CHAPTER VII
DE L'INFLUENCE DES INÉGALITÉS INCONNUES QUI PEUVENT EXISTER ENTRE DES CHANCES QUE L'ON SUPPOSE PARFAITMENT ÉGALES

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Théorie Analytique des Probabilités §34, pp. 410–415

ON THE INFLUENCE OF THE UNKNOWN INEQUALITIES WHICH ARE ABLE TO EXIST AMONG THE CHANCES THAT ONE SUPPOSES PERFECTLY EQUAL

Examination of the cases in which this influence is favorable or contrary. It is contrary to the one who, in the game of *heads* and *tails*, wagers to bring forth *heads* an odd number of times, in an even number of trials. Means to correct this influence. N° 34.

34. I have already considered this influence in n° 1, where one has seen that these inequalities increase the probability of the events composed of the repetition of simple events. I will resume here this important object in the applications of the analysis of probabilities.

There results from the section cited that if, in the game of *heads* and *tails*, there exists an unknown difference between the possibilities to bring forth one or the other, by naming α this difference, so that $\frac{1\pm\alpha}{2}$ is the possibility to bring forth *heads*, and consequently $\frac{1\mp\alpha}{2}$ that to bring forth *tails*, the one of the two signs $+$ and $-$ that one must adopt being unknown, the probability to bring forth *heads* n times consecutively will be

$$\frac{(1+\alpha)^n+(1-\alpha)^n}{2^{n+1}}$$

or

$$\frac{1}{2^n}\left[1+\frac{n(n-1)}{1.2}\alpha^2+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\alpha^4+\cdots\right]. \tag{1}$$

The game of *heads* and *tails* consists, as one knows, in casting into the air a very thin coin, which falls again necessarily on one of the two opposite faces that one names *heads* and *tails*. One is able to diminish the value of α, by rendering these two faces the most equal as it is possible. But it is physically impossible to obtain a perfect equality, and then the one who wagers to bring forth *heads* twice consecutively or *tails* twice consecutively has the advantage over the one who wagers that, in two trials, *heads* and *tails* will alternate, its probability being $\frac{1+\alpha^2}{2}$.

One is able to diminish the influence of the inequality of the two faces of the coin, by submitting them themselves to the chances of hazard. We designate by A this coin, and we imagine a second coin B similar to the first. We suppose that after having projected this second coin, one projects the coin A in order to form a first trial, and we determine the probability that in n similar consecutive trials, the coin A will present the same faces as the coin B. If one names p the probability to bring forth *heads* with the coin A and q the probability to bring forth *tails*; if one designates next by p' and q' the same probabilities for the coin B, $pp' + qq'$ will be the probability that in one trial the coin A will present the same faces as the coin B. Thus $(pp' + qq')^n$ will be the probability that that will take place constantly in n trials. Let

$$p = \frac{1+\alpha}{2}, \qquad q = \frac{1-\alpha}{2},$$
$$p' = \frac{1+\alpha'}{2}, \qquad q' = \frac{1-\alpha'}{2};$$

one will have

$$(pp' + qq')^n = \frac{1}{2^n}(1+\alpha\alpha')^n.$$

But, as one is ignorant of what are the faces that the inequalities α and α' favor, the preceding probability is able to be equally either $\frac{1}{2^n}(1+\alpha\alpha')^n$ or $\frac{1}{2^n}(1-\alpha\alpha')^n$, according as α or α' are of like sign or of contrary signs. The true value of this probability is therefore, α and α' being supposed positives,

$$\frac{1}{2^{n+1}}\left[(1+\alpha\alpha')^n \ + \ \frac{1}{2^n}(1-\alpha\alpha')^n\right]$$

or

$$\frac{1}{2^n}\left[1+\frac{n(n-1)}{1.2}\alpha^2\alpha'^2+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\alpha^4\alpha'^4+\cdots\right].$$

If one compares this formula to formula (1), one sees that it is more near $\frac{1}{2^n}$ more than it, or of the probability which would hold if the faces of the coins were perfectly equal. Thus the inequality of these faces is thence corrected in great part; it would be it even in totality, if α' were null, or if the two faces of the coin B were perfectly equal.

p representing the probability of *heads* with the coin A, and q that of *tails*, the probability to bring forth *heads* an odd number of times in n trials will be

$$\frac{1}{2}[(p+q)^n \mp (p-q)^n],$$

the $-$ sign holding if n is even, and the $+$ sign holding if n is odd. Making $p=\frac{1+\alpha}{2}$, $q=\frac{1-\alpha}{2}$, the preceding function becomes

$$\frac{1}{2}(1\mp\alpha)^n.$$

If n is odd and equal to $2i+1$, this function is

$$\frac{1}{2}(1+\alpha^{2i+1});$$

but, as one is able to suppose equally α positive or negative, it is necessary to take the half of the sum of its two values relative to these suppositions, this which gives $\frac{1}{2}$ for its true value; the inequality of the faces of the coin changes therefore not at all the probability $\frac{1}{2}$ to bring forth *heads* an odd number of times. But, if n is even and equal to $2i$, this probability becomes

$$\frac{1}{2}(1-\alpha^{2i}), \tag{2}$$

$\pm\,\alpha$ being the unknown inequality of the probability between *heads* and *tails*; there is therefore disadvantage to wager to bring forth *heads* or *tails* an odd number of times in $2i$ trials, and consequently there is advantage to wager to bring forth one or the other an even number of times.

One is able to diminish this advantage by changing the wager to bring forth heads an odd number of times in $2i$ trials, in the wager to bring forth in the same number of trials an odd number of resemblances between the faces of the two coins A and B, projected as one has said above. In effect, the probability of a resemblance at each trial is, as one has seen, $pp' + qq'$, and the probability of a dissemblance is $pq' + p'q$. We name P the first of these two quantities and Q the second; the probability to bring forth an odd number of resemblances in $2i$ trials will be

$$\frac{1}{2}\left[(P+Q)^{2i} - (P-Q)^{2i}\right].$$

If one makes, as previously,

$$p = \frac{1+\alpha}{2}, \quad q = \frac{1-\alpha}{2}, \quad p' = \frac{1+\alpha'}{2}, \quad q' = \frac{1-\alpha'}{2},$$

one will have

$$P = \frac{1+\alpha\alpha'}{2}, \quad Q = \frac{1-\alpha\alpha'}{2};$$

the preceding function becomes thus

$$\frac{1}{2}(1 - \alpha^{2i}\alpha'^{2i}).$$

This function remains the same, whatever change that one makes in the signs of α and of α'; it is the true probability to bring forth an odd number of resemblances; but, α and α' being small fractions, one sees that it is nearer $\frac{1}{2}$ more than formula (2); the disadvantage of an odd number is therefore thence diminished.

One sees by that which precedes that one is able to diminish the influence of the unknown inequalities among the chances that one supposes equals, by submitting them themselves to chance. For example, if one puts into an urn the tickets $1, 2, 3, \ldots, n$ following this order, and if next, after having agitated the urn in order to mix the tickets, one draws from it; if there is among the probabilities to exit some tickets a small difference depending on the order following which they have been placed in the urn, one will diminish it considerably by putting these tickets into a second urn, according to their order of exit from the first urn, and by agitating next this second urn, in order to well mix the tickets. Then the order according to which one has placed the tickets in the first urn will have extremely little influence on the extraction of the first ticket which will exit from the second urn. One would diminish further this influence, by considering in the same manner a third urn, a fourth, etc.

We will consider two players A and B playing together, in a manner that at each trial the one who loses gives a token to his adversary, and that the set endures until one of them has won all the tokens of the other. Let p and q be their respective skills, a and b their numbers of tokens at commencement. There results from formula (H) of n$^{\text{o}}$ 10, by making i infinity, that the probability of A in order to win the set is

$$\frac{p^b(p^a - q^a)}{p^{a+b} - q^{a+b}}.$$

If one makes in this expression

$$p = \frac{1 \pm \alpha}{2}, \quad q = \frac{1 \mp \alpha}{2},$$

one will have, by taking the superior sign, the probability relative to the case where A is stronger than B, and, by taking the inferior sign, one will have the probability relative to the case where A is less strong than B. If one is ignorant of who is the strongest of the players, the half-sum of these two probabilities will be the probability of A, that one finds thus equal to

$$\frac{\frac{1}{2}[(1+\alpha)^a - (1-\alpha)^a][(1+\alpha)^b - (1-\alpha)^b]}{(1+\alpha)^{a+b} - (1-\alpha)^{a+b}}; \tag{3}$$

by changing a into b and reciprocally, one will have the probability of B. If one supposes α infinitely small or null, these probabilities become $\frac{a}{a+b}$ and $\frac{b}{a+b}$; they are therefore proportionals to the numbers of tokens of the players; thus, for equality of the game, their stakes must be in this ratio. But then the inequality which is able to exist among them is favorable to the player who has the smallest number of tokens; because, if one supposes a less than b, it is easy to see that the expression (3) is greater than $\frac{a}{a+b}$. If the players agree to double, to triple, etc. their tokens, the advantage of A increases without ceasing, and, in the case of a and b infinite, its probability becomes $\frac{1}{2}$ or the same as that of B.

P being the probability of an event composed of two simple events of which p and $1 - p$ are the respective probabilities, if one supposes that the value of p is susceptible of an unknown inequality z which is able to be extended from $-\alpha$ to $+\alpha$, by naming ϕ the probability of $p + z$, ϕ being a function of z, one will have, for the true probability of the composite event,

$$\frac{\int P'\phi\,dz}{\int \phi\,dz},$$

P' being that which P becomes when one changes p into $p+z$, and the integrals being taken from $z=-\alpha$ to $z=\alpha$.

If one has no other givens in order to determine z but one observed event, formed from the same simple events, by naming Q the probability of this event, $p+z$ and $1-p-z$ being the probabilities of the simple events, the preceding expression gives, by changing ϕ into Q, for the probability of the composite event,

$$\frac{\int P'Q\,dz}{\int Q\,dz},$$

the integrals being taken here from $z=-p$ to $z=1-p$; this which is conformed to that which we have found in the preceding Chapter.