CHAPTER VIII

DES DURÉES MOYENNES DE LA VIE, DES MARIAGES ET DES ASSOCIATIONS QUELCONQUE

Pierre Simon Laplace

Théorie Analytique des Probabilités §§35–37, pp. 416–427

ON THE MEAN DURATION OF LIFE, OF MARRIAGES AND OF ANY ASSOCIATIONS WHATSOEVER

Expression of the probability that the mean duration of life of a great number n of infants will be comprehended within these limits, true mean duration of life, more or less a given very small quantity. There results from it that this probability increases without ceasing in measure as the number of infants increases and that, in the case of an infinite number, this probability is confounded with certitude, the interval of the limits becoming infinitely small or null. Expression of the mean error that one is able to fear by taking for mean duration of life that of a great number of infants. Rule in order to conclude from the Tables of mortality the mean duration of that which remains to live to a person of a given age. N° 35.

Expression of the mean duration of life, if one of the causes of mortality comes to be extinguished. Particular expression in the case where one happens to destroy a maladie that one is able to contract only one time in life. The extinction of the small pox, by means of vaccine, would increase by more than three years the mean duration of life, if the increase of population which would result from it would not be arrested by the deficiency of subsistences. N° 36.

On the mean duration of marriages. Expression of their most probable mean duration and of the probability that the error of this expression is comprehended within some given limits. On the mean duration of the associations formed by any number of individuals. N° 37.

35. We suppose that one has followed with respect to a very great number n of infants the law of mortality, from their birth to their total extinction; one will have their mean life, by making a sum of the durations of all their lives and by dividing it by the number n. If this number were infinite, one would have exactly the duration of mean life. We seek the probability that the mean life of n infants will deviate from that her only within some given limits.

We designate by $\phi\left(\frac{x}{a}\right)$ the probability to die at age x, a being the limit of x, a and x being supposed to contain an infinite number of parts taken for unity. We will consider the power

$$\left[\begin{array}{l}\phi\left(\dfrac{0}{a}\right)+\phi\left(\dfrac{1}{a}\right)c^{-\varpi\sqrt{-1}}+\phi\left(\dfrac{2}{a}\right)c^{-2\varpi\sqrt{-1}}+\cdots\\ \qquad +\phi\left(\dfrac{x}{a}\right)c^{-x\varpi\sqrt{-1}}+\cdots+\phi\left(\dfrac{a}{a}\right)c^{-a\varpi\sqrt{-1}}\end{array}\right]^n.$$

It is clear that the coefficient of $c^{-(l+n\mu)\varpi\sqrt{-1}}$, in the development of this power, is the probability that the sum of the ages to which the n infants will arrive will be $l+n\mu$; by multiplying therefore by $c^{(l+n\mu)\varpi\sqrt{-1}}$ the preceding power, the term independent of the powers of $c^{\pm\varpi\sqrt{-1}}$ in the product will be this probability, which, consequently, is equal to

$$(1)\quad \frac{1}{2\pi}\int d\varpi c^{l\varpi\sqrt{-1}}\left\{c^{\mu\varpi\sqrt{-1}}\left[\begin{array}{l}\phi\left(\dfrac{0}{a}\right)+\phi\left(\dfrac{1}{a}\right)c^{-\varpi\sqrt{-1}}+\cdots+\phi\left(\dfrac{x}{a}\right)c^{-x\varpi\sqrt{-1}}\\ \qquad\qquad +\cdots+\phi\left(\dfrac{a}{a}\right)c^{-a\varpi\sqrt{-1}}\end{array}\right]\right\}^n,$$

the integral being taken from $\varpi=-\pi$ to $\varpi=\pi$.

If one takes in this integral the hyperbolic logarithm of the quantity under the $\int$ sign, raised to the power n, one will have, by developing the exponential into series, this logarithm equal to

$$(2)\quad n\mu\varpi\sqrt{-1}+n\log\left[\int\phi\left(\frac{x}{a}\right)-\varpi\sqrt{-1}\int x\phi\left(\frac{x}{a}\right)-\frac{\varpi^2}{2}\int x^2\phi\left(\frac{x}{a}\right)+\cdots\right];$$

the $\int$ sign is returning here to all the values of x, from $x=0$ to $x=a$. If one makes $\frac{x}{a}=x'$, and if one observes that, the variation of x being unity, one has $adx'=1$, one will have

$$\begin{aligned}\int\phi\left(\frac{x}{a}\right)&=a\int dx'\phi(x'),\\ \int x\phi\left(\frac{x}{a}\right)&=a^2\int x'dx'\phi(x'),\\ \int x^2\phi\left(\frac{x}{a}\right)&=a^3\int x'^2dx'\phi(x'),\\ \cdots\cdots\cdots&\cdots\cdots\cdots,\end{aligned}$$

the integrals relative to x being taken from $x' = 0$ to $x' = 1$. We name k, k', k'', $\ldots$ these successive integrals; the probability that the duration of life of an infant will be comprehended within the limits zero and a is $\int \phi\left(\frac{x}{a}\right)$ or $a \int dx' \phi(x')$; now this probability is certitude itself; one has therefore $ak = 1$. This put, the function (2) becomes

$$n\mu\varpi\sqrt{-1} + n\log\left(1 - \frac{k'}{k} a\varpi\sqrt{-1} - \frac{k''}{k}\frac{a^2\varpi^2}{2} + \cdots\right)$$

or

$$\left(\frac{n\mu}{a} - \frac{nk'}{k}\right) a\varpi\sqrt{-1} - n\frac{kk'' - k'^2}{2k^2} a^2\varpi^2 - \cdots$$

If one makes

$$\mu = \frac{ak'}{k} = \frac{a^2k'}{ak} = a^2k'.$$

the first power of ϖ disappears, and moreover, n being supposed a very great number, one is able to stop at the second power of ϖ; the function (1) becomes thus, by passing again from the logarithms to the numbers,

$$\frac{1}{2\pi}\int d\varpi\, c^{l\varpi\sqrt{-1} - n\frac{kk'' - k'^2}{2k^2} a^2\varpi^2}.$$

If one makes

$$\beta^2 = \frac{k^2}{2(kk'' - k'^2)}, \quad t = \frac{a\varpi\sqrt{n}}{2\beta} - \frac{\beta l\sqrt{-1}}{a\sqrt{n}},$$

this integral becomes, by taking it from $t = -\infty$ to $t = \infty$,

$$\frac{\beta}{a\sqrt{n\pi}} c^{-\frac{\beta^2 l^2}{a^2 n}}.$$

By multiplying it by dl, and making $l = ar\sqrt{n}$, one will have

$$\frac{2}{\sqrt{\pi}}\int \beta dr\, c^{-\beta^2 r^2}$$

for the probability that the sum of the ages to which the n infants will arrive will be comprehended within the limits $na^2k' \pm ar\sqrt{n}$.

The quantity a^2k' or $\int x\phi\left(\frac{x}{a}\right)$ is the sum of the products of each age by the probability of arriving there; it is therefore the true duration of mean life; thus the probability that the sum of the ages to which the n infants cease to live, divided by their number, is comprehended within these limits

True duration of mean life, more or less $\frac{ar}{\sqrt{n}}$,

has for expression

$$\frac{2}{\sqrt{\pi}}\int \beta dr\, c^{-\beta^2 r^2}.$$

The mean value of r, to more or to less, is, by n° 20,

$$\pm\frac{1}{\sqrt{\pi}}\int \beta r\, dr\, c^{-\beta^2 r^2}.$$

the integral being taken from $r = 0$ to r infinity. By multiplying it by $\frac{a}{\sqrt{n}}$, one will have the mean error to fear to more or to less, when one takes for mean duration of life the sum of the ages that the n infants considered above have lived, divided by n, a quotient that we designate by G; this error is therefore

$$\pm\frac{a}{2\beta\sqrt{n\pi}}.$$

One has, very nearly,

$$a^2k' = G,$$

and, as $ak = 1$, one will have

$$\frac{k'}{k} = \frac{G}{a}.$$

If one names next H the sum of the squares of the ages that the n infants have lived, divided by n, one will find, by the analysis of n° 19,

$$\frac{k''}{k}a^2 = H;$$

these values give

$$\beta^2 = \frac{a^2}{2(H - G^2)}:$$

the mean error to fear to more or to less with respect to the duration of life becomes thus

$$\pm \frac{\sqrt{H - G^2}}{\sqrt{2n\pi}}.$$

It is clear that these results hold equally relative to the mean duration of that which remains to live, when, instead of departing from the epoch of birth, one departs from any epoch of life.

One is able to determine easily, by means of the Tables of mortality, formed from year to year, the mean duration of that which remains to live to a person of whom the age is an entire number of years A. For that, one will add all the numbers of the Table which follow the one which corresponds to age A; one will divide the sum by this last number, and one will add $\frac{1}{2}$ to the quotient. In effect, if one designates by (1), (2), (3), ... the numbers of the Table, corresponding to the year A and to the following years, the number of individuals who die in the first year, in departing from year A, will be (1) − (2); but, in this short interval, the mortality is able to be supposed constant; $\frac{1}{2}[(1) - (2)]$ is therefore the sum of the durations of their life, in departing from age A. Similarly $\frac{3}{2}[(2) - (3)]$, $\frac{5}{2}[(3) - (4)]$, ... are the sums of the durations of life, by departing from the same age, of those who die in the second, third, etc., years counted from year A. The reunion of all these sums is $\frac{(1)}{2} + (2) + (3) + (4) + \cdots$; and, by dividing it by (1), one will have the mean duration of that which remains to live to the person of age A. One will form thus a Table of the mean durations of that which remains to life at the different ages. One will be able likewise to conclude these durations from one another, by observing that, if F designates this duration for age A, and F' the corresponding duration at age $A + 1,$ one has

$$F = \frac{(2)}{(1)}\left(F' + \frac{1}{2}\right) + \frac{1}{2}.$$

36. We determine now the mean duration of life which would hold, if one of the causes of mortality were to be extinguished. Let U be the number of infants

who, out of the number n of births, would survive yet to the age x under this hypothesis, u being the one of the infants living at this age out of the same number of births, in the case where that cause of mortality subsists. We name $z\Delta x$ the probability that one individual of age x will perish of this malady in the very short interval of time Δx; $uz\Delta x$ will be, very nearly, by n° 25, the number of individuals u who will perish of this malady in the interval of time Δx, if this number is considerable. Similarly, if one designates by $\phi\Delta x$ the probability that an individual of age x will perish by the other causes of mortality in the interval Δx, $u\phi\Delta x$ will be the number of individuals who will perish by these causes, in the interval of time Δx; this will be therefore the value of $-\Delta u$; I affect Δu with the $-$ sign, because u diminishes in measure as x increases; one has therefore

$$-\Delta u = u\Delta x(\phi + z).$$

One will have similarly

$$-\Delta U = U\phi\Delta x.$$

By eliminating ϕ from these two equations, one will have

$$\frac{\Delta U}{U} = \frac{\Delta u}{u} + z\Delta x.$$

Δx being a very small quantity, one is able to transform the characteristic Δ into the differential characteristic d, and then the preceding equation becomes

$$\frac{dU}{U} = \frac{du}{u} + z\,dx;$$

whence one draws, by integrating and observing that at age zero $U = u = n$,

$$U = uc^{\int z\,dx}, \tag{3}$$

the integral being taken from x null. One is able to obtain this integral, by means of the registers of mortality, in which one takes account of the age of the dead individuals and of the causes of their death. In effect, $uz\Delta x$ being, by that which precedes, the number of those who, arrived to age x, have perished in the interval of time Δx, by the malady of which there is concern, one will have very nearly the integral $\int z\,dx$, by supposing Δx equal to a year, and by taking from the birth of the n infants that one has considered, until the year x, the sum of the fractions which have for numerator the number of individuals who the malady has made perish each year, and for denominator, the number of the n infants who survive yet to the middle of the same year. Thus one will be able to transform, by means of equation (3), a Table of ordinary mortality, into that which would hold if the malady of which there is concern did not exist.

Smallpox has that in particular, namely, that the same individual is never twice attacked, or at least this case is so rare, that, if it exists, one is able to set it aside. We imagine that, out of a very great number n of infants, u arrive to age x, and that, in the number u, y have not had smallpox at all. We imagine further that out of this number y, $iy\ dx$ take this malady in the instant dx, and that out of this number, $iry\ dx$ perish from this malady. By designating, as above, by ϕ the probability to perish at age x by some other causes, one will have evidently

$$du = -u\phi\, dx - iry\, dx.$$

One will have next

$$dy = -y\phi\, dx - iy\, dx.$$

In effect, y diminishes by the number of those who, in the instant dx, take smallpox, and this number is, by the supposition, $iy\, dx$; y diminishes further by the number of individuals comprehended in y, who perish by some other causes, and this number is $y\phi\, dx$.

Now, if from the first of the two preceding equations, multiplied by y, one subtracts the second multiplied by u, and if one divides the difference by y^2, one will have

$$d\frac{u}{y} = i\frac{u}{y}\, dx - ir\, dx,$$

this which gives, by integrating from x null, and observing that at this origin $u = y = n$.

$$\frac{u}{y} = \left(1 - \int ir\, dx\, c^{-\int i\, dx}\right) c^{\int i\, dx}; \tag{4}$$

this equation will make known the number of individuals of age x who have not at all yet had smallpox. One has next

$$z\, dx = \frac{iry\, dx}{u},$$

$uz\, dx$ being, as above, the number of those who perish within the time dx, of the malady that one considers. By substituting, instead of $\frac{y}{n}$, its preceding value, one will have, after having integrated,

$$c^{\int z\,dx} = \frac{1}{1 - \int ir\,dx\,c^{-\int i\,dx}};$$

equation (3) will give therefore

$$U = \frac{u}{1 - \int ir\,dx\,c^{-\int i\,dx}}. \tag{5}$$

This value of U supposes that one knew by observation i and r. If these numbers were constants, it would be easy to determine them; but, as they are able to vary from age to age, the elements of formula (3) are easier to know, and this formula seems to me more proper to determine the law of mortality which would hold, if smallpox was extinct. By applying to it the givens that one has been able to procure with respect to the mortality caused by this malady, at the diverse ages of life, one finds that its extinction by means of the vaccine would increase more than three years the duration of mean life, if besides this duration was not at all restrained by the diminution related to the subsistances, due to a greater increase of population.

37. We consider presently the mean duration of marriages. For that we imagine a great number n of marriages among n young men of age a, and n young women of age a'; and we determine the number of these marriages subsisting after x years elapsed from their origin. We name ϕ the probability that a young man who is married at age a will arrive to age $a+x$; and ψ the probability that a young woman who is married at age a' will arrive to age $a'+x$. The probability that their marriage will subsist after its x^{th} year will be $\phi\psi$; therefore, if one develops the binomial $[\phi\psi + (1-\phi\psi)]^n$, the term $H(\phi\psi)^i(1-\phi\psi)^{n-i}$ of this development will express the probability that, out of n marriages, i will subsist after x years. The greatest term of the development is, by n° 16, the one in which i is equal to the greatest whole number contained in $(n+1)\phi\psi$; and, by the same section, it is extremely probable that the number of the marriages subsisting will deviate only very little to the more or to the less from this number. Thus, by designating by i the number of subsisting marriages, one will be able to suppose, very nearly,

$$i = n\phi\psi.$$

$n\phi$ is quite near the number of the n husbands surviving to the age $a+x$. The

Tables of mortality will make it known in a quite near manner, if they have been formed out of the numerous lists of mortality; because if one designates by p' the number of men surviving to age a, out of the collection of these lists, and by q' the number of the surviving to age $a+x$, one will have, to quite nearly, by nº 29,

$$n\phi = \frac{nq'}{p'}.$$

If one names similarly p'' the number of women surviving to age a' and by q'' the number of the surviviors to age $a'+x$, one will have, to very nearly,

$$n\psi = \frac{nq''}{p''};$$

therefore

$$i = \frac{nq'q''}{p'p''}.$$

One will form thus, from year to year, a Table of values of i. By making next a sum of all the numbers of this Table, and by dividing it by n, one will have the mean duration of the marriages made at age a for the young men and at the age a' for the young women.

We seek now the probability that the error of the preceding value of i will be comprehended within some given limits. We suppose, in order to simplify the calculation, that the two spouses are of the same age, and that the probability of the life of the men is the same as that of the women; then one has

$$a' = a, \quad q'' = q', \quad p'' = p', \quad \phi = \psi;$$

and the preceding expression of i becomes

$$i = \frac{nq'^2}{p'^2}.$$

We imagine that the value of i is $\frac{nq'^2}{p'^2} + s$; s will be the error of this expression of i. One has seen, in nº 30, that if one has observed that, out of a very great number p of individuals of age a, q are arrived to the age $a+x$, the probability that, out of p' other individuals of the age a, $\frac{p'q}{p} + z$ will arrive to the age $a+x$, is

$$\sqrt{\frac{p^3}{2qp'(p-q)(p+p')\pi}}\, c^{-\frac{p^3z^2}{2qp'(p-q)(p+p')}}.$$

If one supposes p and q infinite, one will have evidently

$$\phi = \frac{q}{p},$$

and, if one makes

$$\frac{p'q}{p} + z = q',$$

one will have

$$\phi = \frac{q'}{p'} - \frac{z}{p'},$$

this which gives very nearly, by neglecting the square $\frac{nz^2}{p'^2}$,

$$n\phi^2 = \frac{nq'^2}{p'^2} - \frac{2nq'z}{p'^2};$$

thus the preceding probability of z is at the same time the probability of this expression of $n\phi^2$. We suppose now $i = n\phi^2 + l$; by considering the binomial $[\phi^2 + (1-\phi^2)]^n$, the probability of this expression of i is, by n^o 16,

$$\frac{1}{\sqrt{2n\pi\phi^2(1-\phi^2)}}\, c^{-\frac{l^2}{2n\phi^2(1-\phi^2)}}.$$

But the preceding value of i becomes, by substituting for $n\phi^2$ its value,

$$i = \frac{nq'^2}{p'^2} - \frac{2nq'z}{p'^2} + l;$$

the probability of this last expression of i is equal to the product of those of i and of z, found above; it is therefore equal to

$$\frac{c^{-\frac{z^2}{2p'\phi(1-\phi)} - \frac{l^2}{2n\phi^2(1-\phi^2)}}}{2\pi\sqrt{np'\phi^3(1-\phi)^2(1+\phi)}}.$$

Having supposed previously $i = \frac{nq'^2}{p'^2} + s$, one will have $s = l - \frac{2nq'z}{p'^2}$; by substituting therefore for l its value drawn from this equation and observing that one has to very nearly $\frac{q'}{p'} = \phi$, one will have, for the probability that the value of s will be comprehended within some given limits, the integral expression

$$\frac{\int\int dz\, ds\, c^{-\frac{z^2}{2\phi(1-\phi)p'} - \frac{\left(s+\frac{2nq'z}{p'^2}\right)}{2n\phi^2(1-\phi^2)}}}{2\pi\sqrt{np'\phi^3(1-\phi)^2(1+\phi)}},$$

the integral relative to z being able to be taken from $z = -\infty$ to $z = \infty$. Thence, it is easy to conclude, by the methods exposed previously, that, if one makes

$$k^2 = \frac{p'}{2n\phi^2(1-\phi)[p' + (p' + 4n)\phi]},$$

the preceding integral become

$$\int \frac{k\, ds}{\sqrt{\pi}} c^{-k^2 s^2};$$

thus the probability that the error of the expression $i = \frac{nq'^2}{p'^2}$ will be $\pm\, s$ is

$$\frac{2}{\sqrt{\pi}} \int k\, ds\, c^{-k^2 s^2},$$

the integral being taken from s null.

The preceding analysis is applied equally to the mean duration of a great number of associations formed of three individuals or of four individuals, etc. Let n be this number, and we suppose that all the associates are of the same age a at the moment of association; we designate by p the number of individuals from the Table of mortality of the age a, and by q the number of individuals of the age $a + x$; the number i of the associations existing after x years elapsed from the origin of the associations will be to quite nearly

$$i = \frac{nq^r}{p^r},$$

r being the number of individual of each association. One will find by the same analysis the probability that this number will be contained within the given limits.

The sum of the values of i corresponding to all the values of x, divided by n, will be the mean duration of this kind of associations.