frac{d\mu\sqrt{n}}{2},$$

taken from $\mu=-\infty$ to $\mu=\infty$. By substituting for U its value given by formula (*k*), one will have, by virtue of the preceding theorems, for this integral,

$$\frac{1}{2}n+\frac{\sqrt{n}}{4}L^{(0)}c^{-\frac{2r}{n}}.$$

At the origin where r is null, this value is $\frac{1}{2}n+\frac{\sqrt{n}}{2}L^{(0)}$; thus one will have $L^{(0)}$ at the mean of the number of white balls that urn A contains at the origin.

One is able to obtain quite simply, in the following manner, the mean value of the number of white balls, after r drawings. We imagine that each white ball has a value that we will represent by unity, the black balls being supposed to have no value. It is clear that the price[1] of urn A will be the sum of the products of all the possible numbers of white balls which are able to exist in the urn, multiplied by their respective probabilities; this prize is therefore that which we have named *mean value of the number of white balls*. We name it z, after the r^{th} drawing. At the following drawing, if there exits a white ball, this price diminishes by one unit; now, if one supposes that x is the number of white balls contained in the urn after the r^{th} drawing, the probability to extract a white ball from it will be $\frac{x}{n}$; by naming therefore U the probability of this supposition, the integral $\int \frac{Uxdx}{n}$, extended from $x = 0$ to $x = n$, will be the diminution of z, resulting from the probability to extract a white ball from the urn. If one makes, as above, $\frac{r}{n} = r'$, and if one designates the very small fraction $\frac{1}{n}$ by dr', this diminution will be equal to zdr'; because z is equal to $\int Ux\,dx,$ a sum of the products of the numbers of white balls by their respective probabilities. The price of urn A is increased, if one extracts a white ball from urn B, in order to put it into urn A; now, x being supposed the number of white balls of urn A, $n - x$ will be the one of the white balls of urn B, and the probability to extract a white ball from this last urn will be $\frac{n-x}{n}$; by multiplying this probability by the probability U of x, the integral $\int U \frac{n-x}{n}\,dx$, taken from x null to $x = n$, will be the increase of z. $\int U(n-x)dx$ is the price of urn B; by naming therefore z' this price, $z'dr'$ will be the increase of z; one will have therefore

$$dz = z'dr' - zdr'.$$

The sum of the prices of the two urns is evidently equal to n, a number of white balls that they contain, this which gives $z' = n - z$; substituting this value of z' into the preceding equation, there arrives

$$dz = (n - 2z)dr';$$

whence one draws, by integrating,

$$z = \frac{1}{2}n + \frac{L^{(0)}}{4c^{2r'}},$$

$L^{(0)}$ being an arbitrary constant, this which is conformed to that which precedes.

[1] Translator's note: *Prix*. The value of the urn is set as the number of white balls contained in it.

One is able to extend all this analysis to the case of any number whatsoever of urns; we will limit ourselves here to seek the mean value of the number of white balls that each urn contains after n drawings.

We will consider a number e of urns, disposed circularly, and containing each the number n of balls, the ones white, and the others black, n being supposed a very great number. We suppose that after r drawings, $z_0,\, z_1,\, z_2,\, \dots,\, z_{e-1}$ are the respective prices of the diverse urns. Each drawing consists of extracting at the same time a ball from each urn and to putting it into the following, by departing from one of them in a determined sense. If one makes $\frac{r}{n} = r'$ and $\frac{1}{n} = dr'$, one will have, by the reasoning that we have just made relative to two urns,

$$dz_i = (z_{i-1} - z_i)dr';$$

this equation holds from $i = 1$ to $i = e - 1$. In the case of $i = e$, one has

$$dz_0 = (z_{e-1} - z_0)dr'.$$

By integrating these equations, and supposing that at the origin the respective prices of each urn, or the numbers of white balls that they contain, are

$$\lambda_0,\, \lambda_1,\, \dots,\, \lambda_{e-1},$$

one arrives to this result, which holds from $i = 0$ to $i = e - 1$,

$$z_i = \frac{1}{e}\mathrm{S}c^{-\left(1-\cos\frac{2s\pi}{e}\right)r'}\left\{\begin{array}{l} \quad\lambda_0\cos\left(\dfrac{2si\pi}{e} - ar'\right) \\ + \lambda_1\cos\left[\dfrac{2s(i-1)\pi}{e} - ar'\right] \\ + \lambda_2\cos\left[\dfrac{2s(i-2)\pi}{e} - ar'\right] \\ + \cdots\cdots\cdots\cdots\cdots\cdots \\ + \lambda_{e-1}\cos\left[\dfrac{2s(i-e+1)\pi}{e} - ar'\right] \end{array}\right\}$$

the sign S extending to all the values of s, from $s = 1$ to $s = e$, and a being equal to $\sin\frac{2s\pi}{e}$. The term of this expression, corresponding to $s = e$, is independent of r', and equal to $\frac{1}{e}(\lambda_0 + \lambda_1 + \cdots + \lambda_{e-1})$, that is to say the entire sum of the white balls of the urns divided by their number. This term is the limit of the expression of z_i, whence it follows that after an infinite number of drawings the prices of each urn are equal among them.