SUR

L'APPLICATION DU CALCUL DES PROBABILITÉS
A LA PHILOSOPHIE NATURELLE[1]

Pierre Simon Laplace

Connaissance des Temps for the year 1818; 1815.

When one wishes to know the laws of phenomena, and to attain to a great exactitude, one combines the observations or the experiences in a manner to bring out the unknown elements, and one takes the mean among them. The more observations are numerous, and the less they depart from their mean result, the more this result approaches to the truth. One fulfills this last condition by the choice of the methods, by the precision of the instruments, and by the care that one puts to observe well. Next, one determines by the theory of probabilities the most advantageous mean result, or the one which gives the least taken to error. But this does not suffice; it is yet necessary to appreciate the probability that the error of this result is contained within some given limits; without this, one has only an imperfect knowledge of the degree of exactitude obtained. Formulas proper to this object are therefore a true perfection of the method of natural philosophy, as it is quite important to add to this method. It is one of the things that I have had principally in view in my *Théorie analytique des Probabilités*, where I am arrived to some formulas of this kind which have the remarkable advantage to be independent of the law of the probability of errors, and to contain only quantities given by the same observations and by their analytic expressions. I am going to recall here the principles.

Each observation has for analytic expression a function of the elements which one wishes to determine; and if these elements are nearly known, this function becomes a linear function of their corrections. By equating it to the same observation itself, one forms that which one names the *equation of condition*. If one has a great number of similar observations, one combines them in a manner to form as many final equations as there are elements; and by resolving these

[1] Read to the first class of the Institut, 18 September 1815.

equations, one determines the corrections of the elements. The art consists therefore to combine the equations of condition in the most advantageous manner. For this one must observe that the formation of a final equation, by means of the equations of condition, returns to multiplying each of these by an indeterminate factor, and to reunite these products; but it is necessary to choose the system of factors which give the smallest error to fear. Now it is clear that if one multiplies each error of which an element determined by a system is yet susceptible, by the probability of this error, the most advantageous system will be the one in which the sum of these products, all taken positively, is a minimum; because a positive or negative error can be considered as a loss. By forming therefore this sum of products, the condition of minimum will determine the system of most advantageous factors, and the minimum error to fear respecting each element. I have shown, in the Work cited, that this system is the one of the coefficients of the elements in each equation of condition; so that one forms a first final equation by multiplying respectively each equation of condition by its coefficient of the first element, and by reuniting all these equations thus multiplied. One forms a second final equation by employing the coefficients of the second element, and thus in succession. I have given in the same Work the expression of the minimum of error, whatever be the number of elements. This minimum gives the probability of the errors of which the corrections of these elements are yet susceptible, and which is proportional to the number of which the hyperbolic logarithm is unity, raised to a power of which the exponent is the square of the error taken to less, and divided by the square of the minimum of error, multiplied by the ratio of the circumference to the diameter. The coefficient of the negative square of the error, in this exponent, is able therefore to be considered as the modulus of the probability of the errors, since, the error remaining the same, the probability decreases with rapidity when it increases; so that the result obtained weighs, if I may thus say, towards the truth, so much more as this modulus is greater. I will name, for this reason, this modulus, *weight* of the result. By a remarkable analogy of these weights with those of the bodies, compared to their common center of gravity, it happens that, if a like element is given by diverse systems composed each of a great number of observations, the most advantageous mean result of their whole is the sum of the products of each partial result by its weights, this sum being divided by the sum of all the weights. Moreover, the total weights of the diverse systems is the sum of their partial weights; so that the probability of the errors of the mean result of their whole is proportional to the number which has the unit for hyperbolic logarithm, raised to a power of which the exponent is the square of the error, taken in less, and multiplied by the sum of all the weights. Each weight depends, in truth, on the law of probability of the errors in each system, and nearly always this law is unknown; but I am happily arrived to eliminate the factor which contains it, by means of the sum of the squares of the deviations of the observations of the system, from their mean result. It will be therefore to wish for, in order to complete our knowledge from the results obtained by the whole of a great number of observations, that one wrote, beside each result, the weight which corresponds to it. In order to facilitate the calculation, I develop its analytic expression when one has no more than four elements to determine. But this expression becoming more and more complicated in measure as the number of elements increases, I give a quite simple way to determine the weights of the result, whatever be the number of elements. Then, a regular process to arrive to that which one seeks is preferable to the use of analytic formulas. When one has thus obtained the exponential which represents the law of probability of the errors of a result, the integral of the product of this exponential, by the differential of the error, being taken within some determined limits, it will give the probability that the error of the result is contained within these limits, by multiplying it by the square root of the weight of the result, divided by the circumference of which the diameter is unity. One finds, in the Work[2] cited, some very simple formulas in order to obtain this integral, and Mr. Kramp, in his *Traité des Réfractions astronomiques*, has reduced this genre of integrals into quite convenient Tables.

In order to apply this method with success, it is necessary to vary the circumstances of the observations in a manner to avoid the constant causes of error. It is necessary that the observations be reported faithfully and without bias, by separating only those which contain some evident causes of error. It is necessary that they be numerous, and that they be so many more as there are more elements to determine; because the weight of the mean result increases as the number of observations divided by the number of elements. It is yet necessary that the elements follow, in these observations, a different march; because if the march of two elements were rigorously the same, this which renders their coefficients proportional in the equations of condition, these elements would form only a single unknown, and it would be impossible to distinguish them by these observations. Finally, it is necessary that the observations be precise, so that their deviations from the mean result are of small size. The weight of the result is, thence, much increased, its expression having for divisor the sum of the

[2] TAP, page 109.

squares of these deviations. With these precautions one will be able to make use of the preceding method, and to determine the degree of confidence that the results deduced from a great number of observations merit.

In the Researches which I have read last to the Class on the phenomena of the seas, I have applied this method to the observations of these phenomena. I give here two new applications of them: one is related to the values of the masses of Jupiter, of Saturn and of Uranus; the other is related to the law of variation of gravity. For the first object, I have profited from the immense work that Mr. Bouvard had just finished on the movements of Jupiter and Saturn, from which he has constructed new very precise Tables. He has made use of all the oppositions and all the quadratures observed since Bradley, and which he has discussed anew with the greatest care, this which has given to him for the movement of Jupiter, in longitude, 126 equations of condition. They contain five elements, namely: the mean movement of Jupiter, its mean longitude at a fixed epoch, the longitude of its perihelion to the same epoch, the eccentricity of its orbit; finally the mass of Saturn, of which the action is the principle source of the inequalities of Jupiter. These equations have been reduced, by the most advantageous method, to five final equations of which the resolution has given the value of the five elements. Mr. Bouvard finds thus the mass of Saturn equal to the 3512th part of that of the Sun. One must observe that this mass is the sum of the masses of Saturn, of its satellites and of its ring. My formulas of probability show that there are odds of 11000 against one that the error of this result is not a hundredth of its value, or, that which returns to very nearly the same, that after a century of new observations added to the preceding and discussed in the same manner, the new result will not differ by one hundredth of the one of Mr. Bouvard. There are odds of many thousand against one that this last result is not in error of a fiftieth, because the odds against one increases, by the nature of its analytic expression, with a great rapidity when the interval of the limits of the error increases.

Newton had found, by the observations of Pound out of the greatest elongation of the fourth satellite of Saturn, the mass of this planet equal to the 3012th part of that of the Sun, this which surpasses by a sixth the result of Mr. Bouvard. There are odds of millions of billions against one that the one of Newton is in error, and one will not be surprised at all if one considers the extreme difficulty to observe the greatest elongations of the satellites of Saturn. The ease to observe those of the satellites of Jupiter has rendered much more exact the value of the mass of this planet, that Newton has fixed by the observations of Pound to the 1067th part of that of the Sun. Mr. Bouvard, by the whole of 129 oppositions and quadratures of Saturn, finds it a 1071th of this star, this which differs very little from the value of Newton. My method of probability, applied to the 129 equations of condition of Mr. Bouvard, gives odds 1000000 against one that his result is not in error of one hundredth of its value; there are odds 900 against one that his error is not one hundred fiftieth.

Mr. Bouvard has made the mass of Uranus enter into his equations as indeterminate; he has deduced from them this mass equal to the 17918th part of that of the Sun. The perturbations which it produces in the movement of Saturn being of small size, one must not yet expect from the observations of this movement a great precision in this value. But it is so difficult to observe the elongations of the satellites of Uranus, that one is able to justly fear a considerable error in the value of the mass which results from the observations of Mr. Hershel. It was therefore interesting to see that which, in this regard, the perturbations of the movement of Saturn give. I find that there are odds 213 against one that the error of the result of Mr. Bouvard is not a fiftieth; there are odds 2456 against one that it is not a fourth. After a century of new observations added to the preceding, and discussed in the same manner, these odds numbers will increase further by their squares; one will have therefore then the value of the mass of Uranus, with a great probability that it will be contained within some narrow limits.

I come now to the law of gravity. Since Richer who recognized, first, the diminution of this force at the equator by the deceleration of his clock transported from Paris to Cayenne, one has determined the intensity of gravity, in a great number of places, either by the number of diurnal oscillations of a like pendulum, or by measuring directly the length of the pendulum in seconds. The observations which have to me seemed to merit the most confidence are in number of thirty-seven and extend from 67° of northern latitude to 51° of southern latitude. Although their march is quite regular, they leave however to desire a greater precision still. The length of the isochronous pendulum which results from it follows very nearly the most simple law of variation, that of the square of the sine of the latitude, and the two hemispheres present not at all, in this regard, sensible difference, or at least what can not be attributed to the errors of the observations. But, if there exists among them a slight difference, the observations of the pendulum, by their facility and the precision which one can bring there now, are very proper to demonstrate it. Mr. Mathieu has well wished to discuss, at my request, the observations of which I have just spoken, and he has found that, the length of the pendulum in seconds at the equator being taken for unity, the coefficient of the term proportional to the square of the sine of the latitude is 551 hundred thousandths. My formulas of probability, applied to these observations, give odds 2127 against one that the true coefficient is contained within the limits 5 thousandths and 6 thousandths.

If the Earth is an ellipsoid of revolution, one has its flatness by subtracting from it the coefficient of the law of gravity of 868 hundred thousandths. The coefficient 5 thousandths corresponds thus to the flatness $\frac{1}{272}$; there are therefore odds 4254 against one that the flatness of the Earth is below. There are odds some millions of billions against one that this flatness is less than the one which corresponds to the homogeneity of the Earth, and that the terrestrial layers increase with density in measure as they approach the center of this planet. The great regularity of gravity at its surface proves that they are disposed symmetrically around this point. These two conditions, necessarily following from the fluid state, could not evidently subsist for the Earth, if it had been not at all primitively this state, that an excessive heat has been able to give alone to the whole Earth.

1. Suppose that one has a sequence of equations of condition of the form

$$\epsilon^{(i)} = p^{(i)}z + q^{(i)}z' + r^{(i)}z'' + t^{(i)}z''' + \nu^{(i)}z^{\text{iv}} + \lambda^{(i)}z^{\text{v}} + \cdots - \omega^{(i)}, \tag{1}$$

z, $z', z'', \ldots$ being some m elements of the corrections of the elements which one seeks to determine by the whole of these equations, of which the number is supposed very great; $p^{(i)}$, $q^{(i)}, \ldots$ being some quantities given by the analytic expressions of the observations; $\omega^{(i)}$ being the quantity given by the same observation, and $\epsilon^{(i)}$ being the error of the observation. I have shown in no. 21 of the second Book of my *Théorie analytique des probabilités*,[3] that if n is the number of elements, one will have n final equations the most proper to determine the elements: 1° by multiplying each final equation by its coefficient of z, and by reuniting all the resulting equations with these products, this which gives

$$\text{S}p^{(i)}\epsilon^{(i)} = z\text{S}p^{(i)^2} + z'\text{S}p^{(i)}q^{(i)} + z''\text{S}p^{(i)}r^{(i)} + \cdots - \text{S}p^{(i)}\omega^{(i)},$$

the sign S indicating the sum of the quantities which it affects, from $i = 0$ to $i = s - 1$, s being the number of observations or of equations of condition; 2° by multiplying each equation of condition by its coefficient of z'; this which gives, by reuniting these products,

[3] TAP, page 327.

$$\mathrm{S}q^{(i)}\epsilon^{(i)} = z\mathrm{S}p^{(i)}q^{(i)} + z'\mathrm{S}q^{(i)^2} + z''\mathrm{S}q^{(i)}r^{(i)} + \cdots - \mathrm{S}q^{(i)}\omega^{(i)},$$

and thus consecutively. One will resolve these equations by supposing

$$\mathrm{S}p^{(i)}\epsilon^{(i)} = 0, \quad \mathrm{S}q^{(i)}\epsilon^{(i)} = 0, \quad \mathrm{S}r^{(i)}\epsilon^{(i)} = 0, \quad \ldots,$$

and one will have the most advantageous values of z, z', $z'', \ldots$ There results from the section cited, that the probability of error u of the value of z thus determined, is of the form $\frac{\sqrt{P}\,c^{-Pu^2}}{\sqrt{\pi}}$, c being the number of which the hyperbolic logarithm is unity, and π being the ratio of the circumference to the diameter. By multiplying this probability by udu, and taking the integral from $u = 0$ to u infinity, one will have, by the section cited, that which I have named in this section the *minimum* error to fear; this minimum is therefore $\frac{1}{2\sqrt{\pi P}}$. I have given in the same section the expression of this minimum error; this expression will give therefore the value of P, or of the weight of the result; and one finds that if there is only one correction or element z, one has

$$P = \frac{s\mathrm{S}p^{(i)^2}}{2\mathrm{S}\epsilon^{(i)^2}}.$$

If there are two elements z and z', one will have the value of P, relative to the first element, by changing $\mathrm{S}p^{(i)^2}$ into $\mathrm{S}p^{(i)^2} - \frac{(\mathrm{S}p^{(i)}q^{(i)})^2}{\mathrm{S}q^{(i)^2}}$, by making therefore generally

$$P = \frac{s}{2\mathrm{S}\epsilon^{(i)^2}}\,\frac{A}{B},$$

and designating, for brevity, $\mathrm{S}p^{(i)^2}$ by $p^{(2)}$, $\mathrm{S}p^{(i)}q^{(i)}$ by $\overline{pq}$, $\mathrm{S}q^{(i)^2}$ by $q^{(2)}$, one will have

$$A = p^{(2)}q^{(2)} - \overline{pq}^2,$$
$$B = q^{(2)}.$$

If there are three elements z, z', z'', one will have A by changing, in the value preceding A, $p^{(2)}$ into $p^{(2)} - \frac{\overline{pr}^2}{r^{(2)}}$, $\overline{pq}$ into $\overline{pq} - \frac{\overline{pr}\,\overline{qr}}{r^{(2)}}$, and $q^{(2)}$ into $q^{(2)} - \frac{\overline{qr}^2}{r^{(2)}}$, and multiplying the whole by $r^{(2)}$. One will have B by making the same substitutions and the same multiplication relative to the preceding value of

B; one has thus

$$A = p^{(2)}q^{(2)}r^{(2)} - p^{(2)}\overline{qr}^2 - q^{(2)}\overline{pr}^2 - r^{(2)}\overline{pq}^2 + 2\overline{pq}\,\overline{pr}\,\overline{qr},$$
$$B = q^{(2)}r^{(2)} - \overline{qr}^2.$$

If there are four elements, one will have the values of A and of B by changing, in the two preceding, $p^{(2)}$ into $p^{(2)} - \frac{\overline{pt}^2}{t^{(2)}}$, $\overline{pq}$ into $\overline{pq} - \frac{\overline{pt}\,\overline{qt}}{t^{(2)}}, \dots$ and multiplying the whole by $t^{(2)}$, this which gives

$$\begin{aligned} A = p^{(2)}q^{(2)}r^{(2)}t^{(2)} &- p^{(2)}q^{(2)}\overline{rt}^2 - p^{(2)}r^{(2)}\overline{qt}^2 - p^{(2)}t^{(2)}\overline{qr}^2 \\ &- q^{(2)}r^{(2)}\overline{pt}^2 - q^{(2)}t^{(2)}\overline{pr}^2 - r^{(2)}t^{(2)}\overline{pq}^2 \\ &+ \overline{pq}^2\,\overline{rt}^2 + \overline{pr}^2\,\overline{qt}^2 + \overline{pt}^2\,\overline{qr}^2 \\ &+ 2p^{(2)}\overline{qr}\,\overline{qt}\,\overline{rt} + 2q^{(2)}\overline{pr}\,\overline{pt}\,\overline{rt} \\ &+ 2r^{(2)}\overline{pq}\,\overline{pt}\,\overline{qt} + 2t^{(2)}\overline{pq}\,\overline{pr}\,\overline{qr} \\ &- 2\overline{pq}\,\overline{pr}\,\overline{qt}\,\overline{rt} - 2\overline{pq}\,\overline{pt}\,\overline{qr}\,\overline{rt} - 2\overline{pr}\,\overline{pt}\,\overline{qr}\,\overline{qt}, \\ B = q^{(2)}r^{(2)}t^{(2)} &- q^{(2)}\overline{rt}^2 - r^{(2)}\overline{qt}^2 - t^{(2)}\overline{qr}^2 + 2\overline{qr}\,\overline{qt}\,\overline{rt}. \end{aligned}$$

In continuing thus, one will have the value of P relative to the first element, whatever be the number of elements. By changing p into q and q into p, one will have the value of P relative to the second element; p into r and r into p, one will have the value of P relative to the third element, and thus consecutively.

The value of A becomes more complicated in measure as the number of elements increases; its expression for six elements is of an excessive length, and its numeric calculation would be impractical. It is worth more then to have a simple and regular process in order to arrive there; it is this which one obtains in the following manner:

Suppose that there are six elements, and that thus the equation of condition (1) is of the form

$$\text{(2)} \qquad \epsilon^{(i)} = \lambda^{(i)}z^{\mathrm{v}} + \nu^{(i)}z^{\mathrm{iv}} + t^{(i)}z''' + r^{(i)}z'' + q^{(i)}z' + p^{(i)}z - \omega^{(i)}.$$

By multiplying this equation by $\lambda^{(i)}$, and reuniting the similar products, relative to all the equations of condition that equation (2) represents, one will have

$$\mathrm{S}\lambda^{(i)}\epsilon^{(i)} = z^{\mathrm{v}}\mathrm{S}\lambda^{(i)^2} + z^{\mathrm{iv}}\mathrm{S}\lambda^{(i)}\nu^{(i)} + z'''\mathrm{S}\lambda^{(i)}t^{(i)} + \cdots - \mathrm{S}\lambda^{(i)}\omega^{(i)}.$$

By the conditions of the most advantageous method, one has

$$\mathrm{S}\lambda^{(i)}\epsilon^{(i)} = 0,$$

the preceding equation will give therefore

$$z^{\mathrm{v}} = -z^{\mathrm{iv}}\frac{\mathrm{S}\lambda^{(i)}\nu^{(i)}}{\mathrm{S}\lambda^{(i)^2}} - z'''\frac{\mathrm{S}\lambda^{(i)}t^{(i)}}{\mathrm{S}\lambda^{(i)^2}} - \cdots + \frac{\mathrm{S}\lambda^{(i)}\omega^{(i)}}{\mathrm{S}\lambda^{(i)^2}}.$$

By substituting this value of z^{v} into equation (2), one will have this here

$$(3)\qquad \begin{cases} \epsilon^{(i)} = z^{\mathrm{iv}}\left(\nu^{(i)} - \lambda^{(i)}\dfrac{\mathrm{S}\lambda^{(i)}\nu^{(i)}}{\mathrm{S}\lambda^{(i)^2}}\right) \\ \qquad\quad + z'''\left(t^{(i)} - \lambda^{(i)}\dfrac{\mathrm{S}\lambda^{(i)}t^{(i)}}{\mathrm{S}\lambda^{(i)^2}}\right) + \cdots - \omega^{(i)} + \lambda^{(i)}\dfrac{\mathrm{S}\lambda^{(i)}\omega^{(i)}}{\mathrm{S}\lambda^{(i)^2}}. \end{cases}$$

One has thus, by making successively $i = 0$, $i = 1, \ldots, i = s-1$, a new system of equations of condition, which contains no more than five elements, z^{iv}, z''', $\ldots$

Making, for brevity,

$$\begin{aligned} \nu_1^{(i)} &= \nu^{(i)} - \lambda^{(i)}\frac{\mathrm{S}\lambda^{(i)}\nu^{(i)}}{\mathrm{S}\lambda^{(i)^2}}, \\ t_1^{(i)} &= t^{(i)} - \lambda^{(i)}\frac{\mathrm{S}\lambda^{(i)}t^{(i)}}{\mathrm{S}\lambda^{(i)^2}}, \\ &\cdots, \\ \omega_1^{(i)} &= \omega^{(i)} - \lambda^{(i)}\frac{\mathrm{S}\lambda^{(i)}\omega^{(i)}}{\mathrm{S}\lambda^{(i)^2}}, \end{aligned}$$

equation (3) will become

$$(4)\qquad \epsilon^{(i)} = \nu_1^{(i)}z^{\mathrm{iv}} + t_1^{(i)}z''' + r_1^{(i)}z'' + q_1^{(i)}z' + p_1^{(i)}z - \omega_1^{(i)}.$$

By multiplying this equation by $\nu_1^{(i)}$, and reuniting the similar products, relative to all the equations which this represents, by observing next that one has $\mathrm{S}\nu_1^{(i)}\epsilon^{(i)} = 0$, by virtue of the two equations $\mathrm{S}\lambda^{(i)}\epsilon^{(i)}$=0, $\mathrm{S}\nu^{(i)}\epsilon^{(i)}$=0, which the conditions of the most advantageous method give, one will have

$$0 = z^{\mathrm{iv}}\mathrm{S}\nu_1^{(i)^2} + z'''\mathrm{S}\nu_1^{(i)}t_1^{(i)} + \cdots$$

If one draws from this equation the value of z^{iv}, one will have, in substituting it into equation (4),

$$\epsilon^{(i)} = t_2^{(i)} z''' + r_2^{(i)} z'' + q_2^{(i)} z' + p_2^{(i)} z - \omega_2^{(i)}, \tag{5}$$

by making

$$t_2^{(i)} = t_1^{(i)} - \nu_1^{(i)} \frac{\mathrm{S}\nu_1^{(i)} t_1^{(i)}}{\mathrm{S}\nu_1^{(i)^2}},$$
$$r_2^{(i)} = r_1^{(i)} - \nu_1^{(i)} \frac{\mathrm{S}\nu_1^{(i)} r_1^{(i)}}{\mathrm{S}\nu_1^{(i)^2}},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots,$$

By multiplying further equation (5) by $t_2^{(i)}$, and reuniting the similar products relative to all the equations of condition represented by equation (5), in observing next that one has $\mathrm{S}t_2^{(i)}\epsilon^{(i)} = 0$, by virtue of the equations

$$\mathrm{S}\lambda^{(i)}\epsilon^{(i)}=0, \quad \mathrm{S}\nu^{(i)}\epsilon^{(i)}=0, \quad \mathrm{S}t^{(i)}\epsilon^{(i)}=0,$$

one will have an equation whence one will draw the value of z''', which, substituted into equation (5), will give[4]

$$\epsilon^{(i)} = r_3^{(i)} z'' + q_3^{(i)} z' + p_3^{(i)} z - \omega_3^{(i)}, \tag{6}$$

by making

$$r_3^{(i)} = r_2^{(i)} - t_2^{(i)} \frac{\mathrm{S}t_2^{(i)} r_2^{(i)}}{\mathrm{S}t_2^{(i)^2}}, \quad \cdots$$

By continuing thus, one arrives to an equation of the form

$$\epsilon^{(i)} = p_5^{(i)} z - \omega_5^{(i)}, \tag{7}$$

There results from n° 20 of the second Book of my *Théorie analytique des probabilités*[5] that if the value of z is determined by equation (7) and if u is the error of this value, the probability of this error is

[4] Translator's note: The original lacks superscripts on p, q, r and t in equation (6) and the following displayed equation. These have been inserted.

[5] TAP, page 318.

$$\sqrt{\frac{s\mathrm{S}p_5^{(i)^2}}{2\mathrm{S}\epsilon^{(i)^2}}}\; c^{-\frac{s\mathrm{S}p_5^{(i)^2}}{2\mathrm{S}\epsilon^{(i)^2}}u^2};$$

one has therefore

$$P = \frac{s\mathrm{S}p_5^{(i)^2}}{2\mathrm{S}\epsilon^{(i)^2}}.$$

Now the question is to form the quantity $\mathrm{S}p_5^{(i)^2}$. For this, I observe that the equations of condition, represented by equation (2), give the following six equations, by multiplying them first by their coefficient of z^{v} and adding them, next by multiplying them by their coefficient of z^{iv} and adding them, and thus consecutively:

$$\text{(A)}\qquad \begin{cases} \overline{\lambda\omega} = \lambda^{(2)}z^{\mathrm{v}} + \overline{\lambda\nu}z^{\mathrm{iv}} + \overline{\lambda t}z''' + \overline{\lambda r}z'' + \overline{\lambda q}z' + \overline{\lambda p}z, \\ \overline{\nu\omega} = \overline{\lambda\nu}z^{\mathrm{v}} + \nu^{(2)}z^{\mathrm{iv}} + \overline{\nu t}z''' + \overline{\nu r}z'' + \overline{\nu q}z' + \overline{\nu p}z, \\ \overline{t\omega} = \overline{\lambda t}z^{\mathrm{v}} + \overline{\nu t}z^{\mathrm{iv}} + t^{(2)}z''' + \overline{tr}z'' + \overline{tq}z' + \overline{tp}z, \\ \overline{r\omega} = \overline{\lambda r}z^{\mathrm{v}} + \overline{r\nu}z^{\mathrm{iv}} + \overline{rt}z''' + r^{(2)}z'' + \overline{rq}z' + \overline{rp}z, \\ \overline{q\omega} = \overline{\lambda q}z^{\mathrm{v}} + \overline{q\nu}z^{\mathrm{iv}} + \overline{qt}z''' + \overline{qr}z'' + q^{(2)}z' + \overline{qp}z, \\ \overline{p\omega} = \overline{\lambda p}z^{\mathrm{v}} + \overline{p\nu}z^{\mathrm{iv}} + \overline{pt}z''' + \overline{pr}z'' + \overline{pq}z' + p^{(2)}z. \end{cases}$$

One must observe that, in these equations, one has

$$\lambda^2 = \mathrm{S}\lambda^{(i)^2}, \quad \overline{\lambda\nu} = \mathrm{S}\lambda^{(i)}\nu^{(i)}, \quad \ldots,$$

and thus of the rest.

One will form in the same manner the following five equations:

$$\text{(B)}\qquad \begin{cases} \overline{\nu_1\omega_1} = \nu_1^{(2)}z^{\mathrm{iv}} + \overline{\nu_1 t_1}z''' + \overline{\nu_1 r_1}z'' + \overline{\nu_1 q_1}z' + \overline{\nu_1 p_1}z, \\ \overline{t_1\omega_1} = \overline{t_1\nu_1}z^{\mathrm{iv}} + t_1^{(2)}z''' + \overline{t_1 r_1}z'' + \overline{t_1 q_1}z' + \overline{t_1 p_1}z, \\ \overline{r_1\omega_1} = \overline{r_1\nu_1}z^{\mathrm{iv}} + \overline{r_1 t_1}z''' + r_1^{(2)}z'' + \overline{r_1 q_1}z' + \overline{r_1 p_1}z, \\ \overline{q_1\omega_1} = \overline{q_1\nu_1}z^{\mathrm{iv}} + \overline{q_1 t_1}z''' + \overline{q_1 r_1}z'' + q_1^{(2)}z' + \overline{q_1 p_1}z, \\ \overline{p_1\omega_1} = \overline{p_1\nu_1}z^{\mathrm{iv}} + \overline{p_1 t_1}z''' + \overline{p_1 r_1}z'' + \overline{p_1 q_1}z' + p_1^{(2)}z. \end{cases}$$

One will have the values of $\nu_1^{(2)}$, $\overline{\nu_1 t_1}$, $\ldots$, by means of the coefficients of equations (A), by observing that

$$\nu_1^{(2)} = \nu^{(2)} - \frac{\overline{\lambda\nu}^2}{\lambda^{(2)}}, \quad \overline{\nu_1 t_1} = \overline{\nu t} - \frac{\overline{\lambda\nu}\,\overline{\lambda t}}{\lambda^{(2)}}, \quad \overline{\nu_1 r_1} = \overline{\nu r} - \frac{\overline{\lambda\nu}\,\overline{\lambda r}}{\lambda^{(2)}}, \quad \ldots,$$
$$t_1^{(2)} = t^{(2)} - \frac{\overline{\lambda t}^2}{\lambda^{(2)}}, \quad \ldots, \quad \overline{\nu_1 \omega_1} = \overline{\nu\omega} - \frac{\overline{\lambda\nu}\,\overline{\lambda\omega}}{\lambda^{(2)}}, \quad \ldots$$

One will form in the same manner the following four equations:

$$\text{(C)} \qquad \begin{cases} \overline{t_2\omega_2} = t_2^{(2)} z''' + \overline{t_2 r_2} z'' + \overline{t_2 q_2} z' + \overline{t_2 p_2} z, \\ \overline{r_2\omega_2} = \overline{r_2 t_2} z''' + r_2^{(2)} z'' + \overline{r_2 q_2} z' + \overline{r_2 p_2} z, \\ \overline{q_2\omega_2} = \overline{q_2 t_2} z''' + \overline{q_2 r_2} z'' + q_2^{(2)} z' + \overline{q_2 p_2} z, \\ \overline{p_2\omega_2} = \overline{p_2 t_2} z''' + \overline{p_2 r_2} z'' + \overline{p_2 q_2} z' + p_2^{(2)} z. \end{cases}$$

whence one has

$$t_2^{(2)} = t_1^{(2)} - \frac{\overline{\nu_1 t_1}^2}{\nu_1^{(2)}}, \quad \overline{t_2 r_2} = \overline{t_1 r_1} - \frac{\overline{\nu_1 t_1}\,\overline{\nu_1 r_1}}{\nu_1^{(2)}},$$
$$\overline{t_2\omega_2} = \overline{t_1\omega_1} - \frac{\overline{\nu_1 t_1}\,\overline{\nu_1\omega_1}}{\nu_1^{(2)}}, \quad \ldots$$

As one has no more here than four elements, one can apply to these equations the formulas of nº 1, but one can continue to eliminate and to form thus the value of $p_5^{(2)}$.

2. In order to apply this method to an example, I take the following six equations:

$$
\begin{aligned}
&129z^{\text{v}} + 46,310z^{\text{iv}} + 1,1128z''' + 1,3371z'' + 5722z' + 2602z && = -1002,900.\\
&46,310z^{\text{v}} + 2,1543z^{\text{iv}} + 3,6213z''' + 1,2484z'' - 5459z' + 696,13z && = -343,455.\\
&1,1128z^{\text{v}} + 3,6213z^{\text{iv}} + 57,1911z''' - 3,2252z'' - 39749,1z' - 1959,0z && = -40,335.\\
&1,3371z^{\text{v}} + 1,2484z^{\text{iv}} - 3,2252z''' + 71,8720z'' - 153106,5z' + 6788,2z && = 237,782.\\
&5722z^{\text{v}} - 5459z^{\text{iv}} - 39749,1z''' - 153106,5z'' + 424865729z' - 12729398z && = -738297,8.\\
&2602z^{\text{v}} + 696,13z^{\text{iv}} - 1959,0z''' + 6788,2z'' - 12729398z' + 795938z && = 7212,6.
\end{aligned}
$$

These equations are those in which Mr. Bouvard is arrived by 129 so many oppositions as quadratures of Saturn, and from which he has concluded the corrections of the elements of the movement of this planet. z^{v} is the correction of the mean longitude, in 1750; z^{iv} is the secular correction of the mean movement; z''' is the correction of the equation of the center; z'' is the product of the equation of the center with the correction of the perihelion; z' is the mass of Jupiter and z is that of Uranus. The second decimal is unity.

By means of these equations, which are contained in the system (A), I have formed the following five, contained in the system (B):

$$
\begin{aligned}
&4,9181z^{\text{iv}} + 3,2217z''' + 0,7684z'' - 7513,2z' - 237,97z && = \cdots,\\
&3,2217z^{\text{iv}} + 57,1815z''' - 3,2367z'' - 39798,5z' - 1981,4z && = \cdots,\\
&0,7684z^{\text{iv}} - 3,2367z''' + 71,8581z'' - 153165,8z' + 6761,2z && = \cdots,\\
&-7513,2z^{\text{iv}} - 39798,5z''' - 153165,8z'' + 424611921z' - 12844814z && = \cdots,\\
&-237,97z^{\text{iv}} - 1981,4z''' + 6761,2z'' - 12844814z' + 743454z && = \cdots,
\end{aligned}
$$

From these equations I have drawn the following four, contained in the system (C):

$$
\begin{aligned}
&55,071z''' - 3,7401z'' - 34876,8z' - 1825,5z && = \cdots,\\
&-3,7401z''' + 71,7380z'' - 151992,0z' + 6798,4z && = \cdots,\\
&-34876,8z''' - 151992,0z'' + 413134287z' - 13208352z && = \cdots,\\
&-1825,5z''' + 6798,4z'' - 13208352z' + 731939z && = \cdots,
\end{aligned}
$$

These last equations have led me to the following three:

$$\begin{aligned}
&71,4840z'' - 154360,6z' + 6674,4z &&= \cdots,\\
&-154360,6z'' + 391046641z' - 14364450z &&= \cdots,\\
&6674,4z'' + 14364450z' + 671427z &&= \cdots,
\end{aligned}$$

Finally, I have drawn from this last system of equations the following two:

$$\begin{aligned}
57724487z' + 48067z &= \cdots,\\
48067z' + 48244z &= \cdots,
\end{aligned}$$

I am myself stopped at this system since it is easy to conclude from it the values of P, relative to the two elements z' and z, which I wished particularly to know, and I have found by the formulas of nº 1, for z',

$$P = \frac{s}{2\mathrm{S}\epsilon^{(i)^2}}\left[57724487 - \frac{(48067)^2}{48244}\right],$$

and for z,

$$P = \frac{s}{2\mathrm{S}\epsilon^{(i)^2}}\left[48244 - \frac{(48067)^2}{57724487}\right].$$

The number s of observations is here 129, and Mr. Bouvard has found

$$\mathrm{S}\epsilon^{(i)^2} = 31096,$$

one has therefore, for z',

$$\log P = 5,0778548,$$

and, for z,

$$\log P = 1,9999383.$$

The mass of Jupiter is

$$\frac{1}{1067,09}(1 + z'),$$

and Mr. Bouvard has found $z' = -0,00332$, this which gives the mass of Jupiter equal to $\frac{1}{1070,5}$.

The probability that the error of z' is contained within the limits $\pm U$, equal$s$

$$\frac{\sqrt{P}}{\sqrt{\pi}}\int du\, c^{-Pu^2},$$

the integral being taken within the limits $u = \pm U$. One finds thus the probability that the error of the value of the mass of Jupiter, determined by Mr. Bouvard, is contained within the limits $\pm\frac{1}{150}$ of $\frac{1}{1067{,}09}$, equal to $\frac{900}{901}$, and the probability that this error is contained within the limits $\pm\frac{1}{100}$ of $\frac{1}{1067{,}09}$, equal to $\frac{999307}{999308}$. The mass of Uranus is

$$\frac{1}{19504}(1+z),$$

and Mr. Bouvard has found $z = 0,08848$, this which gives $\frac{1}{17918}$ for the mass of Uranus. The probability that the error of the mass of Uranus thus determined is contained within the limits $\pm\frac{1}{5}$ of $\frac{1}{19504}$, is $\frac{212{,}8}{213{,}8}$.

Relative to the mass of Saturn, Mr. Bouvard has supposed it, in his equations of condition of the movement of Jupiter in longitude, equal to

$$\frac{1+z}{3534,08},$$

and he has found $z = 0,00633$, this which gives $\frac{1}{3512}$ for the mass of Saturn. In applying my formulas to these equations of condition, I find

$$\log P = 4,8851146.$$

The probability that the mass of Saturn thus determined is within the limits $\pm\frac{1}{100}$ of $\frac{1}{3524{,}08}$, equals $\frac{11170}{11171}$.

3. We apply again the formulas of probability to the observations of the pendulum in seconds.

In representing by z' the length of the pendulum at the equator, by $p^{(i)}$ the square of the sine of latitude, and by z its coefficient in the law of gravity, Mr. Mathieu has formed, by comparing to this law the thirty-seven observations of which I have spoken above, thirty-seven equations of condition of the form

$$\epsilon^{(i)} = zp^{(i)} + z' - \omega^{(i)}.$$

In resolving them by the most advantageous method, he has drawn from them two final equations which have given to him the values of z and z', and he has deduced from them, for the expression of the length of the pendulum,

$$(a) \qquad 1,0000043162 + 0,0055188 p^{(i)}.$$

In this expression, the length of the pendulum is compared to none of our linear measures, because the observations, such as Mr. Mathieu has considered them, are, properly speaking, only those of the number of diurnal oscillations which a similar pendulum has made in the diverse places. It is necessary therefore, in order to have in linear measures the length of the pendulum in decimal seconds, to compare this length to these measures, in a given place. It is this which Borda has executed with a care and an extreme precision, at the Observatory of Paris, where he has found this length equal to $0^{\mathrm{m}},741887$. Thence I have concluded, for the general expression of the length of this pendulum,

$$0^{\mathrm{m}},739505 + 0^{\mathrm{m}},0040780 p^{(i)}.$$

Now, in order to have the probability that the coefficient of $p^{(i)}$ or of the law of gravity is contained within the given limits, it is necessary to know the values of $\mathrm{S}p^{(i)}$, $\mathrm{S}p^{(i)^2}$ and $\mathrm{S}\epsilon^{(i)^2}$. Mr. Mathieu has found

$$\begin{aligned} \mathrm{S}p^{(i)} &= 14,255136, \\ \mathrm{S}p^{(i)^2} &= 7,9569564, \\ \mathrm{S}\epsilon^{(i)^2} &= 0,00000093890182. \end{aligned}$$

One has besides here $q^{(i)} = 1$; this which gives $\mathrm{S}q^{(i)} = s$, s being the number of observations which, in the present case, is equal to thirty-seven. This put, I observe that if one names u and u' the simultaneous errors of the values of z and z', determined by the most advantageous method, the probability of these errors is, by n$^{\mathrm{o}}$21 of the second Book of my *Théorie analytique des probabilités*,[6] proportional to the exponential

$$c^{-\frac{(Fu^2 + 2Guu' + Hu'^2)s}{E 2\mathrm{S}\epsilon^{(i)^2}}},$$

and one has, by the same section,

[6] TAP, page 327.

$$
\begin{aligned}
F &= \mathrm{S}p^{(i)^2}\Big[s\mathrm{S}p^{(i)^2} - (\mathrm{S}p^{(i)})^2\Big],\\
G &= \mathrm{S}p^{(i)}\Big[s\mathrm{S}p^{(i)^2} - (\mathrm{S}p^{(i)})^2\Big],\\
H &= s\Big[s\mathrm{S}p^{(i)^2} - (\mathrm{S}p^{(i)})^2\Big],\\
E &= s\mathrm{S}p^{(i)^2} - (\mathrm{S}p^{(i)})^2,
\end{aligned}
$$

this which changes the preceding exponential into this

$$
c^{-\frac{s(u^2\mathrm{S}p^{(i)^2}+2uu'\mathrm{S}p^{(i)}+u'^2)}{2\mathrm{S}\epsilon^{(i)^2}}},
$$

But, if one takes for unity the length of the pendulum at the equator, it will be necessary to divide the formula (*a*) by its first term, and then it becomes quite nearly

$$
(b) \qquad 1 = 0.0055145p^{(i)}.
$$

One sees also that the error of this new coefficient of $p^{(i)}$ is $u - u'$, we will designate it by t, so that $u - u' = t$. In making moreover

$$
\begin{aligned}
P &= \frac{\Big[s\mathrm{S}p^{(i)^2} - (\mathrm{S}p^{(i)})^2\Big]s}{(\mathrm{S}p^{(i)^2} + 2\mathrm{S}p^{(i)} + s)2\mathrm{S}\epsilon^{(i)^2}},\\
t' &= u - \frac{t(\mathrm{S}p^{(i)} + s)}{\mathrm{S}p^{(i)^2} + 2\mathrm{S}p^{(i)} + s},
\end{aligned}
$$

the preceding exponential becomes

$$
c^{-Pt^2-\frac{s(\mathrm{S}p^{(i)^2}+2\mathrm{S}p^{(i)}+s)t'^2}{2\mathrm{S}\epsilon^{(i)^2}}},
$$

By multiplying this exponential by $dt\,dt'$, by integrating it with respect to t', from $t' = -\infty$ to $t' = \infty$, and relative to t, within the given limits; finally, by dividing this double integral by the same double integral, taken relative to t and t' from $-\infty$ to $+\infty$, one will have the probability that the value of t is contained within the given limits. The expression of this probability will be thus

$$
\frac{\sqrt{P}\int dt\, c^{-Pt^2}}{\sqrt{\pi}}
$$

The preceding values of s, $\mathrm{S}p^{(i)^2}$, $\mathrm{S}p^{(i)}$ and $\mathrm{S}\epsilon^{(i)^2}$ give

$$\log P = 7,3884431.$$

By means of this value of $\log P$, one can determine the probability that the true coefficient of $p^{(i)}$, in formula (*b*), is contained within some given limits. I find thus that the probability that it is contained between $0,0050145$ and $0,0060145$ is $\frac{1}{2128,1}$.