CHAPTER IV

DE LA PROBABILITÉ DES ERREURS DES RÉSULTES MOYENS D'UN GRAND NOMBRE D'OBSERVATIONS ET DES RÉSULTATS MOYENS LES PLUS AVANTAGEUX.

Pierre Simon Laplace

Théorie Analytique des Probabilités §§ 18–24, pp. 309–354

ON THE PROBABILITY OF THE ERRORS OF THE MEAN RESULTS OF A GREAT NUMBER OF OBSERVATIONS AND ON THE MOST ADVANTAGEOUS MEAN RESULTS

To determine the probability that the sum of the errors of a great number of observations will be comprehended within some given limits, by supposing that the law of possibility of the errors is known, and the same for each observation, and that the negative errors are as possible as the corresponding positive errors. General expression of this probability. Nº 18.

To determine, under the preceding suppositions, the probability that the sum of the errors of a great number of observations or the sum of their squares, of their cubes, etc., will be comprehended within some given limits, setting aside the sign. General expression of this probability and of the most probable sum. Nº 19.

An element being known quite nearly, to determine its correction by the collection of a great number of observations. Formation of the equations of condition. By disposing them in a manner that, in each of them, the coefficient of the correction of the element has the same sign, and adding them, one forms a final equation which gives a mean correction. Expression of the probability that the error of this mean correction is comprehended within some given limits. The most general manner to form the final equation is to multiply each equation of condition by an indeterminate factor and to add all these products. Expression of the probability that the error of correction given by this final equation is comprehended within some given limits. Expression of the mean error that one is able to fear to the more or to the less. Determination of a system of factors which render this error a minimum. One is led then to the result that gives the method of least squares of the errors of observations. Mean error of its result. Its expression depends on the law of facility of the errors of the observations. Means to render it independent. Nº 20.

To correct, by the collection of a great number of observations, many elements already known quite nearly. Formation of the equations of condition. By multiplying them each by an indeterminate factor and adding the products, one forms a first final equation: a second system of factors gives a second final equation, and thus consecutively until one has as many final equations as there are elements to correct. Expression of the mean errors that one is able to fear with respect to each element corrected by these final equations. Determination of the systems of factors by the condition that these mean errors are of minima. One falls again into the method of least squares of the errors of observations; whence it follows that this method is that which the Calculus of probabilities indicates as being the most advantageous. Expression of the mean errors that it leaves yet to fear, to the more or to the less, with respect to each element. These expressions are independent of the law of facility of the errors of each observation and contain only the givens of the observations. Simple means to compare among them, on the side of precision, diverse astronomical Tables of one same star. Nº 21.

Examination of the case where the possibility of the negative errors is not the same as that of the positive errors. Mean result toward which the sum of the products of the errors of a great number of observations converge, by any factors; probability of this convergence. Nº 22.

Examination of the case where one considers the observations already made. Then the error of the first gives the errors of all the others. The probability of this error, taken *a posteriori* or after the observations already made, is the product of the respective probabilities *a priori* of the errors of each observation. By imagining therefore a curve of which the abscissa is the error of the first observation, and of which this product is the ordinate, this curve will be that of the probabilities *a posteriori* of the errors of the first observation. The error that it is necessary to suppose to it is the abscissa corresponding to the ordinate which divides the area of the curve into two equal parts. The value of this abscissa depends on the unknown law of the probabilities *a priori* of the errors of the observations, and in this ignorance, it is convenient to rest content with the most advantageous result, determined *a priori* by the preceding articles. Research on the law of probabilities *a priori* of the errors, which give constantly the sum of the null errors for the result that it is necessary to choose *a posteriori*. This law gives generally the rule of the minimum of the squares of the errors of the observations. This last rule becomes necessary when one must choose a mean result among many results, each given by a great number of observations of diverse kinds. Nº 23.

Research on the system of corrections of many elements by a great number of observations, which render a minimum, setting aside the sign, the greatest of the errors that it supposes to them. This system is the one which renders a minimum the sum of similar powers, very elevated and even, of each error. It differs little from the system given by the method of least squares of the errors of the observations. Historical notice on the methods of correction of the elements by the observations. Nº 24.

18. We consider now the mean results of a great number of observations of which one knows the law of the facility of errors. We suppose first that, for each observation, the errors are able to be equally

$$-n,\ -n+1,\ -n+2,\ \ldots,\ -1,\ 0,\ 1,\ 2,\ \ldots,\ n-2,\ n-1,\ n.$$

The probability of each error will be $\frac{1}{2n+1}$. If one names s the number of observations, the coefficient of $\mathrm{c}^{l\varpi\sqrt{-1}}$ in the development of the polynomial

$$(\mathrm{c}^{-n\varpi\sqrt{-1}} + \mathrm{c}^{-(n-1)\varpi\sqrt{-1}} + \mathrm{c}^{-(n-2)\varpi\sqrt{-1}} + \cdots + \mathrm{c}^{-\varpi\sqrt{-1}} + 1 + \mathrm{c}^{\varpi\sqrt{-1}} + \cdots + \mathrm{c}^{n\varpi\sqrt{-1}})^s$$

will be the number of combinations in which the sum of the errors is l. This coefficient is the term independent of $\mathrm{c}^{\varpi\sqrt{-1}}$ and of its powers in the development of the same polynomial multiplied by $\mathrm{c}^{-l\varpi\sqrt{-1}}$, and it is clearly equal to the term independent of ϖ in the same development multiplied by $\frac{\mathrm{c}^{l\varpi\sqrt{-1}}+\mathrm{c}^{-l\varpi\sqrt{-1}}}{2}$ or by cos $l\varpi$'; one will have therefore, for the expression of this coefficient,

$$\frac{1}{\pi}\int d\varpi \cos l\varpi(1 + 2\cos\varpi + 2\cos 2\varpi + \cdots + 2\cos n\varpi)^s,$$

the integral being taken from $\varpi = 0$ to $\varpi = \pi$.

One has seen, in n$^\text{o}$ 36 of Book I, that this integral is

$$\frac{(2n+1)^s\sqrt{3}}{\sqrt{n(n+1)2s\pi}}c^{-\frac{\frac{3}{2}l^2}{n(n+1)s}};$$

the total number of combinations of the errors is $(2n+1)^s$; by dividing the preceding quantity by that here, one will have

$$\frac{\sqrt{3}}{\sqrt{n(n+1)2s\pi}}c^{-\frac{\frac{3}{2}l^2}{n(n+1)s}};$$

for the probability that the sum of the errors of the s observations will be l.

If one makes

$$l = 2t\sqrt{\frac{n(n+1)s}{6}},$$

the probability that the sum of the errors will be contained within the limits $+2T\sqrt{\frac{n(n+1)s}{6}}$ and $-2T\sqrt{\frac{n(n+1)s}{6}}$ will be equal to

$$\frac{2}{\pi}\int dt c^{-t^2},$$

the integral being taken from $t = 0$ to $t = T$. This expression holds further in the case of n infinite. Then, by naming $2a$ the interval contained between the limits of the errors of each observation, one will have $n = a$, and the preceding limits would become $\pm\frac{2Ta\sqrt{s}}{\sqrt{6}}$: thus the probability that the sum of the errors will be contained within the limits $\pm ar\sqrt{s}$ is

$$2\sqrt{\frac{3}{2\pi}}\int dr c^{-\frac{3}{2}r^2};$$

it is also the probability that the mean error will be contained within the limits $\pm\frac{ar}{\sqrt{s}}$; because one has the mean error by dividing by s the sum of the errors.

The probability that the sum of the inclination of the orbits of s comets will be contained within some given limits, by supposing all the inclinations equally possible, from zero to a right angle, is evidently the same as the preceding probability; the interval $2a$ of the limits of the errors of each observation is, in this case, the interval $\frac{\pi}{2}$ of the limits of the possible inclinations: then the probability that the sum of the inclinations must be contained within the limits $\pm\frac{\pi r\sqrt{s}}{4}$ is $2\sqrt{\frac{3}{2\pi}}\int dr c^{-\frac{3}{2}r^2}$, this which accords with that which one has found in n$^{\text{o}}$ 13.

We suppose generally that the probability of each positive or negative error is expressed by $\phi\left(\frac{x}{n}\right)$, x and n being of infinite numbers. Then, in the function

$$1 + 2\cos\varpi + 2\cos 2\varpi + 2\cos 3\varpi + \cdots + 2\cos n\varpi,$$

each term, such as $2\cos x\varpi$, must be multiplied by $\phi\left(\frac{x}{n}\right)$; now one has

$$2\phi\left(\frac{x}{n}\right)\cos x\varpi = 2\phi\left(\frac{x}{n}\right) - \frac{x^2}{n^2}\phi\left(\frac{x}{n}\right)n^2\varpi^2 + \cdots$$

By making therefore

$$x' = \frac{x}{n}, \qquad dx' = \frac{1}{n},$$

the function

$$\phi\left(\frac{0}{n}\right) + 2\phi\left(\frac{1}{n}\right)\cos\varpi + 2\phi\left(\frac{2}{n}\right)\cos 2\varpi + \cdots + 2\phi\left(\frac{n}{n}\right)\cos n\varpi$$

becomes

$$2n\int dx'\phi(x') - n^2\varpi^2\int x'^2 dx'\phi(x') + \cdots$$

the integrals must be extended from $x' = 0$ to $x' = 1$. Let then

$$k = 2\int dx'\phi(x'), \quad k'' = \int x'^2 dx'\phi(x'), \quad \ldots$$

The preceding series becomes

$$nk\left(1 - \frac{k''}{k}n^2\varpi^2 + \cdots\right).$$

Now the probability that the sum of the errors of the s observations will be contained within the limits $\pm l$ is, as it easy to be assured of it by the preceding reasonings,

$$\frac{2}{\pi}\int\int d\varpi\, dl \cos l\varpi \left\{ \begin{aligned} &\phi\left(\frac{0}{n}\right) + 2\phi\left(\frac{1}{n}\right)\cos\varpi + 2\phi\left(\frac{2}{n}\right)\cos 2\varpi + \cdots \\ &\qquad\qquad\qquad\qquad + 2\phi\left(\frac{n}{n}\right)\cos n\varpi \end{aligned} \right\}^s,$$

the integral being taken from ϖ null to $\varpi = \pi$; this probability is therefore

$$(u) \qquad 2\frac{(nk)^S}{\pi}\int\int d\varpi\, dl \cos l\varpi \left(1 - \frac{k''}{k}n^2\varpi^2 + \cdots\right)^s.$$

We suppose

$$\left(1 - \frac{k''}{k}n^2\varpi^2 + \cdots\right)^s = c^{-t^2};$$

by taking the hyperbolic logarithms, one will have, very nearly, when s is a great number,

$$s\frac{k''}{k}n^2\varpi^2 = t^2,$$

this which gives

$$\varpi = \frac{t}{n}\sqrt{\frac{k}{k''s}}.$$

If one observes next that, nk or $2\int dx\,\phi\left(\frac{x}{n}\right)$ expressing the probability that the error of an observation is contained within the limits $\pm n$, this quantity must be equal to unity, the function (*u*) will become

$$\frac{2}{n\pi}\sqrt{\frac{k}{k''s}}\int\int dl\,dt\,c^{-t^2}\cos\left(\frac{lt}{n}\sqrt{\frac{k}{k''s}}\right),$$

the integral relative to t must be taken from t null to $t = \pi n\sqrt{\frac{k''s}{k}}$, or to $t = \infty$, n being supposed infinite. Now one has, by nº 25 of Book I,

$$\int dt\cos\left(\frac{lt}{n}\sqrt{\frac{k}{k''s}}\right)c^{-t^2} = \frac{\sqrt{\pi}}{2}c^{-\frac{l^2}{4n^2}\frac{k}{k''s}},$$

by making therefore

$$\frac{l}{n} = 2t'\sqrt{\frac{k''s}{k}},$$

the function (*u*) becomes

$$\frac{2}{\sqrt{\pi}}\int dt'\,c^{-t'^2}$$

Thus, by naming, as above, $2a$ the interval contained between the limits of the errors of each observation, the probability that the sum of the errors of the s observations will be contained within the limits $\pm ar\sqrt{s}$ is

$$\sqrt{\frac{k}{k''s}}\int dr\,c^{-\frac{kr^2}{4k''}}.$$

If $\phi\left(\frac{x}{n}\right)$ is constant, then $\frac{k}{k''} = 6$, and this probability becomes

$$2\sqrt{\frac{3}{2\pi}}\int drc^{-\frac{3}{2}r^2},$$

this which is conformed to that which one has found above.

If $\phi\left(\frac{x}{n}\right)$ or $\phi(x')$ is a rational and entire function of x', one will have, by the method of n$^{\text{o}}$ 15, the probability that the sum of the errors will be contained within the limits $\pm\, ar\sqrt{s}$, expressed by a sequence of powers s, $2s$, $\ldots$ of quantities of the form $s-\mu \pm r\sqrt{s}$, in which μ increases by arithmetic progression, these quantities being continued until they become negatives. By comparing this sequence to the preceding expression of the same probability, one will obtain in a manner very near the value of the sequence, and one will arrive thus out of this kind of sequence to some theorems analogous to those that we have given in n$^{\text{o}}$ 42 of Book I, on the finite differences of the powers of a variable.

If the law of facility of the errors is expressed by a negative exponential which is able to be extended to infinity, and generally if the errors are able to be extended to infinity, then a becomes infinite, and the application of the preceding method is able to offer some difficulties. In all these cases, one will make

$$\frac{x}{h} = x', \qquad \frac{1}{h} = dx',$$

h being any finite quantity whatsoever, and by following exactly the preceding analysis, one will find, for the probability that the sum of the errors of the s observations is contained within the limits $\pm\, hr\sqrt{s}$,

$$\sqrt{\frac{k}{k''s}}\int dr\, c^{-\frac{kr^2}{4k''}}.$$

an expression in which one must observe that $\phi\left(\frac{x}{h}\right)$ or $\phi(x')$ expresses the probability of the error $\pm\, x$, and that one has

$$k = 2\int dx'\phi(x'), \quad k'' = \int x'^2 dx'\phi(x'),$$

the integrals being taken from $x'=0$ to $x'=\infty$.

19. We determine presently the probability that the sum of the errors of a very great number of observations will be contained within some given limits, setting aside the sign of these errors, that is to say, by taking them all positively. For this, we consider the sequence

$$\begin{aligned}\phi\left(\frac{n}{n}\right)\mathrm{c}^{-n\varpi\sqrt{-1}}+\phi\left(\frac{n-1}{n}\right)\mathrm{c}^{-(n-1)\varpi\sqrt{-1}}+\cdots+\phi\left(\frac{0}{n}\right)+\cdots\\+\phi\left(\frac{n-1}{n}\right)\mathrm{c}^{(n-1)\varpi\sqrt{-1}}+\phi\left(\frac{n}{n}\right)\mathrm{c}^{n\varpi\sqrt{-1}},\end{aligned}$$

$\phi\left(\frac{x}{n}\right)$ being ordered on the curve of probability of errors, corresponding to the error $\pm x$, and x being, thus as n, considered as formed of an infinite number of units. If one raises this sequence to the power s, after having changed the sign of the negative exponentials, the coefficient of any one exponential, such as $\mathrm{c}^{(l+\mu s)\varpi\sqrt{-1}}$, will be the probability that the sum of the errors, taken setting aside the sign, is $l+\mu s$; this probability is therefore

$$\frac{1}{2\pi}\int d\varpi\,\mathrm{c}^{-(l+\mu s)\varpi\sqrt{-1}}\left\{\begin{aligned}\phi\left(\frac{0}{n}\right)+2\phi\left(\frac{1}{n}\right)\mathrm{c}^{\varpi\sqrt{-1}}+2\phi\left(\frac{2}{n}\right)\mathrm{c}^{2\varpi\sqrt{-1}}+\cdots\\+2\phi\left(\frac{n}{n}\right)\mathrm{c}^{n\varpi\sqrt{-1}}\end{aligned}\right\}^{s},$$

the integral relative to ϖ being taken from $\varpi=-\pi$ to $\varpi=\pi$; because, in this interval, the integral $\int d\varpi\,\mathrm{c}^{-r\varpi\sqrt{-1}}$ or

$$\int d\varpi(\cos r\varpi-\sqrt{-1}\sin r\varpi)$$

disappears, whatever be r, provided that it is not null.

One has, by developing with respect to the powers of ϖ,

$$(1)\quad\left\{\begin{aligned}&\log\left\{c^{-\mu s\varpi\sqrt{-1}}\left[\phi\left(\frac{0}{n}\right)+2\phi\left(\frac{1}{n}\right)\mathrm{c}^{\varpi\sqrt{-1}}+\cdots+2\phi\left(\frac{n}{n}\right)\mathrm{c}^{n\varpi\sqrt{-1}}\right]^{s}\right\}\\&\quad=s\log\left\{\begin{aligned}&\phi\left(\frac{0}{n}\right)+2\phi\left(\frac{1}{n}\right)+2\phi\left(\frac{2}{n}\right)+\cdots+2\phi\left(\frac{n}{n}\right)\\&+2\varpi\sqrt{-1}\left[\phi\left(\frac{1}{n}\right)+2\phi\left(\frac{2}{n}\right)+\cdots+n\phi\left(\frac{n}{n}\right)\right]\\&-\varpi^{2}\left[\phi\left(\frac{1}{n}\right)+2^{2}\phi\left(\frac{2}{n}\right)+\cdots+n^{2}\phi\left(\frac{n}{n}\right)\right]\\&-\cdots\cdots\cdots\cdots\cdots\cdots\end{aligned}\right\}-\mu s\varpi\sqrt{-1}\end{aligned}\right.$$

By making therefore

$$\frac{x}{n} = x', \qquad \frac{1}{n} = dx',$$

one has

$$2\int dx'\phi(x') = k, \qquad \int x' dx'\phi(x') = k', \qquad \int x'^2 dx'\phi(x') = k'',$$
$$\int x'^3 dx'\phi(x') = k''', \qquad \int x'^4 dx'\phi(x') = k^{\mathrm{iv}}, \qquad \dots\dots\dots\dots\dots\dots\dots\dots,$$

the integrals being taken from x' null to $x' = 1$; the second member of equation (1) becomes

$$s\log nk + s\log\left(1 + \frac{2k'}{k} n\varpi\sqrt{-1} - \frac{k''}{k} n^2\varpi^2 - \cdots\right) - \mu s\varpi\sqrt{-1}.$$

The error of each observation must fall necessarily within the limits $\pm n$, one has $nk = 1$; the preceding quantity becomes thus

$$s\left(\frac{2k'}{k} - \frac{\mu}{n}\right) n\varpi\sqrt{-1} - \frac{(kk'' - 2k')^2 sn^2\varpi^2}{k^2} - \cdots;$$

by making therefore

$$\frac{\mu}{n} = \frac{2k'}{k}$$

and neglecting the powers of ϖ superior to the square, this quantity is reduced to its second term, and the preceding probability becomes

$$\frac{1}{2\pi}\int d\varpi c^{-lw\sqrt{-1} - \frac{kk'' - 2k'^2}{k^2} sn^2\varpi^2}.$$

Let

$$\beta = \frac{k}{\sqrt{kk'' - 2k'^2}}, \qquad \varpi = \frac{\beta t}{n\sqrt{s}}, \qquad \frac{l}{n} = r\sqrt{s}.$$

The preceding integral becomes

$$\frac{1}{2\pi}\frac{c^{-\frac{\beta^2 r^2}{4}}}{n\sqrt{s}}\int \beta dt c^{-\left(t+\frac{l\beta\sqrt{-1}}{2n\sqrt{s}}\right)^2}.$$

This integral must be taken from $t = -\infty$ to $t = \infty$, and then the preceding quantity becomes

$$\frac{\beta}{2\sqrt{\pi}n\sqrt{s}}c^{-\frac{\beta^2 r^2}{4}}.$$

By multiplying it by dl or by $ndr\sqrt{s}$, the integral

$$\frac{1}{2\sqrt{\pi}}\int \beta dr c^{-\frac{\beta^2 r^2}{4}}$$

will be the half probability that the value of l, and, consequently, the sum of the errors of the observations, is contained within the limits $\frac{2k'}{k}as \pm ar\sqrt{s}$, $\pm a$ being the limits of the errors of each observation, limits that we designate by $\pm n$, when we imagine them partitioned into an infinite number of parts.

One sees that the sum of the most probable errors, setting aside the sign, is that which corresponds to $r = 0$. This sum is $\frac{2k'}{k}as$. In the case where $\phi(x)$ is constant, $\frac{2k'}{k} = \frac{1}{2}$; the sum of the most probable errors is therefore then the half of the greatest sum possible, a sum which is equal to sa. But, if $\phi(x)$ is not constant and diminishes in measure as the error x increases, then $\frac{2k'}{k}$ is less than $\frac{1}{2}$, and the sum of the errors, setting aside the sign, is below the half of the greatest sum possible.

One is able, by the same analysis, to determine the probability that the sum of the squares of the errors will be $l + \mu s$; it is easy to see that this probability has for expression the integral

$$\frac{1}{2\pi}\int d\varpi\, \mathrm{c}^{-(l+\mu s)\varpi\sqrt{-1}}\left\{\begin{array}{r}\phi\left(\frac{0}{n}\right) + 2\phi\left(\frac{1}{n}\right)\mathrm{c}^{\varpi\sqrt{-1}} + 2\phi\left(\frac{2}{n}\right)\mathrm{c}^{2^2\varpi\sqrt{-1}} + \cdots\\ + 2\phi\left(\frac{n}{n}\right)\mathrm{c}^{n^2\varpi\sqrt{-1}}\end{array}\right\}^s,$$

taken from $\varpi = -\pi$ to $\varpi = \pi$. By following exactly the preceding analysis, one will have

$$\mu = \frac{2n^2k''}{k},$$

and by making

$$\beta' = \frac{k}{\sqrt{kk^{\text{iv}} - 2k''^2}},$$

the probability that the sum of the squares of the errors of the s observations will be contained within the limits $\frac{2k''}{k}a^2s \pm a^2r\sqrt{s}$ will be

$$\frac{1}{\sqrt{\pi}}\int \beta' dr\, c^{-\frac{\beta'^2r^2}{4}}.$$

The most probable sum is that which corresponds to r null; it is therefore $\frac{2k''}{k}a^2s$. If s is a very great number, the result of the observations will deviate very little from this value, and consequently it will make known very nearly the factor $\frac{a^2k''}{k}$.

20. When one wishes to correct an element already known quite nearly, with the collection of a great number of observations, one forms some equations of condition in the following manner. Let z be the correction of the element, and β the observation; the analytic expression of that here will be a function of the element. By substituting, instead of the element, its approximate value, plus the correction z; by reducing into series with respect to z and neglecting the square of z, this function will take the form $h + pz$; by equating it to the observed quantity β, one will have

$$\beta = h + pz.$$

z would be therefore determined, if the observation was rigorous; but, as it is susceptible to error, by naming ϵ this error, one has exactly, to the quantities near of order z^2,

$$\beta + \epsilon = h + pz;$$

and by making $\beta - h = \alpha$, one has

$$\epsilon = pz - \alpha.$$

Each observation furnishes a similar equation, that one is able to represent for the $(i+1)^{\text{st}}$ observation by that here

$$\epsilon^{(i)} = p^{(i)} z - \alpha^{(i)}.$$

By reuniting all these equations, one has

$$(1) \qquad \mathrm{S}\epsilon^{(i)} = z\mathrm{S}p^{(i)} - \mathrm{S}\alpha^{(i)},$$

the sign S relating to all the values of i, from $i = 0$ to $i = s - 1$, s being the total number of observations. By supposing null the sum of the errors, this equation gives

$$z = \frac{\mathrm{S}\alpha^{(i)}}{\mathrm{S}p^{(i)}};$$

it is this which one names ordinarily *mean result of the observations*.

One has seen, in n° 18, that the probability that the sum of the errors of s observations will be contained within the limits $\pm\, ar\sqrt{s}$ is

$$\sqrt{\frac{k}{k''\pi}}\int dr\, c^{-\frac{kr^2}{4k''}}.$$

We name $\pm\, u$ the error of the result z; by substituting, into equation (1), $\pm\, ar\sqrt{s}$ instead of $\mathrm{S}\epsilon^{(i)}$, and $\frac{\mathrm{S}\alpha^{(i)}}{\mathrm{S}p^{(i)}} \pm u$ instead of z, it gives

$$r = \frac{u\mathrm{S}p^{(i)}}{a\sqrt{s}};$$

the probability that the error of the result z will be contained within the limits $\pm\, u$ is therefore

$$\sqrt{\frac{k}{k''\pi}}\mathrm{S}p^{(i)}\int \frac{du}{a}\, c^{-\frac{ku^2(\mathrm{S}p^{(i)})^2}{4k''a^2 s}}.$$

Instead of supposing null the sum of the errors, one is able to suppose null any linear function of these errors, that we will represent thus,

$$(m) \qquad m\epsilon + m^{(1)}\epsilon^{(1)} + m^{(2)}\epsilon^{(2)} + \cdots + m^{(s-1)}\epsilon^{(s-1)},$$

m, $m^{(1)}$, $m^{(2)}$, … being entire positive or negative numbers. By substituting into this function (m), instead of ϵ, $\epsilon^{(1)}$, $\epsilon^{(2)}$, …, their values given by the equations of condition, it becomes

$$z\mathrm{S}m^{(i)}p^{(i)} - \mathrm{S}m^{(i)}\alpha^{(i)};$$

by equating therefore to zero the function (*m*), one has

$$z = \frac{\mathrm{S}m^{(i)}\alpha^{(i)}}{\mathrm{S}m^{(i)}p^{(i)}}.$$

Let u be the error of this result, so that one has

$$z = \frac{\mathrm{S}m^{(i)}\alpha^{(i)}}{\mathrm{S}m^{(i)}p^{(i)}} + u;$$

the function (*m*) becomes

$$u\mathrm{S}m^{(i)}p^{(i)}.$$

We determine the probability of the error u, when the observations are in great number.

For this, we will consider the product

$$\int \phi\left(\frac{x}{a}\right) \mathrm{c}^{mx\varpi\sqrt{-1}} \times \int \phi\left(\frac{x}{a}\right) \mathrm{c}^{-m^{(1)}x\varpi\sqrt{-1}} \times \cdots \times \int \phi\left(\frac{x}{a}\right) \mathrm{c}^{m^{(s-1)}xn\varpi\sqrt{-1}},$$

the $\int$ sign extending to all the values of x, from the extreme negative value of x to its positive extreme value. $\phi\left(\frac{x}{a}\right)$ is, as in the preceding sections, the probability of an error x in each observation; x being supposed, thus as a, formed from an infinity of parts taken for unity. It is clear that the coefficient of any exponent $\mathrm{c}^{l\varpi\sqrt{-1}}$, in the development of this product, will be the probability that the sum of the errors of the observations, multiplied respectively by m, $m^{(1)}$, $\ldots$, that is to say the function (*m*), will be equal to l; by multiplying therefore the preceding product by $\mathrm{c}^{-l\varpi\sqrt{-1}}$, the term independent of $\mathrm{c}^{\varpi\sqrt{-1}}$ and of its powers, in this new product, will express this probability. If one supposes, as we have done it here, the probability of the positive errors the same as that of the negative errors, one will have, for the sum $\int\phi\left(\frac{x}{a}\right)\mathrm{c}^{mx\varpi\sqrt{-1}}$, to reunite the terms multiplied, one by $\mathrm{c}^{mx\varpi\sqrt{-1}}$, and the other by $\mathrm{c}^{-mx\varpi\sqrt{-1}}$, then this sum takes the form $2\int\phi\left(\frac{x}{a}\right)\cos mx\varpi$. It is likewise of it of all the similar sums. Thence it follows that the probability that the function (*m*) will be equal to l is equal to

$$(i)\qquad \frac{1}{2\pi}\int d\varpi \left\{ \begin{aligned} &\mathrm{c}^{-l\varpi\sqrt{-1}} \times 2\int\phi\Big(\frac{x}{a}\Big)\cos mx\varpi \\ &\quad\times 2\int\phi\Big(\frac{x}{a}\Big)\cos m^{(i)}x\varpi \times\cdots\times 2\int\phi\Big(\frac{x}{a}\Big)\cos m^{(s-1)}x\varpi \end{aligned} \right\},$$

the integral being taken from $\varpi = -\pi$ to $\varpi = \pi$. One has, by reuniting the cosines into series,

$$\int\phi\Big(\frac{x}{a}\Big)\cos mx\varpi = \int\phi\Big(\frac{x}{a}\Big) - \frac{1}{2}m^2a^2\varpi^2\int\frac{x^2}{a^2}\phi\Big(\frac{x}{a}\Big) + \cdots$$

If one makes $\frac{x}{a} = x'$ and if one observes that, the variation of x being unity, one has $dx' = \frac{1}{a}$, one will have

$$\int\phi\Big(\frac{x}{a}\Big) = a\int dx'\phi(x').$$

We name, as in the preceding sections, k the integral $2\int dx'\phi(x')$, taken from x' null to its extreme positive value; we name similarly k'' the integral $\int x'^2dx'\phi(x')$, taken within the same limits, and thus consecutively; we will have

$$2\int\phi\Big(\frac{x}{a}\Big)\cos mx\varpi = ak\bigg(1 - \frac{k''}{k}m^2a^2\varpi^2 + \frac{k^{\text{iv}}}{12k}m^4a^4\varpi^4 - \cdots\bigg).$$

The logarithm of the second member of this equation is

$$-\frac{k''}{k}m^2a^2\varpi^2 + \frac{kk^{\text{iv}} - 6k''^2}{12k^2}m^4a^4\varpi^4 - \cdots + \log ak;$$

ak or $2a\int dx'\phi(x')$ expresses the probability that the error of each observation will be contained within its limits, this which is certain; one has therefore $ak = 1$; this which reduces the preceding logarithm to

$$-\frac{k''}{k}m^2a^2\varpi^2 + \frac{kk^{\text{iv}} - 6k''^2}{12k^2}m^4a^4\varpi^4 - \cdots$$

Thence it is easy to conclude that the product

$$2\int\phi\Big(\frac{x}{a}\Big)\cos mx\varpi \times 2\int\phi\Big(\frac{x}{a}\Big)\cos m^{(i)}x\varpi \times\cdots\times\ \times 2\int\phi\Big(\frac{x}{a}\Big)\cos m^{(s-1)}x\varpi$$

is

$$\left(1+\frac{kk^{\text{iv}}-6k''^2}{12k^2}a^4\varpi^4\text{S}m^{(i)4}+\cdots\right)c^{-\frac{k''}{k}a^2\varpi^2\text{S}m^{(i)2}};$$

the preceding integral (*i*) is reduced therefore to

$$\frac{1}{2\pi}\int d\varpi\left(1+\frac{kk^{\text{iv}}-6k''^2}{12k^2}a^4\varpi^4\text{S}m^{(i)4}+\cdots\right)c^{-lw\sqrt{-1}-\frac{k''}{k}a^2\varpi^2\text{S}m^{(i)2}}.$$

By making $sa^2\varpi^2=t^2$, this integral becomes

$$\frac{1}{2a\pi\sqrt{s}}\int dt\left(1+\frac{kk^{\text{iv}}-6k''^2}{12k^2}\frac{\text{S}m^{(i)4}}{s^2}t^4+\cdots\right)c^{-\frac{lw\sqrt{-1}}{a\sqrt{s}}-\frac{k''}{k}\frac{\text{S}m^{(i)2}}{s}t^2};$$

$\text{S}m^{(i)2}$, $\text{S}m^{(i)4}$, $\ldots$ are evidently some quantities of order s; thus $\frac{\text{S}m^{(i)4}}{s^2}$ is of order $\frac{1}{s}$; by neglecting therefore the terms of this last order vis-à-vis of unity, the last integral is reduced to

$$\frac{1}{2a\pi\sqrt{s}}\int dt c^{-\frac{lw\sqrt{-1}}{a\sqrt{s}}-\frac{k''}{k}\frac{\text{S}m^{(i)2}}{s}t^2}.$$

The integral relative to ϖ must be taken from $\varpi=-\pi$ to $\varpi=\pi$, the integral relative to t must be taken from $t=-a\pi\sqrt{s}$ to $t=a\pi\sqrt{s}$, and in these cases the exponential under the $\int$ sign is insensible to these two limits, either because s is a great number, or because a is here supposed divided into an infinity of parts taken for unity; one is able therefore to take the integral from $t=-\infty$ to $t=\infty$. We make

$$t'=\sqrt{\frac{k''\text{S}m^{(i)2}}{ks}}\left(t+\frac{l\sqrt{-1}k\sqrt{s}}{2ak''\text{S}m^{(i)2}}\right);$$

the preceding integral function becomes

$$\frac{c^{-\frac{kl^2}{4k''a^2\text{S}m^{(i)2}}}}{2a\pi\sqrt{\frac{k''}{k}\text{S}m^{(i)2}}}\int dt'c^{-t'^2}.$$

The integral relative to t' must be taken, as the integral relative to t, from $t'=-\infty$ to $t'=\infty$, this which reduces the preceding quantity to that here,

$$\frac{c^{-\frac{kl^2}{4k''a^2\mathrm{S}m^{(i)2}}}}{2a\sqrt{\pi}\sqrt{\frac{k''}{k}\mathrm{S}m^{(i)2}}}$$

If one makes $l = ar\sqrt{s}$ and if one observes that, the variation of l being unity, one has $adr = 1$, one will have

$$\frac{\sqrt{s}}{2\sqrt{\frac{k''\pi}{k}\mathrm{S}m^{(i)2}}}\int dr\, c^{-\frac{kr^2 s}{4k''\mathrm{S}m^{(i)2}}},$$

for the probability that the function (*m*) will be contained within the limits zero and $ar\sqrt{s}$, the integral being taken from r null.

We have need here of knowing the probability of the error u of the element determined by making null the function (*m*). This function being supposed equal to l or to $ar\sqrt{s}$, one will have, by that which precedes,

$$u\mathrm{S}m^{(i)}p^{(i)} = ar\sqrt{s};$$

by substituting this value into the preceding integral function, it becomes

$$\frac{\mathrm{S}m^{(i)}p^{(i)}}{2a\sqrt{\frac{k''\pi}{k}\mathrm{S}m^{(i)2}}}\int du\, c^{-\frac{ku^2(\mathrm{S}m^{(i)}p^{(i)})^2}{4k''a^2\mathrm{S}m^{(i)2}}};$$

it is the expression of the probability that the value of u will be contained within the limits zero and u, it is also the expression of the probability that u will be contained within the limits zero and $-u$. If one makes

$$u = 2at\sqrt{\frac{k''}{k}}\frac{\sqrt{\mathrm{S}m^{(i)2}}}{\mathrm{S}m^{(i)}p^{(i)}},$$

the preceding probability becomes

$$\frac{1}{\sqrt{\pi}}\int dt\, c^{-t^2}.$$

Now, the probability remains the same, t remains the same, and the interval of the two limits of u are tightened so much more as $a\sqrt{\frac{k''}{k}}\frac{\sqrt{\mathrm{S}m^{(i)2}}}{\mathrm{S}m^{(i)}p^{(i)}}$ is smaller. This interval remaining the same, the value of t, and consequently the probability that the error of the element falls within this interval, is so much greater as the same quantity $a\sqrt{\frac{k''}{k}}\frac{\sqrt{\mathrm{S}m^{(i)2}}}{\mathrm{S}m^{(i)}p^{(i)}}$ is smaller; it is necessary therefore to choose the system of factors $m^{(i)}$, which renders this quantity a minimum; and as a, k, k'' are the same in all these systems, it is necessary to choose the system which renders $\frac{\sqrt{\mathrm{S}m^{(i)2}}}{\mathrm{S}m^{(i)}p^{(i)}}$ a minimum.

One is able to arrive to the same result in this manner. We resume the expression of the probability that u will be contained within the limits zero and u. The coefficient of du in the differential of this expression is the ordinate of the curve of probabilities of the errors u of the element, errors represented by the abscissa u of this curve, that one is able to extend to infinity on each side of the ordinate which corresponds to u null. This put, each error, either positive, or negative, must be considered as a disadvantage or a real loss, in any game whatsoever; now, by the principles of the theory of probabilities, exposed at the beginning of this Book, one evaluates this disadvantage by taking the sum of all the products of each disadvantage by its probability; the mean value of the error to fear to the plus is therefore the sum of the products of each error by its probability; it is consequently equal to the integral

$$\frac{\int u\,du\,\mathrm{S}m^{(i)}p^{(i)}c^{-\frac{ku^2(\mathrm{S}m^{(i)}p^{(i)})^2}{4k''a^2\mathrm{S}m^{(i)2}}}}{2a\sqrt{\frac{k''\pi}{k}}\mathrm{S}m^{(i)2}},$$

taken from u null to u infinity; thus this error is

$$a\sqrt{\frac{k''}{k\pi}}\frac{\sqrt{\mathrm{S}m^{(i)2}}}{\mathrm{S}m^{(i)}p^{(i)}}.$$

This quantity, taken with the $-$ sign, gives the mean error to fear to the less. It is clear that the system of factors $m^{(i)}$ that it is necessary to choose must be such that these errors are some minima and consequently such that $\frac{\sqrt{\mathrm{S}m^{(i)2}}}{\mathrm{S}m^{(i)}p^{(i)}}$ is a minimum.

If one differentiates this function with respect to $m^{(i)}$, one will have, by equating its differential to zero, by the condition of the minimum,

$$\frac{m^{(i)}}{\mathrm{S}m^{(i)2}} = \frac{p^{(i)}}{\mathrm{S}m^{(i)}p^{(i)}}.$$

This equation holds, whatever be i, and, as the variation of i does not change the fraction $\frac{\mathrm{S}m^{(i)2}}{\mathrm{S}m^{(i)}p^{(i)}}$ at all, by naming μ this fraction, one will have

$$m=\mu p, \quad m^{(1)}=\mu p^{(1)}, \quad \ldots, \quad m^{(s-1)}=\mu p^{(s-1)},$$

and one is able, whatever be p, $p^{(1)}$, …, to take μ such that the numbers m, $m^{(1)}$, … are entire numbers, as the preceding analysis supposes it. Then one has

$$z=\frac{\mathrm{S}p^{(i)}\alpha^{(i)}}{\mathrm{S}p^{(i)2}},$$

and the mean error to fear becomes

$$\pm\frac{a\sqrt{\frac{k''}{k\pi}}}{\mathrm{S}p^{(i)2}}.$$

It is, in all the hypotheses that one is able to make on the factors m, $m^{(1)}$, …, the smallest possible error.

If one makes the values of m, $m^{(1)}$, … equal to ± 1, the mean error to fear will be smaller when the sign $\pm$ will be determined, in a manner that $m^{(i)}p^{(i)}$ is positive, this which returns to supposing $1=m=m^{(1)}=\cdots$, and to prepare the equations of condition, so that the coefficient of z in each of them is positive; this is that which one does in the ordinary method. Then the mean result of the observations is

$$z=\frac{\mathrm{S}\alpha^{(i)}}{\mathrm{S}p^{(i)}},$$

and the mean error to fear to the plus or to the less is

$$\pm\frac{a\sqrt{\frac{k''s}{k\pi}}}{\mathrm{S}p^{(i)}};$$

but this error surpasses the preceding, which, as one has seen is the smallest possible. One is able to be convinced of it besides in this manner. It suffices to show that one has the inequality

$$\frac{\sqrt{s}}{\mathrm{S}p^{(i)}}>\frac{1}{\sqrt{\mathrm{S}p^{(i)2}}}$$

or

$$s\mathrm{S}p^{(i)2} > (\mathrm{S}p^{(i)})^2.$$

In effect, $2pp^{(1)}$ is less than $p^2 + p^{(1)2}$, since $(p^{(1)} - p)^2$ is a positive quantity; one is able therefore, in the second member of the preceding inequality, to substitute, for $2pp^{(1)}$, $p^2 + p^{(1)2} - f$, f being a positive quantity. By making some similar substitutions for all the similar products, this second member will be equal to the first, less a positive quantity.

The result

$$z = \frac{\mathrm{S}p^{(i)}\alpha^{(i)}}{\mathrm{S}p^{(i)2}},$$

to which corresponds the minimum of mean error to fear, is the one which the method of least squares of the errors of the observations gives; because, the sum of these squares being

$$(pz - \alpha)^2 + (p^{(1)}z - \alpha^{(1)})^2 + \cdots + (p^{(s-1)}z - \alpha^{(s-1)})^2,$$

the condition of the minimum of this function, by making z vary, gives for this variable the preceding expression; this method must therefore be employed in preference, whatever be the law of facility of the errors, a law on which depends the ratio $\frac{k''}{k}$.

This ratio is $\frac{1}{6}$, if $\phi(x)$ is a constant; it is less than $\frac{1}{6}$, if $\phi(x)$ is variable, and such that it diminishes in measure as x increases, as it is natural to suppose. By adopting the mean law of errors, that we have given in nº 15 and following which $\phi(x)$ is equal to $\frac{1}{2a}\log\frac{a}{x}$, one has

$$\frac{k''}{k} = \frac{1}{18}.$$

As for the limits $\pm a$, one is able to take for these limits the deviations of the mean result, which would make to reject an observation.

But one is able, by the same observations, to determine the factor $a\sqrt{\frac{k''}{k}}$ of the expression of the mean error. Indeed, one has seen, in the preceding section, that the sum of the squares of the errors of the observations is very nearly $2s\frac{a^2k''}{k}$, and that, if they are in great number, it becomes extremely probable that the observed sum will not deviate from this value by a sensible quantity; one is able therefore to equate them; now the observed sum is equal to $\mathrm{S}\epsilon^{(i)2}$ or to

$\mathrm{S}(p^{(i)}z - \alpha^{(i)})^2$, by substituting for z its value $\frac{\mathrm{S}p^{(i)}\alpha^{(i)}}{\mathrm{S}p^{(i)2}}$; one finds thus

$$2s\frac{a^2k''}{k} = \frac{\mathrm{S}p^{(i)2}.\mathrm{S}\alpha^{(i)2} - (\mathrm{S}p^{(i)}\alpha^{(i)})^2}{\mathrm{S}p^{(i)2}}.$$

The preceding expression of the mean error to fear respecting the result z becomes then

$$\pm\sqrt{\frac{\mathrm{S}p^{(i)2}.\mathrm{S}\alpha^{(i)2} - (\mathrm{S}p^{(i)}\alpha^{(i)})^2}{\mathrm{S}p^{(i)2}\sqrt{2s\pi}}},$$

an expression in which there is nothing which is not given by the observations and by the coefficients of the equations of condition.

21. We suppose now that one has two elements to correct by the collection of a great number of observations. By naming z and z' the respective corrections of these elements, one will form, as in the preceding section, some equations of condition, which will be contained under this general form

$$\epsilon^{(i)} = p^{(i)}z + q^{(i)}z' - \alpha^{(i)},$$

$\epsilon^{(i)}$ being, as in that section, the error of the $(i+1)^{\text{st}}$ observation. If one multiplies respectively by m, $m^{(1)}$, …, $m^{(s-1)}$ these equations, and if one adds together these products, one will have a first final equation

$$\mathrm{S}m^{(i)}\epsilon^{(i)} = z.\mathrm{S}m^{(i)}p^{(i)} + z'.\mathrm{S}m^{(i)}q^{(i)} - \mathrm{S}m^{(i)}\alpha^{(i)}.$$

By multiplying further the same equations respectively by n, $n^{(1)}$, …, $n^{(s-1)}$ and adding these products, one will have a second final equation

$$\mathrm{S}n^{(i)}\epsilon^{(i)} = z.\mathrm{S}n^{(i)}p^{(i)} + z'.\mathrm{S}n^{(i)}q^{(i)} - \mathrm{S}n^{(i)}\alpha^{(i)},$$

the sign S extending here, as in the preceding section, to all the values of i, from $i = 0$ to $i = s - 1$.

If one supposes null the two functions $\mathrm{S}m^{(i)}\epsilon^{(i)}$, $\mathrm{S}n^{(i)}\epsilon^{(i)}$, functions which we will designate respectively by (*m*) and (*n*), the two preceding final equations will give the corrections z and z' of the two elements. But these corrections are susceptible of errors, relative to that of which the supposition that we have just made is itself susceptible. We imagine therefore that the functions (*m*) and (*n*), instead of being nulls, are respectively l and l', and we name u and u' the errors corresponding to the corrections z and z', determined by that which precedes; the two final equations will become

$$l = u.\mathrm{S}m^{(i)}p^{(i)} + u'.\mathrm{S}m^{(i)}q^{(i)},$$
$$l' = u.\mathrm{S}n^{(i)}p^{(i)} + u'.\mathrm{S}n^{(i)}q^{(i)}.$$

It is necessary now to determine the factors $m, m^{(1)}, \dots, n, n^{(1)}, \dots,$ in a manner that the mean error to fear respecting each element is a minimum. For this, we will consider the product

$$\int \phi\left(\frac{x}{a}\right) c^{-(m\varpi+n\varpi')x\sqrt{-1}} \times \int \phi\left(\frac{x}{a}\right) c^{-(m^{(1)}\varpi+n^{(1)}\varpi')x\sqrt{-1}} \times \cdots \times \int \phi\left(\frac{x}{a}\right) c^{-(m^{(s-1)}\varpi+n^{(s-1)}\varpi')x\sqrt{-1}},$$

the sign $\int$ referring to all the values of x, from $x = -a$ to $x = a$; $\phi\left(\frac{x}{a}\right)$ being, as in the preceding section, the probability of the error x, thus as of the error $-x$. The preceding function becomes, by reuniting the two exponentials relative to x and to $-x$,

$$2\int \phi\left(\frac{x}{a}\right)\cos(mx\varpi + nx\varpi') \times 2\int \phi\left(\frac{x}{a}\right)\cos(m^{(1)}x\varpi + n^{(1)}x\varpi') \times \cdots \times 2\int \phi\left(\frac{x}{a}\right)\cos(m^{(s-1)}x\varpi + n^{(s-1)}x\varpi'),$$

the sign $\int$ extending here to all the values of x, from $x = 0$ to $x = a$, x being supposed, thus as a, divided into an infinity of parts taken for unity. Presently it is clear that the term independent of the exponentials, in the product of the preceding function by $c^{-l\varpi\sqrt{-1}+l'\varpi'\sqrt{-1}}$, is the probability that the sum of the errors of each observation, multiplied respectively by $m, m^{(1)}, \dots,$ or the function (*m*), will be equal to l, at the same time as the function (*n*), sum of the errors of each observation, multiplied respectively by $n, n^{(1)}, \dots,$ will be equal to l'; this probability is therefore

$$\frac{1}{4\pi^2}\int\int d\varpi d\varpi' c^{-l\varpi\sqrt{-1}-l'\varpi'\sqrt{-1}} \left\{ \begin{array}{c} 2\int \phi\left(\frac{x}{a}\right)\cos(mx\varpi + nx\varpi') \times \cdots \\ \times 2\int \phi\left(\frac{x}{a}\right)\cos(m^{(s-1)}x\varpi + n^{(s-1)}x\varpi'), \end{array} \right\},$$

the integrals being taken from ϖ and ϖ' equal to $-\pi$, to ϖ and ϖ' equal to π. This put:

By following exactly the analysis of the preceding section, one finds that the preceding function is reduced to very nearly to

$$\frac{1}{4\pi^2}\int\int d\varpi d\varpi' c^{-l\varpi\sqrt{-1}-l'\varpi'\sqrt{-1}-\frac{k''}{k}a^2[\varpi^2\text{S}m^{(i)2}+2\varpi\varpi'.\text{S}m^{(i)}n^{(i)}+\varpi'^2.\text{S}n^{(i)2}]},$$

k and k'' having here the same signification as in the section cited. One sees further, by the same section, that the integrals are able to be extended from $a\varpi=-\infty$, $a\varpi'=-\infty$, to $a\varpi=\infty$ and $a\varpi'=\infty$. If one makes

$$\begin{aligned} t &= a\varpi+\frac{a\varpi'.\text{S}m^{(i)}n^{(i)}}{\text{S}m^{(i)2}}+\frac{kl\sqrt{-1}}{2k''a.\text{S}m^{(i)2}} \\ t' &= a\varpi'-\frac{k}{2k''a}\frac{(l\text{S}m^{(i)}n^{(i)}-l'\text{S}m^{(i)2})\sqrt{-1}}{\text{S}m^{(i)2}.\text{S}n^{(i)2}-(\text{S}m^{(i)}n^{(i)})^2}; \end{aligned}$$

if one makes next

$$E=\text{S}m^{(i)2}.\text{S}n^{(i)2}-(\text{S}m^{(i)}n^{(i)})^2,$$

the preceding double integral becomes

$$c^{-\frac{k}{4k''a^2E}[l^2\text{S}n^{(i)2}-2ll'\text{S}m^{(i)}n^{(i)}+l'^2\text{S}m^{(i)2}]}\times\int\int\frac{dt\,dt'}{4\pi^2a^2}c^{-\frac{k''l^2}{k}\text{S}m^{(i)2}-\frac{k''t'^2E}{k\text{S}m^{(i)2}}}.$$

By taking the integrals within the positive and negative infinite limits, as those relative to $a\varpi$ and $a\varpi'$, one will have

$$(o)\qquad \frac{1}{\frac{4k''\pi}{k}a^2\sqrt{E}}c^{-\frac{k}{4k''a^2}\frac{l^2\text{S}n^{(i)2}-2ll'\text{S}m^{(i)}n^{(i)}+l'^2\text{S}m^{(i)2}}{E}}.$$

It is necessary now, in order to have the probability that the values of l and of l' will be contained within the given limits, to multiply this quantity by $dl\ dl'$, and to integrate next within these limits. By naming X this quantity, the probability of which there is concern will be therefore $\int\int X\,dl\,dl'$. But, in order to have the probability that the errors u and u' of the corrections of the elements will be contained within the given limits, it is necessary to substitute within this integral, instead of l and l', their values in u and u'. Now, if one differentiates the expressions of l and of l', by supposing l' constant, one has

$$\begin{gathered}dl = du\mathrm{S}m^{(i)}p^{(i)} + du'\mathrm{S}m^{(i)}q^{(i)},\\ 0 = du\mathrm{S}n^{(i)}p^{(i)} + du'\mathrm{S}n^{(i)}q^{(i)},\end{gathered}$$

this which gives

$$dl = \frac{du(\mathrm{S}m^{(i)}p^{(i)}.\mathrm{S}n^{(i)}q^{(i)} - \mathrm{S}n^{(i)}p^{(i)}.\mathrm{S}m^{(i)}q^{(i)})}{\mathrm{S}n^{(i)}q^{(i)}}.$$

If one differentiates next the expression of l', by supposing u constant, one has

$$dl' = du'\mathrm{S}n^{(i)}q^{(i)};$$

one will have therefore

$$dl\,dl' = (\mathrm{S}m^{(i)}p^{(i)}.\mathrm{S}n^{(i)}q^{(i)} - \mathrm{S}n^{(i)}p^{(i)}.\mathrm{S}m^{(i)}q^{(i)})du\,du'.$$

By making next

$$\begin{aligned}F &= \mathrm{S}n^{(i)2}(\mathrm{S}m^{(i)}p^{(i)})^2 - 2\mathrm{S}m^{(i)}n^{(i)}.\mathrm{S}m^{(i)}p^{(i)}.\mathrm{S}n^{(i)}p^{(i)} + \mathrm{S}m^{(i)2}.(\mathrm{S}n^{(i)}p^{(i)})^2,\\ G &= \mathrm{S}n^{(i)2}.\mathrm{S}m^{(i)}p^{(i)}.\mathrm{S}m^{(i)}q^{(i)} + \mathrm{S}m^{(i)2}.\mathrm{S}n^{(i)}p^{(i)}.\mathrm{S}n^{(i)}q^{(i)}\\ &\qquad - \mathrm{S}m^{(i)}n^{(i)}.\mathrm{S}n^{(i)}p^{(i)}.\mathrm{S}m^{(i)}q^{(i)} + \mathrm{S}m^{(i)}p^{(i)}.\mathrm{S}n^{(i)}q^{(i)}.\\ H &= \mathrm{S}n^{(i)2}.(\mathrm{S}m^{(i)}q^{(i)})^2 - 2\mathrm{S}m^{(i)}n^{(i)}.\mathrm{S}m^{(i)}q^{(i)}.\mathrm{S}n^{(i)}q^{(i)} + \mathrm{S}m^{(i)2}.(\mathrm{S}n^{(i)}q^{(i)})^2,\\ I &= \mathrm{S}m^{(i)}p^{(i)}.\mathrm{S}n^{(i)}q^{(i)} - \mathrm{S}n^{(i)}p^{(i)}.\mathrm{S}m^{(i)}q^{(i)},\end{aligned}$$

the function (o) becomes

$$\int\int \frac{k}{4k''\pi}\frac{1}{\sqrt{E}}\frac{du\,du'}{a^2}c^{-\frac{k(Fu^2+2Guu'+Hu'^2)}{4k''a^2E}}.$$

We integrate first this function from $u' = -\infty$ to $u' = \infty$. If one makes

$$t = \frac{\sqrt{\frac{kH}{4k''}}\left(u' + \frac{Gu}{H}\right)}{a\sqrt{E}},$$

and if one takes the integral from $t = -\infty$ to $t = \infty$, one will have, by considering of it only the variation of u',

$$\int\sqrt{\frac{k}{4k''\pi}}\frac{du}{a}\frac{1}{\sqrt{H}}c^{-\frac{ku^2}{4k''a^2}\frac{FH-G^2}{EH}}.$$

Now one has

$$\frac{FH - G^2}{E} = I^2;$$

the preceding integral becomes therefore

$$\int \frac{1}{\sqrt{H}} \frac{du}{a} \sqrt{\frac{k}{4k''\pi}} c^{-\frac{k}{4k''}\frac{I^2u^2}{a^2H}}.$$

One will have, by the preceding section, the mean error to fear, to the more or to the less, respecting the correction of the first element, by multiplying the quantity under the sign $\int$ by $\pm u$, and taking the integral from $u = 0$ to $u = \infty$, this which gives, for this error,

$$\pm \frac{a\sqrt{H}}{I\sqrt{\frac{k\pi}{k''}}},$$

the $+$ sign indicating the mean error to fear to the plus, and the $-$ sign the mean error to fear to the less.

We determine presently the factors $m^{(i)}$ and $n^{(i)}$, in a manner that this error is a minimum. By making $m^{(i)}$ vary alone, one has

$$\begin{aligned} d\log\frac{\sqrt{H}}{I} &= dm^{(i)}\frac{-p^{(i)}\mathrm{S}n^{(i)}q^{(i)} + q^{(i)}\mathrm{S}n^{(i)}p^{(i)}}{I} \\ &+ dm^{(i)}\frac{\left\{\begin{array}{l} q^{(i)}\mathrm{S}n^{(i)2}.\mathrm{S}m^{(i)}q^{(i)} - n^{(i)}.\mathrm{S}m^{(i)}q^{(i)}.\mathrm{S}n^{(i)}q^{(i)} \\ \quad - q^{(i)}.\mathrm{S}m^{(i)}n^{(i)}.\mathrm{S}n^{(i)}q^{(i)} + m^{(i)}(\mathrm{S}n^{(i)}q^{(i)})^2 \end{array}\right\}}{H} \end{aligned}$$

It is easy to see that this differential disappears, if one supposes, in the coefficients of $dm^{(i)}$,

$$m^{(i)} = \mu p^{(i)}, \quad n^{(i)} = \mu q^{(i)},$$

μ being an arbitrary coefficient independent of i, and by means of which one is able to render $m^{(i)}$ and $n^{(i)}$ some whole numbers; the preceding supposition renders therefore null the differential of $\frac{\sqrt{H}}{I}$, taken with respect to $m^{(i)}$. One will see in the same manner that this supposition renders null the differential of the same quantity, taken with respect to $n^{(i)}$. Thus this supposition renders a minimum the mean error to fear respecting the correction of the first element; and one will see in the same manner that it renders further a minimum mean error to fear respecting the correction of the second element, an error that one obtains by changing in the expression of the preceding H into F. Under this supposition, the corrections of the two elements are

$$z = \frac{\mathrm{S}q^{(i)2}.\mathrm{S}p^{(i)}\alpha^{(i)} - \mathrm{S}p^{(i)}q^{(i)}.\mathrm{S}q^{(i)}\alpha^{(i)}}{\mathrm{S}p^{(i)2}.\mathrm{S}q^{(i)2} - (\mathrm{S}p^{(i)}q^{(i)})^2},$$
$$z' = \frac{\mathrm{S}p^{(i)2}.\mathrm{S}q^{(i)}\alpha^{(i)} - \mathrm{S}p^{(i)}q^{(i)}.\mathrm{S}p^{(i)}\alpha^{(i)}}{\mathrm{S}p^{(i)2}.\mathrm{S}q^{(i)2} - (\mathrm{S}p^{(i)}q^{(i)})^2}.$$

It is easy to see that these corrections are those that the method of least squares of the errors of the observations gives, or of the minimum of the function

$$\mathrm{S}(p^{(i)}z + q^{(i)}z' - \alpha^{(i)})^2;$$

whence it follows that this method holds generally, whatever be the number of elements to determine; because it is clear that the previous analysis can be extended to any number of elements.

By substituting for $a\sqrt{\frac{k''}{k\pi}}$ the quantity $\sqrt{\frac{\mathrm{S}\epsilon^{(i)2}}{2s\pi}}$, to which one is able, by n° 20, to suppose it equal, ϵ, $\epsilon^{(1)}$, ... being that which remains in the equations of condition after having substituted there the corrections given by the method of least squares of the errors, the mean error to fear respecting the first element is

$$\pm\frac{\sqrt{\frac{\mathrm{S}\epsilon^{(i)2}}{2s\pi}}\sqrt{\mathrm{S}q^{(i)2}}}{\sqrt{\mathrm{S}p^{(i)2}.\mathrm{S}q^{(i)2} - (\mathrm{S}p^{(i)}q^{(i)})^2}}.$$

The mean error to fear to the more or to the less respecting the second element is

$$\pm\frac{\sqrt{\frac{\mathrm{S}\epsilon^{(i)2}}{2s\pi}}\sqrt{\mathrm{S}p^{(i)2}}}{\sqrt{\mathrm{S}p^{(i)2}.\mathrm{S}q^{(i)2} - (\mathrm{S}p^{(i)}q^{(i)})^2}}.$$

whence one sees that the first element is more or less well determined as the second, according as $\mathrm{S}q^{(i)2}$ is smaller or greater than $\mathrm{S}p^{(i)2}$.

If the r first equations of condition do not contain q at all, and if the $s-r$ last do not contain p at all, then $\mathrm{S}p^{(i)}q^{(i)} = 0$, and the preceding formulas coincide with that of the preceding section.

One is able to obtain thus the mean error to fear respecting each element determined by the method of least squares of the errors, whatever be the number of elements, provided that one considers a great number of observations. Let z, z', z'', z''', $\ldots$ be the corrections of each element, and we represent generally the equations of condition by the following:

$$\epsilon^{(i)} = p^{(i)}z + q^{(i)}z' + r^{(i)}z'' + t^{(i)}z''' + \cdots - \alpha^{(i)}.$$

In the case of a single element, the mean error to fear is, as one has seen,

$$(a) \qquad \pm\sqrt{\frac{\mathrm{S}\epsilon^{(i)2}}{2s\pi}}\,\frac{1}{\sqrt{\mathrm{S}p^{(i)2}}}.$$

When there are two elements, one will have the mean error to fear respecting the first element by changing, in the function (*a*), $\mathrm{S}p^{(i)2}$ into $\mathrm{S}p^{(i)2} - \frac{(\mathrm{S}p^{(i)}q^{(i)})^2}{\mathrm{S}q^{(i)2}}$, this which gives, for this error,

$$(a') \qquad \pm\frac{\sqrt{\frac{\mathrm{S}\epsilon^{(i)2}}{2s\pi}}\sqrt{\mathrm{S}q^{(i)2}}}{\sqrt{\mathrm{S}p^{(i)2}.\mathrm{S}q^{(i)2} - (\mathrm{S}p^{(i)}q^{(i)})^2}}.$$

When there are three elements, one will have the error to fear respecting the first element, by changing, in this expression (*a*′), $\mathrm{S}p^{(i)2}$ into $\mathrm{S}p^{(i)2} - \frac{(\mathrm{S}p^{(i)}r^{(i)})^2}{\mathrm{S}r^{(i)2}}$, $\mathrm{S}p^{(i)}q^{(i)}$ into $\mathrm{S}p^{(i)}q^{(i)} - \frac{\mathrm{S}p^{(i)}r^{(i)}.\mathrm{S}q^{(i)}r^{(i)}}{\mathrm{S}r^{(i)2}}$, and $\mathrm{S}q^{(i)2}$ into $\mathrm{S}q^{(i)2} - \frac{(\mathrm{S}q^{(i)}r^{(i)})^2}{\mathrm{S}r^{(i)2}}$; this which gives for this error

$$(a'') \qquad \pm\frac{\sqrt{\frac{\mathrm{S}\epsilon^{(i)2}}{2s\pi}}\sqrt{\mathrm{S}q^{(i)2}.\mathrm{S}r^{(i)2} - (\mathrm{S}q^{(i)}r^{(i)})^2}}{\sqrt{\left\{\begin{array}{c}\mathrm{S}p^{(i)2}.\mathrm{S}q^{(i)2}.\mathrm{S}r^{(i)2} - \mathrm{S}p^{(i)2}(\mathrm{S}q^{(i)}r^{(i)})^2 - \mathrm{S}q^{(i)2}(\mathrm{S}p^{(i)}r^{(i)})^2\\ - \mathrm{S}r^{(i)2}(\mathrm{S}p^{(i)}q^{(i)})^2 + 2\mathrm{S}p^{(i)}q^{(i)}.\mathrm{S}p^{(i)}r^{(i)}.\mathrm{S}q^{(i)}r^{(i)}\end{array}\right\}}}.$$

In the case of four elements, one will have the error to fear respecting the first element, by changing in this expression (*a*″), $\mathrm{S}p^{(i)2}$ into $\mathrm{S}p^{(i)2} - \frac{(\mathrm{S}p^{(i)}t^{(i)})^2}{\mathrm{S}t^{(i)2}}$, $\mathrm{S}p^{(i)}q^{(i)}$ into $\mathrm{S}p^{(i)}q^{(i)} - \frac{\mathrm{S}p^{(i)}t^{(i)}.\mathrm{S}q^{(i)}t^{(i)}}{\mathrm{S}t^{(i)2}}$, etc. By continuing thus, one will have the mean error to fear respecting the first element, whatever be the number of elements. By changing, in the expression of this error, that which is relative to the first element, into that which is relative to the second and reciprocally, one will have the mean error to fear respecting the second element, and thus of the others.

Thence the result a simple way to compare among them diverse astronomical Tables, on the side of precision. These Tables are able always to be supposed reduced to the same form, and then they differ only by the epochs, the mean movements and the coefficients of their arguments; because, if one of them, for example, contains an argument which is not found at all in the others, it is clear that that returns to suppose, in that here, this coefficient null. Now, if one compared these Tables to the totality of the good observations, by rectifying them through this comparison, these Tables, thus rectified, would satisfy, by that which precedes, the condition that the sum of the squares of the errors that they would permit to subsist yet be a minimum. The Tables which would approach most to fulfill this condition would merit therefore preference, whence it follows that by comparing these diverse Tables to a considerable number of observations, the presumption of exactitude must be in favor of that in which the sum of the squares of the errors is smaller than in the others.

22. To here we have supposed the facilities of the positive errors the same as those of the negative errors. We consider now the general case in which these facilities are able to be different. We name a the interval in which the errors of each observation are able to be extended, and we suppose it divided into an infinite number $n + n'$ of equal parts and taken for unity, n being the number of the parts which correspond to the negative errors, and n' being the number of the parts which correspond to the positive errors. On each point of the interval a we raise an ordinate which expresses the probability of the corresponding error, and we designate by $\phi\left(\frac{x}{n+n'}\right)$ the ordinate corresponding to the error x. This put, we will consider the sequence

$$\begin{gathered}\phi\left(\frac{-n}{n+n'}\right)c^{-qn\varpi\sqrt{-1}} + \phi\left[\frac{-(n-1)}{n+n'}\right]c^{-q(n-1)\varpi\sqrt{-1}} + \cdots\\ + \phi\left(\frac{-1}{n+n'}\right)c^{-q\varpi\sqrt{-1}} + \phi\left(\frac{0}{n+n'}\right) + \phi\left(\frac{1}{n+n'}\right)c^{q\varpi\sqrt{-1}} + \cdots\\ + \phi\left(\frac{n'-1}{n+n'}\right)c^{-q(n'-1)\varpi\sqrt{-1}} + \phi\left(\frac{n'}{n+n'}\right)c^{qn'\varpi\sqrt{-1}}.\end{gathered}$$

We represent this sequence by $\int\phi\left(\frac{x}{n+n'}\right)c^{qx\varpi\sqrt{-1}}$, the $\int$ sign extending to all the values of x, from $x = -n$ to $x = n'$. The term independent of $c^{\varpi\sqrt{-1}}$ and of its powers, in the development of the function

$$c^{-(l+\mu)\varpi\sqrt{-1}}\int\phi\left(\frac{x}{n+n'}\right)c^{qx\varpi\sqrt{-1}}\int\phi\left(\frac{x}{n+n'}\right)c^{q^{(1)}x\varpi\sqrt{-1}}\cdots\int\phi\left(\frac{x}{n+n'}\right)c^{q^{(s-1)}x\varpi\sqrt{-1}},$$

will be, by nº 21, the probability that the function

$$(m)\qquad q\epsilon+q^{(1)}\epsilon^{(1)}+\cdots+q^{(s-1)}\epsilon^{(s-1)}$$

will be equal to $l+\mu$; this probability is therefore

$$(1)\qquad \frac{1}{2\pi}\int d\varpi\, c^{-l\varpi\sqrt{-1}}c^{-\mu\varpi\sqrt{-1}}\int\phi\left(\frac{x}{n+n'}\right)c^{qx\varpi\sqrt{-1}}\times\cdots,$$

the integral being taken from $\varpi=-\pi$ to $\varpi=\pi$. The logarithm of the function

$$(2)\qquad c^{-\mu\varpi\sqrt{-1}}\int\phi\left(\frac{x}{n+n'}\right)c^{qx\varpi\sqrt{-1}}\times\int\phi\left(\frac{x}{n+n'}\right)c^{q^{(1)}x\varpi\sqrt{-1}}\cdots$$

is

$$-\mu\varpi\sqrt{-1}+\log\left[\int\phi\left(\frac{x}{n+n'}\right)c^{qx\varpi\sqrt{-1}}\right]+\cdots$$

n and n' being supposed infinite numbers, if one makes

$$\frac{x}{n+n'}=x',\quad \frac{1}{n+n'}=dx';$$

if, moreover, one supposes

$$k=\int dx'\phi(x'),\quad k'=\int x'dx'\phi(x'),\quad k''=\int x'^2dx'\phi(x'),\quad \ldots,$$

the integrals being taken from $x'=-\frac{n}{n+n'}$ to $x'=\frac{n}{n+n'}$, one will have

$$\int\phi\left(\frac{x}{n+n'}\right)c^{qx\varpi\sqrt{-1}}=(n+n')k\left\{\begin{array}{l}1+\dfrac{k'}{k}q(n+n')\varpi\sqrt{-1}\\ \quad-\dfrac{k''}{2k}q^2(n+n')^2\varpi^2+\cdots\end{array}\right\}.$$

The error of each observation must fall within the limits $-n$ and $+n'$, and the probability that this will hold being $\int\phi\left(\frac{x}{n+n'}\right)$ or $(n+n')k$, this quantity must be equal to unity. Thence it is easy to conclude that the logarithm of the function (2) is, by making $\mu' = \frac{\mu}{n+n'}$,

$$\left(\frac{k'}{k}\mathrm{S}q^{(i)} - \mu'\right)(n+n')\varpi\sqrt{-1} - \frac{kk''-{k'}^2}{2k^2}\mathrm{S}q^{(i)2}(n+n')^2\varpi^2 + \cdots,$$

the sign S embracing all the values of i, from i null to $i = s-1$. One will make disappear the first power of ϖ, by making

$$\mu' = \frac{k'}{k}\mathrm{S}q^{(i)},$$

and if one considers only the second power, this which one is able to do by that which precedes, when s is a very great number, one will have, for the logarithm of the function (2),

$$-\frac{kk''-{k'}^2}{2k^2}\mathrm{S}q^{(i)2}(n+n')^2\varpi^2.$$

By passing again from the logarithms to the numbers, the function (2) is transformed into the following

$$c^{-\frac{kk''-{k'}^2}{2k^2}(n+n')^2\varpi^2\mathrm{S}q^{(i)2}};$$

the integral (1) becomes thus

$$\frac{1}{2\pi}\int d\varpi\, c^{-lw\sqrt{-1}}\, c^{-\frac{kk''-{k'}^2}{2k^2}(n+n')^2\varpi^2\mathrm{S}q^{(i)2}}.$$

We suppose

$$l = (n+n')r\sqrt{\mathrm{S}q^{(i)2}},$$
$$t = \sqrt{\frac{(kk''-{k'}^2)\mathrm{S}q^{(i)2}}{2k^2}}(n+n')\varpi - \frac{r\sqrt{-1}}{2}\sqrt{\frac{2k^2}{kk''-{k'}^2}}.$$

The variation of l being unity, one will have

$$1 = (n+n')dr\sqrt{\mathrm{S}q^{(i)2}};$$

the preceding integral becomes thus, after having it integrated from $t = -\infty$ to $t = \infty$,

$$\frac{kdr}{\sqrt{2(kk''-k'^2)\pi}}c^{-\frac{k^2r^2}{2(kk''-k'^2)}}$$

Thus the probability that the function (*m*) will be contained within the limits

$$\frac{ak'}{k}\mathrm{S}q^{(i)} \pm ar\sqrt{\mathrm{S}q^{(i)2}},$$

is equal to

$$\frac{2}{\sqrt{\pi}}\int\frac{kdr}{\sqrt{2(kk''-k'^2)}}c^{-\frac{k^2r^2}{2(kk''-k'^2)}},$$

the integral being taken from r null.

$\frac{ak'}{k}$ is the abscissa of the ordinate which passes through the center of gravity of the area of the curve of the probabilities of the errors of each observation; the product of this abscissa by $\mathrm{S}q^{(i)}$ is therefore the mean result toward which the function (*m*) converges without ceasing. If one supposes $1 = q = q^{(1)} = \cdots$ the function (*m*) becomes the sum of the errors, and then $\mathrm{S}q^{(i)}$ becomes s; therefore by dividing the sum of the errors by s, in order to have the mean error, this error converges without ceasing toward the abscissa of the center of gravity, in a manner that by taking on both sides any interval whatsoever as small as one will wish, the probability that the mean error will fall within this interval will finish, by multiplying indefinitely the observations, with differing from certainty only by a quantity less than every given magnitude.

23. We have just investigated the mean result that some observations numerous and not yet made must indicate with most advantage, and the law of probability of the errors of this result. We will consider presently the mean result of observations already made and of which one knows the respective deviations. For this, we imagine a number s of observations of the same kind, that is to say such that the law of errors is the same for all. We name A the result of the first, $A + q$ the one of the second, $A + q^{(1)}$ the one of the third, and thus in sequence; q, $q^{(1)}$, $q^{(2)}$, $\ldots$ being positive and increasing quantities, this which one is always able to obtain by a convenient disposition of the observations. We designate further by $\phi(z)$ the probability of the error z for each observation, and we suppose that $A + x$ is the true result. The error of the first observation is then $-x$; $q - x$, $q^{(1)} - x$, $\ldots$ are the errors of the second, of the third, etc. The probability of the simultaneous existence of all these errors is the product of their

respective probabilities; it is therefore

$$\phi(-x)\phi(q-x)\phi(q^{(1)}-x)\cdots$$

Now, x being susceptible of an infinity of values, by considering them as so many causes of the observed event, the probability of each of them will be, by n° 1,

$$\frac{dx\,\phi(-x)\phi(q-x)\phi(q^{(1)}-x)\cdots}{\int dx\,\phi(-x)\phi(q-x)\phi(q^{(1)}-x)\cdots},$$

the integral of the denominator being taken for all the values of which x is susceptible. We name $\frac{1}{H}$ this denominator. This put, we imagine a curve of which x is the abscissa, and of which the ordinate y is

$$H\phi(-x)\phi(q-x)\phi(q^{(1)}-x)\cdots,$$

this curve will be that of the probabilities of the values of x. The value that it is necessary to choose for the mean result is that which renders the mean error to fear a minimum. Each error, either positive, or negative, must be considered as a disadvantage, or a real loss in the game, one has the mean disadvantage, by taking the sum of the products of each disadvantage by its probability; the mean value of the error to fear is therefore the sum of the products of each error, setting aside the sign, by its probability. We determine the abscissa that it is necessary to choose in order that this sum is a minimum. For this, we give to the abscissas for origin the first extremity of the preceding curve, and we name x' and y' the coordinates of the curve, by departing from this origin. Let l be the value that it is necessary to choose. It is clear that, if the true result were x', the error of the result l would be, setting aside the sign, $l-x'$, as much as x' would be less than l; or y' is the probability that x' is the true result; the sum of the errors to fear, setting aside the sign, multiplied by their probability, is therefore for all the values of x' less than l, $\int(l-x')y'dx'$, the integral being taken from $x'=0$ to $x'=l$. One will see in the same manner that, for the values of x' superior to l, the sum of the errors to fear, multiplied by their probability, is $\int(x'-l)y'dx'$, the integral being taken from $x'=l$ to the abscissa x' corresponding to the last extremity of the curve; the entire sum of the errors to fear, setting aside the sign, multiplied by their respective probabilities, is therefore

$$\int (l - x')y'dx' + \int (x' - l)y'dx'.$$

The differential of this function, taken with respect to l, is

$$dl\int y'dx' - dl\int y'dx';$$

because one has the differential of $\int (l - x')y'dx'$, by differentiating first the value of l under the $\int$ sign, and by adding to this differential the increase which results from the variation of the limit of the integral, a limit which is changed into $l + dl$. This increase is equal to the element $(l - x')y'dx'$, to the limit where $x' = l$; it is therefore null, and $dl\int y'dx'$ is the differential of the integral $\int (l - x')y'dx'$. One will see in the same manner that $-dl\int y'dx'$ is the differential of the integral $\int (x' - l)y'dx'$. The sum of these differentials is null relative to the abscissa l, for which the mean error to fear is a minimum; one has therefore, relative to this abscissa,

$$\int y'dx' = \int y'dx',$$

the first integral being taken from $x' = 0$ to $x' = l$, and the second being taken from $x' = l$ to the extreme value of x'.

It follows thence that the abscissa which renders the mean error to fear a minimum is that of which the ordinate divides the area of the curve into two equal parts. This point enjoys further the property to be the one on the side of which it is also probable that the true result falls, that beyond, and by this reason it is able further to be named *middle of probability*. Some celebrated geometers have taken for the middle that it is necessary to choose the one which renders the observed result the most probable, and consequently the abscissa which corresponds to the greatest ordinate of the curve; but the middle that we adopt is evidently indicated by the theory of probabilities.

If one puts $\phi(x)$ under the form of an exponential, and if one designates it by $c^{-\psi(x')}$, so that it is able equally to agree to the positive and negative errors, one will have

$$y = Hc^{-\psi(x^2)-\psi(x-q)^2-\psi(x-q^{(1)})^2-\cdots} \tag{1}$$

If one makes $x = a + z$, and if one develops the exponent of c with respect to the powers of z, y will take this form

$$y = Hc^{-M-2Nz-Pz^2-Qz^3-\cdots},$$

an expression in which one has

$$\begin{aligned}
&M = \psi(a^2) + \psi(a-q)^2 + \psi(a-q^{(1)})^2 + \cdots,\\
&N = a\psi'(a^2) + (a-q)\psi'(a-q)^2 + (a-q^{(1)})\psi'(a-q^{(1)})^2 + \cdots,\\
&P = \psi'(a^2) + \psi'(a-q)^2 + \psi'(a-q^{(1)})^2 + \cdots + 2a^2\psi''(a^2)\\
&\qquad\qquad + 2(a-q)^2\psi''(a-q)^2 + a(a-q^{(1)})^2\psi''(a-q^{(1)})^2 + \cdots,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,
\end{aligned}$$

$\psi'(t)$ being the coefficient of dt in the differential of $\psi(t)$, $\psi''t)$ being the coefficient of dt in the differential of $\psi'(t)$, and thus consecutively.

We suppose the number s of observations very great, and we determine a by the equation $N = 0$ which gives the condition of the maximum of y; then one has

$$y = Hc^{-M-Pz^2-Qz^3-\cdots}.$$

M, P, Q, $\ldots$ are of order s; now, if z is very small of order $\frac{1}{\sqrt{s}}$, Qz^3 becomes of order $\frac{1}{\sqrt{s}}$, and the exponential $c^{-Qz^3-\cdots}$ is able to be reduced to unity. Thus, in the interval from $z = 0$ to $z = \frac{r}{\sqrt{s}}$, one is able to suppose

$$r = Hc^{-M-Pz^2}.$$

Farther on, and when z is of order $s^{-\frac{m}{2}}$, m being smaller than unity, Pz^2 becomes of order s^{1-m}; consequently c^{-Pz^2} becomes, thus as y, insensible; so that one is able, in all extent of the curve, to suppose

$$y = Hc^{-M-Pz^2}.$$

The value of a given by the equation $N = 0$, or

$$0 = a\psi'(a^2) + (a-q)\psi'(a-q)^2 + (a-q^{(1)})\psi'(a-q^{(1)})^2 + \cdots,$$

is then the abscissa x corresponding to the ordinate which divides the area of the curve into equal parts. The condition that the entire area of the curve must represent certitude or unity gives

$$\frac{1}{H} = \int dz\, c^{-M-Pz^2},$$

the integral being taken from $z = -\infty$ to $z = \infty$, this which gives

$$H = \frac{c^M \sqrt{P}}{\sqrt{\pi}}.$$

The mean error to fear to the plus or to the less, by taking a for the mean result of the observations, is $\pm \int zydz$, the integral being taken from z null to z infinity, this which gives for this error

$$\pm \frac{1}{2\sqrt{\pi P}}.$$

But the entire ignorance where one is of the law $c^{-\psi(x^2)}$ of the errors of each observation does not permit forming the equation

$$0 = a\psi'(a^2) + (a-q)\psi'(a-q)^2 + \cdots,$$

Thus, knowledge of the values of q, $q^{(1)}$, $\ldots$ sheds *a posteriori* no light on the mean result a of the observations, it is necessary to be held to the most advantageous result determined *a priori*, and that one has seen to be the one which furnishes the method of least squares of the errors.

We seek the function $\psi(x^2)$ which gives constantly the rule of the arithmetic means, admired by the observers. For this, we imagine that, out of the s observations, the first i coincide, thus as the $s-i$ last. The equation $N = 0$ becomes then

$$0 = ia\psi'(a^2) + (s-i)(a-q)\psi'(a-q)^2.$$

The rule of the arithmetic mean gives

$$a = \frac{s-i}{s}q;$$

the preceding equation becomes thus

$$\psi'\left[\left(\frac{s-i}{s}\right)^2 q^2\right] = \psi\left(\frac{i^2}{s^2}q^2\right).$$

This equation must hold whatever be $\frac{i}{s}$ and q, it is necessary that $\psi'(t)$ is independent of t, this which gives

$$\psi'(t) = k,$$

k being a constant. By integrating, one has

$$\psi(t) = kt - L,$$

L being an arbitrary constant; hence,

$$c^{-\psi(x^2)} = c^{L-kx^2}.$$

Such is therefore the function which is able alone to give generally the rule of the arithmetic means. The constant L must be determined in a manner that the integral $\int dx\, c^{L-kx^2}$, taken from $x = -\infty$ to $x = \infty$, is equal to unity; because it is certain that the error x of an observation must fall within these limits; one has therefore

$$c^L = \sqrt{\frac{k}{\pi}};$$

consequently the probability of the error is $\sqrt{\frac{k}{\pi}} c^{-kx^2}$.

In truth, this expression gives infinity for the limit of the errors, this which is not admissible; but, seeing the rapidity with which this kind of exponential diminishes in measure as x increases, one is able to take k rather great for which beyond the admissible limit of the errors their probabilities are insensible and able to be supposed null.

The preceding law of errors gives, for the general expression (1) of y,

$$y = \sqrt{\frac{sk}{\pi}} e^{-ksu^2},$$

by determining H in a manner that the entire integral $\int y dx$ is unity, and making

$$x = \frac{\mathrm{S}q^{(i)}}{s} + u.$$

The ordinate which divides the area of the curve into two equal parts is that which corresponds to $u = 0$ and consequently to

$$x = \frac{\mathrm{S}q^{(i)}}{s};$$

this is therefore the value of x that it is necessary to choose for the mean result of the observations; now this value is that which the rule of the arithmetic means gives; the preceding law of errors of each observation gives therefore constantly the same results as this rule, and one has seen that it is the only law which enjoys this property.

By adopting this law, the probability of the error $\epsilon^{(i)}$ of the $(i+1)^{\text{st}}$ observation is

$$\sqrt{\frac{k}{\pi}} e^{-k\epsilon^{(i)2}};$$

now one has seen in n$^{\text{o}}$ 20 that, z being the correction of an element, this observation furnishes the equation of condition

$$\epsilon^{(i)} = p^{(i)}z - \alpha^{(i)}.$$

The probability of the value of $p^{(i)}z - \alpha^{(i)}$ is therefore

$$\sqrt{\frac{k}{\pi}} e^{-k(p^{(i)}z-\alpha^{(i)})^2};$$

the probability of the simultaneous existence of the s values $pz - \alpha$, $p^{(1)}z - \alpha^{(1)}$, $\ldots$, $p^{(s-1)}z - \alpha^{(s-1)}$ will be therefore

$$\left(\sqrt{\frac{k}{\pi}}\right)^{s-1} e^{-k\mathrm{S}(p^{(i)}z-\alpha^{(i)})^2}.$$

This probability varies with z; one will have the probability of any value whatsoever of z by multiplying this quantity by dz and dividing the product by the integral of this product; taken from $z = -\infty$ to $z = \infty$. Let

$$z = \frac{\mathrm{S}p^{(i)}q^{(i)}}{\mathrm{S}p^{(i)2}} + u;$$

this probability becomes

$$du\sqrt{\frac{k\mathrm{S}p^{(i)2}}{\pi}} e^{-ku^2\mathrm{S}p^{(i)2}},$$

so that, if one describes a curve of which the coefficient of du is the ordinate and of which u is the abscissa, this curve, extended from $u = -\infty$ to $u = \infty$, is able to be considered as the curve of the probabilities of the errors u, of which the result

$$z = \frac{\mathrm{S}p^{(i)}\alpha^{(i)}}{\mathrm{S}p^{(i)2}}$$

is susceptible. The ordinate which divides the area of the curve into two equal parts is that which corresponds to $u = 0$, and consequently to z equal to $\frac{\mathrm{S}p^{(i)}\alpha^{(i)}}{\mathrm{S}p^{(i)2}}$; this result is therefore the one that it is necessary to choose; now it is the same as the one which the method of least squares of errors of observations gives; the preceding law of errors of each observation leads therefore to the same results as this method.

The method of least squares of errors becomes necessary when there is concern to take a mean of many given results, each, by the collection of a great number of observations of diverse kinds. We suppose that a like element is given: 1° by the mean result of s observations of a first kind and that it is, by these observations, equal to A; 2° by the mean result of s' observations of a second kind and that it is equal to $A + q$; 3° by the mean result of s'' observations of a third kind and that it is equal to $A + q'$, and thus of the remaining. If one represents by $A + x$ the true element, the error of the result of the observations s will be $-x$; by supposing therefore β equal to

$$\sqrt{\frac{k}{k''}}\frac{\sqrt{\mathrm{S}p^{(i)2}}}{2a},$$

if one makes use of the method of least squares of errors in order to determine the mean result, or to

$$\sqrt{\frac{k}{k''}}\frac{\mathrm{S}p^{(i)2}}{2a\sqrt{s}},$$

if one makes use of the ordinary method; the probability of this error will be, by n° 20,

$$\frac{\beta}{\sqrt{\pi}}c^{-\beta^2x^2}.$$

The error of the result of the s' observations will be $q - x$, and, by designating by β' for these observations that which we have named β for the s observations, the probability of this error will be

$$\frac{\beta'}{\sqrt{\pi}}c^{-\beta'^2(x-q)^2}.$$

Similarly, the error of the result of the s'' observations will be $q'-x$, and by naming for them β''that which we have named β for the s observations, the probability of this error will be

$$\frac{\beta''}{\sqrt{\pi}}c^{-\beta''^2(x-q')^2}.$$

and thus consecutively. The product of all these probabilities will be the probability that $-x$, $q-x$, $q'-x$, $\ldots$ will be the errors of the mean results of the observations s, s', s'', $\ldots$ By multiplying it by dx and taking the integral from $x=-\infty$ to $x=\infty$, one will have the probability that the mean results of the observations s', s'', $\ldots$ will surpass respectively by q, q', $\ldots$ the mean result of the s observations.

If one takes the integral within the determined limits, one will have the probability that, the preceding condition being fulfilled, the error of the first result will be contained within these limits; by dividing this probability by that of the condition itself, one will have the probability that the error of the first result will be contained within some given limits, when one is certain that the condition effectively holds; this probability is therefore

$$\frac{\int dx c^{-\beta^2x^2-\beta'^2(x-q)^2-\beta''^2(x-q')^2-\cdots}}{\int dx c^{-\beta^2x^2-\beta'^2(x-q)^2-\beta''^2(x-q')^2-\cdots}},$$

the integral of the numerator being taken within the given limits, and that of the denominator being taken from $x=-\infty$ to $x=\infty$. One has

$$\begin{aligned}&\beta^2x^2+\beta'^2(x-q)^2+\beta''^2(x-q')^2+\cdots\\&\quad=(\beta^2+\beta'^2+\beta''^2+\cdots)x^2-2x(\beta'^2q+\beta''^2q'+\cdots)+\beta'^2q^2+\beta''^2q'^2+\cdots\end{aligned}$$

Let

$$x=\frac{\beta'^2q+\beta''^2q'+\cdots}{\beta^2+\beta'^2+\beta''^2+\cdots}+t;$$

the preceding probability will become

$$\frac{\int dt\, c^{-(\beta^2+\beta'^2+\beta''^2+\cdots)t^2}}{\int dt\, c^{-(\beta^2+\beta'^2+\beta''^2+\cdots)t^2}},$$

the integral of the numerator being taken within some given limits, and that of the denominator being taken from $t = -\infty$ to $t = \infty$. This last integral is

$$\frac{\sqrt{\pi}}{\sqrt{\beta^2 + \beta'^2 + \beta''^2 + \cdots}}.$$

By making therefore

$$t' = t\sqrt{\beta^2 + \beta'^2 + \beta''^2 + \cdots},$$

the preceding probability becomes

$$\frac{1}{\sqrt{\pi}} \int dt'\, c^{-t'^2}.$$

The most probable value of t' is that which corresponds to t' null, whence it follows that the most probable value of x is that which corresponds to $t = 0$; thus the correction of the first result, which the collection of all the observations s, s', s'', $\ldots$ have with most probability, is

$$\frac{\beta'^2 q + \beta''^2 q' + \cdots}{\beta^2 + \beta'^2 + \beta''^2 + \cdots}.$$

This correction, added to the result A, gives, for the result that it is necessary to choose,

$$\frac{A\beta^2 + (A+q)\beta'^2 + (A+q')\beta''^2 + \cdots}{\beta^2 + \beta'^2 + \beta''^2 + \cdots}.$$

The preceding correction is that which renders a minimum the function

$$(\beta x)^2 + [\beta'(x-q)]^2 + [\beta''(x-q')]^2 + \cdots$$

Now the greatest ordinate of the curve of probabilities of the first result is, as one has just seen it, $\frac{\beta}{\sqrt{\pi}}$; that of the curve of probabilities of the second result is $\frac{\beta'}{\sqrt{\pi}}$, and thus consecutively; the mean that it is necessary to choose among the diverse results is therefore the one which renders a minimum the sum of the squares of the error of each result multiplied by the greatest ordinate of its curve of probability. Thus the law of the minimum of the squares of the errors becomes necessary, when one must take a mean among some results given each by a great number of observations.

24. One has seen previously that, in all the manners to combine the equations of condition in order to form some final linear equations, necessary to the determination of the elements, the most advantageous is that which results from the method of least squares of errors of the observations, at least when the observations are in great number. If, instead of considering the minimum of the squares of the errors, one considered the minimum of other powers of the errors, or even of each other function of the errors, the final equations would cease to be linear, and their resolution would become impractical, if the observations were in great number. However there is a case which merits a particular attention, in this that it gives the system in which the greatest error, setting aside the sign, is less than in every other system. This case is the one of the minimum of the infinite and even powers of the errors. We consider here only the correction of a single element, and, z expressing this correction, we represent, as previously, the equations of condition by the following,

$$\epsilon^{(i)} = p^{(i)}z - \alpha^{(i)},$$

i being able to vary from zero to $s-1$, s being the number of observations. The sum of the powers $2n$ of the errors will be $\mathrm{S}(\alpha^{(i)} - p^{(i)}z)^{2n}$, the sign S extending to all the values of i. One is able to suppose in this sum all the values of $p^{(i)}$ positive; because, if one of them was negative, it would become positive by changing, as one is able to do it, the signs of the two terms of the binomial raised to the power $2n$, to which it corresponds. We will suppose therefore the quantities $\alpha - pz$, $\alpha^{(1)} - p^{(1)}z$, $\alpha^{(2)} - p^{(2)}z, \ldots$ disposed in a manner that the quantities p, $p^{(1)}$, $p^{(2)}, \ldots$ are positive and increasing. This put, if $2n$ is infinite, it is clear that the greatest term of the sum $\mathrm{S}(\alpha^{(i)} - p^{(i)}z)^{2n}$ will be the entire sum, unless there was one or many other terms which were equal to it, and this is that which must take place in the case of the minimum of the sum. In effect, if there was only a single greatest quantity, setting aside the sign, such as $\alpha^{(i)} - p^{(i)}z$, one would be able to diminish it by making z vary conveniently, and then the sum $\mathrm{S}(\alpha^{(i)} - p^{(i)}z)^{2n}$ would diminish and would not be a minimum. It is necessary moreover that, if $\alpha^{(i)} - p^{(i)}z$ and $\alpha^{(i')} - p^{(i')}z$ are, setting aside the sign, the two greatest quantities and equal between them, they are of contrary sign. In effect, the sum

$$(\alpha^{(i)} - p^{(i)}z)^{2n} + (\alpha^{(i')} - p^{(i')}z)^{2n}$$

must be then a minimum, its differential

$$-2ndz[p^{(i)}(\alpha^{(i)} - p^{(i)}z)^{2n-1} + p^{(i')}(\alpha^{(i')} - p^{(i')}z)^{2n-1}]$$

must be null, this which is able to be, when n is infinite, only in the case where $\alpha^{(i)} - p^{(i)}z$ and $\alpha^{(i')} - p^{(i')}z$ are infinitely little different and of contrary sign. If there are three greatest quantities, and equals among them, setting aside the sign, one will see in the same manner that their signs are not able to be the same.

Now, we consider the sequence

$$(o) \qquad \begin{cases} \alpha^{(s-1)} - p^{(s-1)}z,\ \alpha^{(s-2)} - p^{(s-2)}z,\ \alpha^{(s-3)} - p^{(s-3)}z,\ \dots,\ \alpha - pz, \\ -\alpha + pz,\ \dots,\ -\alpha^{(s-3)} + p^{(s-3)}z,\ -\alpha^{(s-2)} + p^{(s-2)}z, -\alpha^{(s-1)} + p^{(s-1)}z. \end{cases}$$

If one supposes $x = -\infty$, the first term of the sequence surpasses the following, and continues to surpass them by making z increase, to the moment where it becomes equal to one of them. The one here, by the increase of z, becomes greatest of all, and in measure as one makes z increase, it continues always to surpass those which precede it. In order to determine this term, one will form the sequence of quotients

$$\frac{\alpha^{(s-1)} - \alpha^{(s-2)}}{p^{(s-1)} - p^{(s-2)}},\ \frac{\alpha^{(s-1)} - \alpha^{(s-3)}}{p^{(s-1)} - p^{(s-3)}},\ \dots,\ \frac{\alpha^{(s-1)} - \alpha}{p^{(s-1)} - p},\ \frac{\alpha^{(s-1)} + \alpha}{p^{(s-1)} + p},\ \dots,\ \frac{\alpha^{(s-1)} + \alpha^{(s-1)}}{p^{(s-1)} + p^{(s-1)}}.$$

We suppose that $\frac{\alpha^{(s-1)} - \alpha^{(r)}}{p^{(s-1)} - p^{(r)}}$ is the smallest of these quotients by having regard to the sign, that is to say by regarding a greater negative quantity as smaller than another lesser negative quantity. If there are many least and equal quotients, we will consider the one which relates to the most distant term of the first in the sequence (*o*); this term will be the greatest of all, to the moment where, by the increase of z, it becomes equal to one of the following, which begins then to be the greatest. In order to determine this new term, one will form a new sequence of quotients

$$\frac{\alpha^{(r)} - \alpha^{(r-1)}}{p^{(r)} - p^{(r-1)}},\ \frac{\alpha^{(r)} - \alpha^{(r-2)}}{p^{(r)} - p^{(r-2)}},\ \dots,\ \frac{\alpha^{(r)} - \alpha}{p^{(r)} - p},\ \frac{p^{(r)} + \alpha}{p^{(r)} + p},\ \dots,$$

the term of the sequence (*o*) to which the least of these quotients correspond will be the new term. One will continue thus to that which one of the two terms which become equal and the greatest is in the first half of the sequence (*o*), and the other in the second half. Let $\alpha^{(i)} - p^{(i)}z$ and $-\alpha^{(i')} + p^{(i')}z$ be these two terms; then the value of z which corresponds to the system of the minimum of the greatest of the errors, setting aside the sign, is

$$\frac{\alpha^{(i)} + \alpha^{(i')}}{p^{(i)} - p^{(i')}}.$$

If there are many elements to correct, the equations of condition which determine their corrections contain many unknowns, and the investigation of the system of correction, in which the greatest error is, setting aside the sign, smaller than in every other system, becomes more complicated. I have considered this case in a general manner in Book III of the *Mécanique céleste*. I will observe only here that then the sum of the $2n$ powers of the errors of the observations is, as in the case of a single unknown, a minimum when $2n$ is infinite; whence it is easy to conclude that, in the system of which there is concern, it must have as many errors, plus one, equals, and greatest, setting aside the sign, as there are elements to correct. One imagines that the results corresponding to $2n$ equal to a great number must differ little from those which $2n$ infinite gives. It is not necessarily the same for this if the $2n$ power is quite elevated, and I have recognized through many examples that, in the same case where this power does not surpass the square, the results differ little from those that the system of the minimum of the greatest squares gives, this which is a new advantage of the method of least squares of the errors of observations.

Since a long time, geometers take an arithmetic mean among their observations, and, in order to determine the elements that they wish to know, they choose the most favorable circumstances for this object, namely, those in which the errors of the observations alter the least that it is possible the value of these elements. But Cotes is, if I do not deceive myself, the first who has given a general rule in order to make many observations agree in the determination of an element, proportionally to their influence. By considering each observation as a function of the element and regarding the error of the observation as an infinitely small differential, it will be equal to the differential of the function, taken with respect to that element. The more the coefficient of the differential of the element will be considerable, the less it will be necessary to make the element vary, in order that the product of its variation by this coefficient is equal to the error of the observation; this coefficient will express therefore the influence of the observation on the value of the element. This put, Cotes represents all the values of the element, given by each observation, by the parts of an indefinite straight line, all these parts having a common origin. He imagines next, at their other extremities, some weights proportional to the influences respective of the observations. The distance from the common origin of the parts to the common center of gravity of all these weights is the value that he chose for the element.

We take the equation of condition of n$^{\text{o}}$ 20,

$$\epsilon^{(i)} = p^{(i)}z - \alpha^{(i)},$$

$\epsilon^{(i)}$ being the error of the $(i+1)^{\text{st}}$ observation, and z being the correction of the element already known quite nearly; $p^{(i)}$, that one is able always to suppose positive, will express the influence of the corresponding observation. $\frac{\alpha^{(i)}}{p^{(i)}}$ being the value of z resulting from the observation, the rule of Cotes reverts to multiplying this value by $p^{(i)}$, to make a sum of all the products relative to the diverse values, and to divide it by the sum of all the $p^{(i)}$, this which gives

$$z = \frac{\mathrm{S}\alpha^{(i)}}{\mathrm{S}p^{(i)}}.$$

This was in effect the correction adopted by the observers, having the usage of the method of least squares of the errors of the observations.

However, one does not see that, since this excellent geometer, one has employed his rule, to Euler, who in his first piece on Jupiter and Saturn, appears to me to be serves himself the first of the equations of the condition in order to determine the elements of the elliptic movement of these two planets. Near the same time, Tobie Mayer made use of it in this good researches on the libration of the Moon, and next in order to form his lunar Tables. Since, the best astronomers have followed this method, and the success of the Tables which they have constructed by his means has verified the advantage of it.

When one has only one element to determine, this method leaves no embarrassment; but, when one must correct at the same time many elements, it is necessary to have as many final equations formed by the reunion of many equations of condition, and by means of which one determines by elimination the corrections of the elements. But what is the most advantageous manner to combine the equations of condition, in order to form the final equations? It is here that the observers abandoned themselves to some arbitrary gropings, which must have led them to some different results, although deduced from the same observations. In order to avoid these gropings, Mr. Legendre had the simple idea to consider the sum of the squares of the errors of the observations, and to render it a minimum, this which furnishes directly as many final equations, as there are elements to correct. This scholarly geometer is the first who has published this method; but one owes to Mr. Gauss the justice to observe that he had had, many years before this publication, the same idea of which he made a habitual usage, and that he had communicated to many astronomers. Mr. Gauss, in his *Theory of elliptic movement*, has sought to connect this method to the Theory of Probabilities, by showing that the same law of errors of the observations, which give generally the rule of the arithmetic mean among many observations, admitted by the observers, gives similarly the rule of the least squares of the errors of the observations, and it is this which one has seen in n^o 23. But, as nothing proves that the first of these rules gives the most advantageous result, the same uncertainty exists with respect to the second. The research on the most advantageous manner to form the final equations is without doubt one of the most useful of the Theory of Probabilities: its importance in Physics and Astronomy carries me to occupy myself with it. For this, I will consider that all the ways to combine the equations of condition, in order to form a final linear equation, returns to multiply them respectively by some factors which were null relative to the equations that one employed not at all, and to make a sum of all these products, this which gives a first final equation. A second system of factors give a second final equation, and thus consecutively, to this that one has as many final equations as elements to correct. Now it is clear that it is necessary to choose the system of factors, such that the mean error to fear to the plus or to the less respecting each element is a minimum; the mean error being the sum of the products of each error by its probability. When the observations are in small number, the choice of these systems depends on the law of errors of each observation. But, if one considers a great number of observations, this which holds most often in the astronomical researches, this choice becomes independent of this law, and one has seen, in that which precedes, that Analysis leads then directly to the results of the method of least squares of the errors of the observations. Thus this method which offered first only the advantage to furnish, without groping, the final equations necessary to the correction of the elements, gives at the same time the most precise corrections, at least when one wishes to employ only final equations which are linear, an indispensable condition, when one considers at the same time a great number of observations; otherwise, the elimination of the unknowns and their determination would be impractical.