CHAPTER V

APPLICATION DU CALCUL DES PROBABILITÉS A LA RECHERCHE DES PHÉNOMÈNES ET DE LEURS CAUSES

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APPLICATION OF THE CALCULUS OF PROBABILITIES TO THE RESEARCH OF PHENOMENA AND OF THEIR CAUSES

One is able, by the analyses of the preceding Chapters, applied to a great number of observations, to determine the probability of the existence of the phenomena of which the extent is rather small in order to be comprehended within the limits of the errors of each observation. Formulas which express that the probabilities of the existence of the phenomena and of its extent are comprehended within some given limits. Application to the diurnal variation of the barometer and to the rotation of the Earth, deduced from the experiments on the fall of bodies. The same analysis is applicable to the most delicate questions of Astronomy, of political Economics, of Medicine, etc., and to the solution of the problems with respect to chance, too complicated in order to be resolved directly by analysis. *A floor being divided into small rectangular squares by some lines parallel and perpendicular lines among them, to determine the probability that, by projecting at random a needle, it will fall again on a joint of these squares.* Nº 25.

25. The phenomena of nature present themselves most often accompanied with so many strange circumstances, so great a number of perturbing causes mix their influence that it is very difficult, when they are very small, to understand them. One is able then to arrive there only by multiplying the observations, so that, the strange effects coming to destroy themselves, the mean result of the observations no longer permit perceiving but these phenomena. One imagines, by that which precedes, that that holds rigorously only in the case of an infinite number of observations. In every other case, the phenomena are indicated by the mean results only in a probable manner, but which is so much more as the observations are in greater number. The research of this probability is therefore very important for Physics, Astronomy and generally for all the natural sciences. One will see that it returns to the methods that we have just exposed. In the

preceding Chapter, the existence of a phenomenon was certain; its extent alone has been the object of the Calculus of Probabilities: here the existence of a phenomenon and its extent are the object of this calculus.

We take for example the diurnal variation of the barometer, that one observes between the tropics, and which becomes sensible even in our climates, when one chooses and when one multiplies the observations conveniently. One has recognized in general, toward 9^{h} in the morning, the barometer is more elevated than toward 4^{h} in the evening; next it rises again toward the 11^{h} in the evening, and it descends again toward 4^{h} of the morning, it order to arrive again to its maximum height toward 9^{h}. We suppose that one has observed the height of the barometer toward 9^{h} in the morning and toward 4^{h} in the evening, during the number s days, and, in order to avoid the too great influence of the perturbing causes, we choose these days in a manner that, in the interval of 9^{h} and 4^{h}, the barometer has not varied beyond 4^{mm}. We suppose next that by making the sum of the s heights in the morning and the sum of the s heights of the evening, the first of these sums surpasses the second by the quantity q; this difference will indicate a constant cause which tends to raise the barometer toward 9^{h} in the morning, and to lower it toward 4^{h} in the evening. In order to determine with what probability this cause is indicated, we imagine that this cause not exist at all, and that the observed difference q results from accidental perturbing causes and from the errors of the observations. The probability that then the observed difference between the sums of the heights in the morning and in the evening must be below q is, by n° 18, equal to

$$\sqrt{\frac{k}{4k''\pi}}\int dr\, c^{-\frac{kr^2}{4k''}},$$

the integral being taken from $r=-\infty$ to $r=\frac{a}{r\sqrt{s}}$, k and k'' being some constants dependent on the law of probability of the differences between the heights in the morning and in the evening, and $\pm a$ being the limits of these differences, a being here equal to 4^{mm}, $\frac{k}{k''}$ being at least equal to 6, as one has seen in n° 20, $\frac{4k}{k''}$ is not able to be supposed less than $\frac{3}{2}$; by making therefore $s=400$, and supposing the extent of the diurnal variation of 1^{mm}, this which is very nearly that which Mr. Ramond has found in our climates, by the comparison of a very great number of observations, one will have $q=400^{\text{mm}}$. Thus $r=5$, and $\frac{kr^2}{4k''}$ is at least equal to 37.5; by making therefore

$$t^2 = \frac{kr^2}{4k''},$$

the preceding probability becomes at least

$$1 - \frac{\int dt\, c^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from $t = \sqrt{37,5}$ to $t = \infty$. This integral is, quite nearly, by n$^{\text{o}}$ 27 of Book I,

$$1 - \frac{c^{-37.5}}{2\sqrt{37,5\pi}},$$

and it approaches so to unity or to certitude, that it is extremely probable that, if there existed no constant cause at all of the observed excess of the sum of the barometric heights of the morning over those of the heights of the evening, this excess would be smaller than 400^{mm}; it indicates therefore with an extreme probability the existence of a constant cause which has produced it.

The phenomenon of a diurnal variation being thus well established, we determine the most probable value of its extent, and the error that one is able to commit with respect to its evaluation. We suppose for this that that value is $\frac{q}{s} \pm \frac{ar}{\sqrt{s}}$; the probability that the extent of the diurnal variation from the morning to the evening will be comprehended within these limits is, by n$^{\text{o}}$ 18,

$$2\sqrt{\frac{k}{4k''\pi}}\int dr\, c^{-\frac{kr^2}{4k''}},$$

the integral being taken from $r = 0$.

One is able to eliminate $\frac{k''}{k}$ by observing that, by n$^{\text{o}}$ 20, this fraction is nearly equal to $\frac{\text{S}\epsilon^{(i)2}}{2as^2}$, $\pm\,\epsilon^{(i)2}$ being the difference from $\frac{q}{s}$ to the observed extent the $(i+1)^{\text{st}}$ day, and the sign S extending to all the values of i, from $i = 0$ to $i = s - 1$; by making therefore

$$ar = t\sqrt{\frac{2\text{S}\epsilon^{(i)2}}{s}},$$

the probability that the extent of the diurnal variation from the morning to the evening is comprehended within the limits $\frac{q}{s} \pm \frac{t}{\sqrt{s}}\sqrt{\frac{2\mathrm{S}\epsilon^{(i)2}}{s}}$ will be $\frac{2}{\sqrt{\pi}}\int dt\, c^{-t^2}$, the integral being taken from t null.

The diurnal variation of the heights of the barometer depends uniquely on the Sun; but these heights are still affected by the aerial tides that the attraction of the Sun and of the Moon produce on our atmosphere, and of which I have given the theory in Book IV of the *Mécanique céleste*. It is therefore necessary to consider at the same time these two variations, and to determine their magnitudes and their respective epochs, by forming some equations of condition analogous to those of which the astronomers make use, in order to correct the elements of the celestial movements. These variations being principally sensible at the equator, and the perturbing causes being extremely small, one will be able, by means of excellent barometers, to determine them with a great precision, and I do not doubt at all that one recognizes then, in the collection of a very great number of observations, the laws that indicate the theory of the weight in the atmospheric tides, and that manifest themselves in a manner so remarkable in the observations of the tides of the Ocean, that I have discussed with extent, in the Book cited of the *Mécanique céleste*.

One sees, by that which precedes, that one is able to recognize the very small effect of a constant cause, by a long sequence of observations of which the errors are able to exceed this effect itself. But then it is necessary to take care to vary the circumstances of each observation, in a manner that the mean result of their collection is not at all altered sensibly by it and is nearly entirely the effect of the cause of which there is concern; it is necessary to multiply the observations, to that which the analysis indicates a very great probability that the error of this result will be comprehended within some very narrowed limits.

We suppose, for example, that one wishes to recognize by observation the small deviation to the east, produced by the rotation of the Earth, in the fall of a body. I have shown, in Book X of the *Mécanique céleste*, that if, from the summit of a quite high tower, one abandons a body to its weight, it will fall onto a horizontal plane passing through the foot of the tower, at a small distance to the east from the point of contact of this plane with a ball suspended by a wire of which the point of suspension is the one of the departure of the body. I have given, in the Book cited, the expression of this deviation, and there results from it that by setting aside the resistance of the air, it is uniquely toward the east; that it is proportional to the cosine of the latitude and to the square root of the cube of the height, and that at the latitude of Paris it is elevated to 5^{mm},1, when the height of the tower is 50$^{\text{m}}$. The resistance of the air changes this last result; I have given similarly the expression of it in this case, in the Book cited.

One has already made a great number of experiments in order to confirm, by this means, the movement of the rotation of the Earth, which besides is demonstrated by so many other phenomena that this confirmation becomes useless. The small errors of these very delicate experiments have often exceeded the effect that one would wish to determine, and it is only by multiplying considerably the experiments that one is able thus to establish its existence and to fix its value. We will submit this object to the analysis of probabilities.

If one takes for origin of the coordinates the point of contact of the plane and the ball suspended by a wire of which the summit of the suspension is the one of the departure of a ball that one makes fall; if one next marks on this plane the diverse points where the ball will touch the plane in each experiment; by determining the common center of gravity of these points, the line drawn from the origin of the coordinates to this center will determine the sense and the mean quantity of which the ball deviated from this origin, and both will be determined with so much more exactitude as the experiments will be more numerous and more precise.

We will consider now the common axis of the abscissas the line drawn from the origin of the coordinates to the east, and we designate by x, $x^{(1)}$, $x^{(2)}$, $\ldots$, $x^{(s-1)}$, y, $y^{(1)}$, $y^{(2)}$, $\ldots$, $y^{(s-1)}$ the respective coordinates of the points determined by the experiments of which the number is s. In expressing by X and Y the coordinates of the center of gravity of all these points, one will have

$$X = \mathrm{S}\frac{x^{(i)}}{s}, \qquad Y = \mathrm{S}\frac{y^{(i)}}{s},$$

the sign S extending to all the values of i, from $i = 0$ to $i = s - 1$. This put, by designating by $\pm a$ the limits of the errors of each experiment, in the sense of x, the probability that the mean deviation of the ball, from the point of origin of the coordinates, is comprehended within the limits $X \pm \frac{ar}{\sqrt{s}}$, will be, by n$^{\text{o}}$ 18,

$$2\sqrt{\frac{k}{4k''\pi}}\int dr\, c^{-\frac{kr^2}{4k''}},$$

k and k'' being some constants which depend on the law of facility of the errors of each experiment in the sense of x.

Similarly, $\pm a'$ being the limits of the errors of each experiment in the sense of y, the probability that the mean value of the deviation in the sense of y is comprehended within the limits $Y \pm \frac{a'r}{\sqrt{s}}$ will be

$$2\sqrt{\frac{\overline{k}}{4\overline{k}''\pi}}\int dr\, c^{-\frac{\overline{k}r^2}{4\overline{k}''}},$$

$\overline{k}$ and $\overline{k}''$ being some constants depending on the law of errors of the experiments in the sense of y. The fractions $\frac{k}{4k''}$ and $\frac{\overline{k}}{4\overline{k}''}$ being, by that which precedes, greater than $\frac{3}{2}$, one will be able to judge the degree of approximation and probability of the values of X and of Y, and to determine the probability of the deviation to the south and to the north, indicated by the observations.

The preceding analysis is able further to be applied to the research of the small inequalities of the celestial movements, of which the extent is comprehended within the limits either of the errors of observations, or of the perturbations produced by accidental causes. It is nearly thus that Tycho Brahe recognized that the equation of the times, relative to the Sun and to the planets, were not at all applicable to the Moon, and that it was necessary to subtract the part depending on the anomaly of the Sun, and even a much greater quantity, this which led Flamsteed to the discovery of the lunar inequality that one names *annual equation*. It is further in the results of a great number of observations that Mayer recognized that the equation of precession, relative to the planets and to the stars, was not at all applicable to the Moon; he evaluated to around $12''$ decimals the quantity of which it was necessary then to diminish it, a quantity that Mason raised next to nearly $24''$, by the comparison of all the observations of Bradley, and that Mr. Bürg has reduced to $21''$, by means of a much greater number of observations of Maskelyne. This inequality, although indicated by the observations, was neglected by the greatest number of astronomers, because it did not appear to result from the theory of universal gravitation. But having submitted its existence to the Calculus of Probabilities, it appeared to me indicated with a probability so strong, that I believed a duty to seek the cause of it. I saw well that it was able to result only from the ellipticity of the terrestrial spheroid, that one had neglected until then in the theory of the lunar movement, as having to produce only some insensible terms, and I conclude that it was extremely probable that these terms became sensible by the successive integrations of the differential equations. Having determined these terms by a particular analysis, that I have exposed in Book VII of the *Mécanique céleste*, I discovered first the inequality of the movement of the Moon in latitude, and which is proportional to the sine of its longitude: by its mean, I recognized that the theory of gravity gives effectively the observed diminution by the astronomers cited, in the inequality of the precession, applicable to the lunar movement in longitude. The quantity of this diminution and the coefficient of the inequality in latitude of which I just spoke are therefore very nearly proper to determine the flatness of the Earth. Having made part of my researches to Mr. Bürg who occupied himself then with his *Tables of the Moon*, I prayed him to determine the coefficients of the two inequalities with a particular care. By a remarkable concurrence, the coefficients that he has determined accord to give to the Earth the flatness $\frac{1}{305}$, a flatness which differs little from the mean concluded from the measures of the degrees of the meridian and from the pendulum, but which, seeing the influence of the errors of the observations and of the perturbing causes on these measures, appears to me more exactly determined by the lunar inequalities. Mr. Burckhardt, who has just formed new Tables of the Moon, very precise, on the collection of observations of Bradley and of Maskelyne, has found the same coefficient as Mr. Bürg for the lunar inequality in latitude: he found $\frac{1}{34}$ to add to the coefficient of the inequality in longitude, this which reduces the flatness to $\frac{1}{301}$, by this inequality. The very slight difference of these results proves that by fixing at $\frac{1}{304}$ this flatness, the error is insensible.

The analysis of Probabilities has led me similarly to the cause of the great irregularities of Jupiter and of Saturn. The difficulty to recognize the law of it and to restore them to the theory of universal attraction had made conjecture that they were due to the momentary actions of the comets; but a theorem to which I was arrived on the mutual attraction of the planets made me reject this hypothesis, indicating to me the mutual attraction of the two planets as the true cause of these irregularities. According to this theorem, if the movement of Jupiter accelerated by virtue of some great inequality with very long period, the one of Saturn must be decelerated in the same manner, and this deceleration is to the acceleration of Jupiter as the product of the mass of this last planet by the square root of the great axis of its orbit is to the similar product relative to Saturn. Thus, by taking for unity the deceleration of Saturn, the corresponding acceleration of Jupiter must be $0,40884$; now Halley had found, by the comparison of the modern observations to the old, that the acceleration of Jupiter corresponded to the deceleration of Saturn, and that it was $0,44823$ of this deceleration. These results, so well in accord with the theory, lead me to think that there exists, in the movement of these planets, two great inequalities corresponding and of contrary sign, which produced these phenomena. I have recognized that the mutual action of the planets was not able to occasion in their mean movements some variations always increasing or periodic, but of a period independent of their mutual configuration; it was therefore in the relation of the mean movements of Jupiter and of Saturn that I had to seek that of which there is concern. Now, by examining this relation, it is easy to recognize that twice the mean movement of Jupiter surpasses only by a very small quantity five times the one of Saturn; thus the inequalities which depend on this difference, and of which the period is around nine centuries, are able to become very great by the successive integrations which give to them for divisor the square of the very small coefficient of the time in the argument of these inequalities. By fixing toward the epoch of Tycho Brahe the origin of this argument, I saw that Halley had had to find, by the comparison of the modern observations to the old, the alterations that he had observed, while the comparison of the modern observations among them must have presented some contrary and parallel alterations to those that Lambert had remarked. The existence of the inequalities of which I just spoke appeared to me therefore extremely probable, and I hesitated not at all to undertake the long and painful calculation, necessary in order to assure myself of it completely. The result of this calculation, not only confirmed them, but it made known to me many other inequalities, of which the collection has carried the Tables of Jupiter and Saturn to the degree of precision of the same observations.

One sees thence how it is necessary to be attentive to the indications of nature, when they are the result of a great number of observations, although besides they be inexplicable by known means. I engage thus the astronomers to follow with a particular attention the lunar inequality with long period, which depends principally on the movement of the perigee of the Moon, added to the double of the mean movement of its nodes; an inequality of which I have spoken in Book VII of the *Mécanique céleste*, and that already the observations indicate with much likelihood. The preceding cases are not the only ones in which the observations have redressed the analysts. The movement of the lunar perigee and the acceleration of the movement of the Moon, which was not given at all by the approximations, has made felt the necessity to rectify these approximations. Thus one is able to say that nature itself has concurred to the analytic perfection of the theories based on the principle of universal gravitation, and it is, in my sense, a most strong proof of the truth of this admirable principle.

One is able further, by the Analysis of Probabilities, to verify the existence or the influence of certain causes of which one has believed to notice the action on organized beings. Of all the instruments that we are able to use in order to understand the imperceptible agents of nature, the most sensible are the nerves, especially when their sensibility is enhanced by some particular circumstances. It is by their means that one has discovered the weak electricity that the contact of two heterogeneous metals develop, this which has opened a vast field to the researches of the physicians and the chemists. The singular phenomena, which result from the extreme sensitivity of the nerves in some individuals, have given birth to diverse opinions on the existence of a new agent that one has named *animal magnetism*, with respect to the action of ordinary magnetism and the influence of the Sun and of the Moon in some nervous affections; finally on the impressions that the proximity of metals or of running water are able to give birth. It is natural to think that the action of these causes is very weak, and able to be troubled easily by a great number of accidental circumstances; thus, of that which, in some cases, it is not at all manifested, one must not conclude that it never exists. We are so averted to know all the agents of nature that it would be less philosophical to deny the existence of the phenomena, uniquely because they are inexplicable in the actual state of our knowledge. Alone we must examine them with an attention so much more scrupulous as it appeared more difficult to admit them, and it is here that the Analysis of the Probabilities becomes indispensable in order to determine to what point it is necessary to multiply the observations or the experiments in order to have in favor of the existence of the agents that they seem to indicate a probability superior to all the reasons that one is able to have besides to reject it.

The same analysis is able to be extended to the diverse results of Medicine and of political Economy, and even to the influence of moral causes; because the action of these causes, when it is repeated a great number of times, offers in its results so much regularity as physical causes.

One is able to determine further by the Analysis of Probabilities, compared to a great number of experiences, the advantage and the disadvantage of the players, in the cases of which the complication renders impossible their direct research. Such is the advantage to the hand, in the game of piquet: such are further the respective probabilities to bring forth the different faces of a right rectangular prism, of which the length, the width and the height are unequal, when the prism projected into the air falls again on a horizontal plane.

Finally one would be able to make use of the Calculus of Probabilities in order to rectify curves or to square their surfaces. Without doubt, the geometers will not employ this means; but, as it gives me place to speak of a particular kind of combinations of chance, I will expose it in a few words.

We imagine a plane divided by parallel lines, equidistant by the quantity a; we conceive moreover a very narrow cylinder, of which $2r$ is the length, supposed equal or less than a. One requires the probability that in casting it on it, it will encounter one of the divisions of the plane.

We erect on any point of one of these divisions a perpendicular extended to the following division. We suppose that the center of the cylinder be on this perpendicular and at the height y above the first of these two divisions. In making the cylinder rotate about its center and naming by ϕ the angle that the cylinder makes with the perpendicular, at the moment where it encounters this division, 2ϕ will be the part of the circumference described by each extremity of the cylinder, in which it encounters the division; the sum of all these parts will be therefore $4\int \phi dy$, or $4\phi y - 4\int y d\phi$; now one has $y = r\cos\ \phi$; this sum is therefore

$$4\phi y - 4r\sin\phi + \text{const.}$$

In order to determine this constant, we will observe that the integral must be extended from y nul to $y = r$, and consequently from $\phi = \frac{\pi}{2}$ to $\phi = 0$, this which gives

$$\text{const.} = 4r;$$

thus the sum of which there is concern is $4r$. From $y = a - r$ to $y = a$, the cylinder is able to encounter the following division, and it is clear that the sum of all the parts relative to this encounter is again $4r$; $8r$ is therefore the sum of all the parts relative to the encounter of one or of the other of the divisions by the cylinder, in the movement of its center the length of the perpendicular. But the number of all the arcs that it describes in rotating in entirety with respect to itself, at each point of this perpendicular, is $2a\pi$; this is the number of all the possible combinations; the probability of the encounter of one of the divisions of the plane by the cylinder is therefore $\frac{4r}{a\pi}$. If one casts this cylinder a great number of times, the ratio of the number of times where the cylinder will encounter one of the divisions of the plane to the total number of casts will be, by n° 16, very nearly, the value of $\frac{4r}{a\pi}$, that which will make known the value of the circumference 2π. One will have, by the same section, the probability that the error of this value will be contained within some given limits, and it is easy to see that the ratio $\frac{8r}{a\pi}$ which, for a given number of projections, renders the least error to fear, is unity, this which gives the length of the cylinder equal to the interval of the divisions, multiplied by the ratio of the circumference to four diameters.

We imagine now the preceding plane divided again by some lines perpendicular to the preceding, and equidistant by a quantity b equal or greater than the length $2r$ of the cylinder. All these lines will form with the first a sequence of rectangles of which b will be the length and a the height. We will consider one of these rectangles; we suppose that in its interior one draws at the distance r from each side some lines which are parallel to them. They will form first an interior rectangle, of which $b-2r$ will be the length, and $a-2r$ the height; next two small rectangles, of which r will be the height, and $b-2r$ the length; then two other small rectangles of which r will be the length and $a-2r$ the height; finally, four small squares of which the sides will be equal to r.

As long as the center of the cylinder will be placed in the interior rectangle, the cylinder, in rotating on its center, will never encounter the sides of the large rectangle.

When the center of the cylinder will be placed in the interior of one of the rectangles of which r is the height and $b-2r$ the length, it is easy to see, by that which precedes, that the product of $8r$ by the length $b-2r$ will be the number of corresponding combinations, in which the cylinder will encounter one or the other of the sides b of the great rectangle. Thus $8r(b-2r)$ will be the total number of combinations corresponding to the cases in which, the center of the cylinder being placed in one or the other of these small rectangle, the cylinder encounters the outline of the great rectangle. By the same reason, $8r(a-2r)$ will be the total number of combinations in which, the center of the cylinder being placed in the interior of the small rectangles of which r and $a-2r$ are the dimensions, the cylinder encounters the outline of the great rectangle.

There now remains for us to consider the four small squares. Let ABCD be one of them. From the angle A common to this square and to the great rectangle, as center, and from the radius r, we describe a quarter circumference terminating itself at the points B and D. As long as the center of the cylinder will be comprehended within the quarter circle formed by this arc, the cylinder, in turning, will encounter the outline of the rectangle in all its positions; the number of combinations in which this will take place is therefore equal to the product of 2π by the area of the quarter circle, and consequently it is equal to $\frac{\pi^2 r^2}{2}$. If the center of the cylinder is in the part of the square which is outside of the quarter circle, the cylinder, in turning around its center, will be able to encounter one or the other of the two sides AB and AD extended, without ever encountering both at the same time. In order to determine the number of combinations relative to this encounter, I conceive on any point of side AB, distant by x from point A, a perpendicular y of which the extremity is beyond the quarter circle. I place the center of the cylinder on this extremity, from which I let down four straight lines equal to r, and of which two descend onto the side AB extended, if that is necessary, and two others onto the side AD similarly prolonged. I name 2ϕ the angle comprehended between the first two lines, and $2\phi'$ the angle contained between the second two. It is clear that the cylinder, in turning on its center, will encounter the side AB extended as often as one of its halves will be within the angle 2ϕ, and that it will encounter the side AD extended as often as one of its halves will be within the angle $2\phi'$; the total number of all combinations in which the cylinder will encounter one or the other of these sides is therefore $4(\phi + \phi')$; thus this number, comparatively to the part of the square exterior to the quarter circle, is

$$4\int(\phi + \phi')dx\,dy;$$

now one has evidently

$$x = r\cos\phi', \qquad y = r\cos\phi;$$

the preceding integral becomes thus

$$4r^2\int\int(\phi + \phi')d\phi\,d\phi'\sin\phi\sin\phi',$$

and it is easy to see that the integral relative to ϕ' must be taken from $\phi' = 0$ to $\phi' = \frac{\pi}{2} - \phi$, and that the integral relative to ϕ must be taken from $\phi = 0$ to $\phi = \frac{\pi}{2}$, that which gives $\frac{1}{2}r^2(12 - \pi^2)$ for this integral. In adding to it $\frac{\pi^2 r^2}{2}$, one will have the number of combinations relative to the square, and in quadrupling this number and joining it to the preceding numbers of combinations relative to the encounter of the outline of the great rectangle by the cylinder, one will have, for the total number of combinations,

$$8(a + b)r - 8r^2.$$

But the total number of possible combinations is evidently equal to 2π multiplied by the area ab of the great rectangle; the probability of the encounter of the divisions of the plane by the cylinder is therefore

$$\frac{4(a + b)r - 4r^2}{ab\pi}.$$