MÉMOIRE
SUR LES
APPROXIMATIONS DES FORMULES
QUI SONT FONCTIONS DE TRÈS GRANDS NOMBRES

ET SUR

LEUR APPLICATION AUX PROBABILITÉS[1]

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Mémoires de l'Académie royale des Sciences, 1st Series, T. X, year 1809; 1810

Analysis often leads to some formulas of which the numerical calculus, when we substitute into it some very large numbers, becomes impractical, because of the multiplicity of the terms and of the factors of which they are composed. This inconvenience takes place principally in the theory of the probabilities, where we consider events repeated a great number of times. It is therefore useful then to be able to transform these formulas into series accordingly more convergent as the substituted numbers are more considerable. The first transformation of this kind is due to Stirling, who reduced in the most fortunate manner, into a similar series, the middle term of the binomial raised to a high power; and the theorem to which he arrived can be set at the rank of the most beautiful things which we have found in analysis. That which struck the geometers above all, and especially Moivre, who had long occupied himself in this object, was the introduction of the square root of the circumference of which the radius is unity, in a research which seems estranged from this transcendent. Stirling had arrived there by means of the expression of the circumference as a fraction of which the numerator and the denominator are the products in infinite number, an expression that Wallis had given. This indirect means left room for a direct and general method to obtain,

[1] Read on 9 April 1810.

not only the approximation of the middle term of the binomial, but yet that of many other more complicated formulas and which present themselves at each step in the analysis of chances. This is that which I myself have proposed in diverse memoirs published in the volumes of the Académe des Sciences for the years 1778 and 1782. The method which I have presented in these memoires transform generally into convergent series the integrals of the equations linear in the ordinary or partial differences, finite and infinitely small, when we substitute some large numbers into these integrals. It is extended yet to many other similar formulas, such as the very elevated differences of the functions. These series have most often for factor the square root of the circumference, and this is the reason for which this transcendent is presented to Stirling; but, sometimes, they contain some higher transcendents of which the number is infinite.

Among the formulas which I have transformed in this manner, one of the most remarkable is that of the finite difference of the power of a variable. But we have need frequently, in the questions of the probabilities, to consider only a part of these terms and to stop when the variable, by its successive diminutions, becomes negative. This case holds, for example, in the problem where we seek the probability that the mean inclination of the orbits of any number of comets is contained within some given limits, all the inclinations being equally possible, a problem of which the solution serves to acknowledge if these orbits participate in the primitive tendency of the orbits of the planets and of the satellites for the sake of bringing themselves together in the plane of the solar equator. In resolving this problem, by the method which I have given for this kind of questions in the volume of the Académie des Sciences of the year 1778, the probability of which there is question is expressed by the finite difference of the power of a variable which decreases uniformly, the degrees of the power and of the difference being the same number of the orbits which we consider, and the formula must be stopped when the variable becomes negative. The numerical calculation of this formula is impractical for the comets already observed; because it is necessary to consider nearly fifty very composite terms and which, being alternately positives and negatives, destroy themselves nearly entirely; so that, in order to have the final result of them together, it would be necessary to calculate them separately with a precision superior to that which we can obtain by means of the most extended Tables of logarithms. This difficulty has stopped me for a long time: I have finally arrived to conquer it by considering the problem under a new point of view, which has led me to express the sought probability, by a convergent series, in the general case where facilities of the inclinations follow any law. This problem is identical with the one in which we seek the probability that the mean of the errors of a great number of observations will be contained within some given limits, and there results from my solution that, by multiplying indefinitely the observations, their mean result converges towards a fixed term, in a manner that, by taking on both sides of this term any interval as small as we will wish, the probability that the result will fall in this interval will stop by differing from certitude only by a quantity less than every assignable size. This mean term is confounded with the truth if the positive and negative errors are equally possible, and generally this term is the abscissa of the curve of facility of the errors corresponding to the ordinate of the center of gravity of the area of this curve, the origin of the abscissas being that of the errors.

By comparing the two solutions of the problem obtained by the methods of which I just spoke, we have, by some convergent series, the value of the finite differences of the elevated powers of a variable and those of many other similar functions, by stopping them at the point where the variable becomes negative; but, this manner being indirect, I have sought a direct method to obtain these approximations, and I have come by the aid of equations in the finite and infinitely small partial differences on which these functions depend, that which leads to diverse curious theorems. These approximations are deduced yet very simply from the reciprocal passage of the imaginary results to the real results, of which I have given diverse examples in the Mémoires cited in the Académie des Sciences and very recently in Book VIII of the *Journal de l'École Polytechnique*.[2] It is analogous to the one from the positive integers, to the negative numbers and to the fractional numbers, passage from which the geometers have known to deduce, by induction, many important theorems; employed as they do with reserve, it becomes a second way to discover, and it indicates more and more the generality of analysis. I dare to hope that these researches, which serve to supplement those which I have given earlier on the same object, may interest the geometers.

In order to apply these researches to the orbits of the comets, I have considered all those which we have observed until 1807 inclusively. Their number is elevated to ninety-seven and, among them, fifty-two have a direct movement and forty-five a retrograde movement; the mean inclination of their orbits to the ecliptic differs very little from the mean of all the possible inclinations or from a half right angle. We find by the formulas of this memoir that, by supposing the inclinations, at the same time the direct and retrograde movements, equally easy, the probability that the observed results must be brought together more from

[2] *Mémoir sur divers points d'analyse*. (1809)

their mean state is much too weak in order to indicate in these stars a primitive tendency to be moved all onto a same plane and in the same sense. But, if we apply the same formulas to the movements of rotation and of revolution of the planets and of the satellites, we see that this double tendency is indicated with a probability much superior to that of the greatest number of historical facts on which we permit no doubt.

I.

We suppose all the inclinations to the ecliptic equally possible from zero to the right angle, and we demand the probability that the mean inclination of n orbits will be contained within some given limits.

We designate the right angle by h, and we represent by k the law of facility of the inclinations of an orbit. Here k will be constant from the null inclination to the inclination h. Beyond this limit, the facility is null; we can therefore generally represent the facility by $k(1 - l^h)$, provided that we make its second term begin only at the inclination h and that we suppose l equal to unity in the result of the calculation.

This put, we name $t, t_1, t_2, \dots$ the inclinations of the n orbits, and we suppose their sum equal to s, we will have

$$t + t_1 + t_2 + \cdots + t_{n-1} = s.$$

The probability of this combination is evidently the product of the probabilities of the inclinations $t, t_1, t_2, \dots$ and consequently it is equal to $k^n(1 - l^h)^n$. By taking the sum of all the probabilities relative to each of the combinations in which the preceding equation holds, we will have the probability that the sum of the inclinations of the orbits will be equal to s. In order to have this sum of the probabilities, we will observe that the preceding equation gives

$$t = s - t_1 - t_2 - \cdots - t_{n-1}.$$

If we suppose first $t_2, t_3, \dots, t_{n-1}$ constants, the variations of t depend only on those of t_1 and they can be extended from t null in which case t_1 is equal to $s - t_2 - \cdots - t_{n-1}$, to

$$t = s - t_2 - \cdots - t_{n-1},$$

that which renders t_1 null. The sum of all the probabilities relative to these variations is evidently

$$k^n(1 - l^h)^n(s - t_2 - \cdots - t_{n-1}).$$

It is necessary next to multiply this function by dt_2 and to integrate it from t_2 null to $t_2 = s - t_3 - \cdots - t_{n-1}$, this which gives

$$\frac{k^n(1-l^h)^n}{1.2}(s - t_3 - \cdots - t_{n-1})^2.$$

By continuing thus to the last variable, we will have the function

$$\frac{k^n(1-l^h)^n s^{n-1}}{1.2.3\ldots(n-1)}.$$

It is necessary next to multiply this function by ds and to integrate it within the given limits, which we will represent by $s - e$ and $s{+}e'$, and we will have

$$\frac{k^n(1-l^h)^n}{1.2.3\ldots n}[(s+e')^n - (s-e)^n]$$

for the probability that the sum of the errors will be contained within these limits. But we must make here an important observation. Any term, such as $Ql^{rh}(s-e)^n$, can have place only as long as a number r of the variables t, t_1, $\ldots, t_{n-1}$ begin to surpass h; because it is only in this way that the factor l^{rh} can be introduced. It is necessary then to increase each of them by the quantity h in the equation

$$t + t_1 + t_2 + \cdots + t_{n-1} = s,$$

that which happens to make these variables depart from zero, by diminishing s from rh. The term $Ql^{rh}(s-e)^n$ becomes thus $Ql^{rh}(s - rh - e)^n$. Moreover, as the variables t, t_1, $\ldots$ are necessarily positives, this term must be rejected when $s - rh - e$ begins to become negative. By this manner, the preceding function becomes, by making $l = 1$ in it,

$$\frac{k^n}{1.2.3\ldots n}\Big[(s+e')^n - n(s+e'-h)^n \\ + \frac{n(n-1)}{1.2}(s+e'-2h)^n \\ - \cdots \\ - (s-e)^n + n(s-e-h)^n \\ - \frac{n(n-1)}{1.2}(s-e-2h)^n \\ + \cdots\Big],$$

by rejecting the terms in which the quantity under the sign of the power is negative. This artifice, extended to some arbitrary laws of facilities, gives a general method to determine the probability that the error of any number of observations will be contained in some given limits. [*See* the *Mémoires de l'Académie des Sciences*, year 1778, page 240 and the following.[3]]

In order to determine k, we will make $n = 1$, $s + e' = h$ and $s - e$ null. The preceding formula becomes then kh; but this quantity must be equal to unity, since it is certain that the inclination must fall between zero and h. We have therefore

$$k = \frac{1}{h},$$

that which changes the preceding formula into this one

$$(a) \qquad \left\{ \begin{aligned} &\frac{1}{1.2.3\ldots n.h^n}\Big[(s+e')^n - n(s+e'-h)^n \\ &\qquad + \frac{n(n-1)}{1.2}(s+e'-2h)^n \\ &\qquad - \cdots \\ &\qquad - (s-e)^n + n(s-e-h)^n \\ &\qquad - \frac{n(n-1)}{1.2}(s-e-2h)^n \\ &\qquad + \cdots\Big]. \end{aligned} \right.$$

[3] *Mémoire sur les probabilités*, §VII.

If we make $s+e'=nh$ and $s-e=0$, the probability that the sum of the inclinations will be contained between zero and nh, being certitude or unity, the preceding formula gives

$$n^n - n(n-1)^n + \frac{n(n-1)}{1.2}(n-2)^n - \cdots = 1.2.3\ldots n,$$

that which we know besides.

II.

We apply this formula to the inclinations of the orbits of the planets. The sum of the inclinations of the other orbits to that of the Earth was, in decimal degrees,[4] of $91^{\mathrm{G}}.4187$ at the beginning of 1801. If we make the inclinations vary from zero to the half circumference, we make vanish the consideration of the retrograde movements; because the direct movement changes itself into retrograde when the inclination surpasses a right angle. Thus the preceding formula will give the probability that the sum of the inclinations of the orbits of the ten other planets to the ecliptic will not surpass $91^{\mathrm{G}}.4187$, by making $n=10^{\mathrm{G}}$ in it, $h=200^{\mathrm{G}}$, $s+e'=91^{\mathrm{G}}.4187$, $s-e=0$. We find then this probability equal to $1-\frac{1.0972}{(10)^{10}}$; consequently, the probability that the sum of the inclinations must surpass that which has been observed is equal to $1-\frac{1.0972}{(10)^{10}}$. This probability approaches so to certitude that the observed result becomes unlikely under the supposition where all inclinations are equally possible. This result indicates therefore with a very great probability the existence of a primitive cause which has determined the orbits of the planets to draw nearer to the plane of the ecliptic or, more naturally, to the plane of the solar equator. It is likewise of the sense of the movement of the eleven planets, which is that of the rotation of the Sun. The probability that this ought not to take place is $1-\frac{1}{2^{10}}$. But if we consider that the eighteen satellites observed until now make their revolutions in the same sense as their respective planets, and that the observed rotations, in the number of thirteen in the planets, the satellites and the ring of Saturn, are yet directed in the same sense, we will have $1-\frac{1}{2^{42}}$ for the probability that this ought not take place under the hypothesis of an equal possibility in the direct and retrograde movements. Thus the existence of a common cause which has directed these movements in the sense of the rotation of the Sun is indicated by the observations with an extreme probability.

[4] Also known as Grads. $400^{\mathrm{G}} = 360° = 2\pi$ radians.

We see now if this cause has influenced on the movement of the comets. The number of these which we have observed until 1807 inclusively, by counting for the same the diverse apparitions of the one of 1759, is of ninety-seven, of which fifty-two have a direct movement and forty-five a retrograde movement. The sum of the inclinations of the orbits of the first is $2622^{\mathrm{G}}.944$ and that of the inclination of the orbits of the others is $2490^{\mathrm{G}}.089$. The mean inclination of all these orbits is $51^{\mathrm{G}}.87663$. If in formula (*a*) of the preceding article we suppose $e' = e$ and $s = \frac{1}{2}nh$, it becomes

$$\begin{aligned}\frac{1}{1.2.3\ldots n.2^n}\Big[\left(n+\frac{2e}{h}\right)^n &- n\left(n+\frac{2e}{h}-2\right)^n \\ &+ \frac{n(n-1)}{1.2}\left(n+\frac{2e}{h}-4\right)^n \\ &- \cdots \\ &- \left(n-\frac{2e}{h}\right)^n + n\left(n-\frac{2e}{h}-2\right)^n - \cdots\Big].\end{aligned}$$

In the present case, $n = 97$, $h = 100^{\mathrm{G}}$, $e = 182^{\mathrm{G}}.033$, and then it gives the probability that the sum of the inclinations must be contained within the limits $50^{\mathrm{G}} \pm 1^{\mathrm{G}}.87664$; but the considerable number of terms of this formula and the precision with which it is necessary to have each of them renders the calculation of it impractical. It is therefore indispensable to seek a method of approximation for this kind of analytic expressions or to resolve the problem in another manner. This is that which I have made by the following method.

III.

I imagine the interval h divided into an infinite number $2i$ of parts that I take for unity, and I consider the function

$$e^{-i\varpi\sqrt{-1}} + e^{-(i-1)\varpi\sqrt{-1}} + \cdots + e^{-\varpi\sqrt{-1}} + 1 + e^{\varpi\sqrt{-1}} + \cdots + e^{(i-1)\varpi\sqrt{-1}} + e^{i\varpi\sqrt{-1}},$$

by designating now by e the number of which the hyperbolic logarithm is unity.

By raising it to the power n, the coefficient of $e^{l\varpi\sqrt{-1}}$ of the development of this power will express the number of the combinations in which the sum of the inclinations of the orbits is equal to l. This power can be put under the form

$$(1 + 2\cos\varpi + 2\cos 2\varpi + \cdots + 2\cos i\varpi)^n.$$

By multiplying it by $d\varpi e^{-l\varpi\sqrt{-1}}$, the term multiplied by $e^{l\varpi\sqrt{-1}}$ in the development of the power will become independent of ϖ in the product; whence it is easy to conclude that we will have the coefficient of this term by taking the integral

$$(a') \qquad \frac{1}{\pi}\int d\varpi \cos l\varpi \,(1 + 2\cos\varpi + \cdots 2\cos i\varpi)^n.$$

from ϖ null to $\varpi = \pi$, π being the semi-circumference or 200^{G}, because the terms of the integral dependent on ϖ all becoming null at the same time, and, for the first time, only within these limits.

Now we have

$$1 + 2\cos\varpi + \cdots + 2\cos i\varpi = \frac{\cos i\varpi - \cos(i+1)\varpi}{1-\cos\varpi} = \cos i\varpi + \frac{\cos\frac{1}{2}\varpi \sin i\varpi}{\sin\frac{1}{2}\varpi}$$

Let $i\varpi = t$, we will have

$$\cos i\varpi + \frac{\cos\frac{1}{2}\varpi \sin i\varpi}{\sin\frac{1}{2}\varpi} = \cos t + \frac{\cos\frac{t}{2i}\sin t}{\sin\frac{t}{2i}}.$$

The second member of this equation becomes, because of i infinite,

$$2i\,\frac{\sin t}{t}.$$

Moreover, if we make

$$l = ir\sqrt{n},$$

we will have

$$\cos l\varpi = \cos rt\sqrt{n}.$$

The function (a') becomes therefore

$$(a'') \qquad \frac{(2i)^n}{i\pi}\int dt \cos rt\sqrt{n}\left(\frac{\sin t}{t}\right)^n.$$

We have, by reducing $\sin t$ into series,

$$\log\left(\frac{\sin t}{t}\right)^n = n\log\left(1-\frac{1}{1.2.3}t^2+\frac{1}{1.2.3.4.5}t^4-\cdots\right)$$
$$= -\frac{nt^2}{6}-\frac{n}{180}t^4-\cdots$$

this which gives

$$\left(\frac{\sin t}{t}\right)^n = e^{-\frac{nt^2}{6}-\frac{n}{180}t^4-\cdots} = e^{-\frac{nt^2}{6}}\left(1-\frac{n}{180}t^4-\cdots\right),$$

e being the number of which the hyperbolic logarithm is unity. The function (*a''*) takes then this form

$$(a''') \qquad \frac{(2i)^n}{i\pi}\int dt\cos rt\sqrt{n}\, e^{-\frac{nt^2}{6}}\left(1-\frac{n}{180}t^4-\cdots\right).$$

We consider the different terms of this function. We have first, by reducing into series $\cos rt\sqrt{n}$ and making $t' = t\sqrt{\frac{n}{6}}$,

$$\int dt\cos rt\sqrt{n}\, e^{-\frac{nt^2}{6}} = \sqrt{\frac{6}{n}}\int dt'\, e^{-t'^2}\left(1-\frac{r^2}{1.2}6t'^2+\frac{r^4}{1.2.3.4}6^2t'^4-\cdots\right).$$

The integral must be taken from t null to t infinity, because i being infinite, t or $i\varpi$ become infinite at the limit $\varpi = \pi$; the integral relative to t' must therefore be taken from t' null to t' infinity. In this case, we have, as I have shown in the *Mémoires cités de l'Académie des Sciences* for the year 1778[5],

$$\int dt'\, e^{-t'^2} = \frac{1}{2}\sqrt{\pi}.$$

We have next, by integrating by parts,

$$\int t'^2\, dt'\, e^{-t'^2} = -\frac{1}{2}t'\, e^{-t'^2} + \frac{1}{2}\int dt'\, e^{-t'^2}.$$

By taking the integral from t' null to t' infinity, this second member is reduced to $\frac{1}{2}.\frac{1}{2}\sqrt{\pi}$. Generally we have, in the same limits,

[5] *Mémoire sur les probabilités*, §XXIII.

$$\int t'^{2m}\,dt'\,e^{-t'^2} = \frac{1.3.5.(2m-1)}{2^m}\frac{1}{2}\sqrt{\pi}.$$

We will have therefore

$$\int dt\cos rt\sqrt{n}\,e^{-\frac{nt^2}{6}} = \sqrt{\frac{6}{n}}\,\frac{1}{2}\sqrt{\pi}\left(1-\frac{3}{2}r^2+\frac{\left(\frac{3}{2}\right)^2 r^4}{1.2}-\cdots\right) = \sqrt{\frac{6}{n}}\,\frac{1}{2}\sqrt{\pi}\,e^{-\frac{3}{2}r^2}.$$

The first term of the function (a''') becomes thus

$$\frac{(2i)^n}{2i\sqrt{\pi}}\sqrt{\frac{6}{n}}\;e^{-\frac{3}{2}r^2}.$$

We consider presently the term

$$\int nt^4 dt\,e^{-\frac{t^2n}{6}}\cos rt\sqrt{n}.$$

By integrating by parts, this term becomes

$$-3t^3 e^{-\frac{t^2n}{6}}\cos rt\sqrt{n}+3\int e^{-\frac{t^2n}{6}}\,d(t^3\cos rt\sqrt{n}).$$

But we have

$$3\int e^{-\frac{t^2n}{6}}\,d(t^3\cos rt\sqrt{n}) = 9\int t^2 dt\,e^{-\frac{t^2n}{6}}\cos rt\sqrt{n}+3r\,\frac{d}{dr}\int t^2\,dt\,e^{-\frac{t^2n}{6}}\cos rt\sqrt{n};$$

we have next

$$\int t^2 dt\,e^{-\frac{t^2n}{6}}\cos rt\sqrt{n} = -\frac{3t}{n}e^{-\frac{t^2n}{6}}\cos rt\sqrt{n}+\frac{3}{n}\int e^{-\frac{t^2n}{6}}\,d(t\cos rt\sqrt{n})$$

and

$$\begin{aligned}\int e^{-\frac{t^2n}{6}}\,d(t\cos rt\sqrt{n}) &= \int dt\cos rt\sqrt{n}\,e^{-\frac{t^2n}{6}}+r\frac{d}{dr}\int dt\,e^{-\frac{t^2n}{6}}\cos rt\sqrt{n}\\ &= \sqrt{\frac{6}{n}}\,\frac{1}{2}\sqrt{\pi}\left(e^{-\frac{3}{2}r^2}+r\frac{d}{dr}e^{-\frac{3}{2}r^2}\right).\end{aligned}$$

By reuniting these values and taking the integral from t null to t infinity, we will have

$$\int nt^4\,dt\,e^{-\frac{t^2n}{6}}\cos rt\sqrt{n} = \frac{3^3}{n}\sqrt{\frac{3\pi}{2n}}\,e^{-\frac{3}{2}r^2}(1-6r^2+3r^4).$$

We can obtain easily from this other way the integral

$$\int dt\cos rt\sqrt{n}\,t^{2f}\,e^{-\frac{t^2n}{6}}.$$

For this, we will substitute, in place of $\cos rt\sqrt{n}$, its value $\frac{e^{rt\sqrt{-n}}+e^{-rt\sqrt{-n}}}{2}$. We consider first the integral

$$\frac{1}{2}\int dt\,e^{rt\sqrt{-n}}\,t^{2f}\;e^{-\frac{nt^2}{6}};$$

we put it under this form

$$\frac{1}{2}e^{-\frac{3}{2}r^2}\int dt\,e^{-\frac{1}{6}(t\sqrt{n}-3r\sqrt{-1})^2}t^{2f}.$$

We make

$$\frac{t\sqrt{n}-3r\sqrt{-1}}{\sqrt{6}} = t',$$

this integral will become

$$\frac{1}{2}e^{-\frac{3}{2}r^2}\sqrt{\frac{6}{n}}\int dt'\;e^{-t'^2}\,\frac{(t'\sqrt{6}+3r\sqrt{-1})^{2f}}{n^f}.$$

But it must be taken from $t' = -\frac{3r\sqrt{-1}}{\sqrt{6}}$ to infinity. The part $\frac{1}{2}e^{-rt\sqrt{-n}}$ from $\cos\,rt\sqrt{n}$ will give likewise the integral

$$\frac{1}{2}e^{-\frac{3}{2}r^2}\sqrt{\frac{6}{n}}\int dt'\;e^{-t'^2}\,\frac{(t'\sqrt{6}-3r\sqrt{-1})^{2f}}{n^f},$$

the integral being taken from $t' = \frac{3r\sqrt{-1}}{\sqrt{6}}$ to infinity. Thence it is easy to conclude that the integral $\int dt\,t^{2f}\,\cos rt\sqrt{n}\,e^{-\frac{nt^2}{6}}$ is equal to

$$\frac{1}{2}e^{-\frac{3}{2}r^2}\sqrt{\frac{6}{n}}\int dt'\, e^{-t'^2}\,\frac{(t'\sqrt{6}-3r\sqrt{-1})^{2f}}{n^f},$$

the integral being taken from $t'=-\infty$ to $t'=+\infty$, or, that which reverts to the same, to the real part of the integral

$$e^{-\frac{3}{2}r^2}\sqrt{\frac{6}{n}}\int dt'\, e^{-t'^2}\,\frac{(t'\sqrt{6}+3r\sqrt{-1})^{2f}}{n^f},$$

the integral being taken from t' null to t' infinity. By making $2f=4$, we have

$$\int dt\, t^4 \cos rt\sqrt{n}\, e^{-\frac{nt^2}{6}} = \frac{e^{-\frac{3}{2}r^2}}{n^2\sqrt{n}} 3^3(1-6^2+3r^4)\sqrt{\frac{3}{2}\pi},$$

this which coincides with the preceding result.

The function (a''') will be thus reduced into the series descending according to the powers of n,

$$\frac{(2i)^n}{2i\sqrt{\pi}}\sqrt{\frac{6}{n}}\, e^{-\frac{3}{2}r^2}\left[1-\frac{3}{20n}(1-6r^2+r^4)-\cdots\right].$$

We will have the sum of all the functions contained between $-l$ and l, by observing that 1 is the differential of l; now this differential is $idr\sqrt{n}$; we can therefore substitute $dr\sqrt{n}$ in the place of $\frac{1}{i}$. The sum of all the functions of which there is question is thus, by doubling the integral,

$$(2i)^n\sqrt{\frac{6}{\pi}}\int dr\, e^{-\frac{3}{2}r^2}\left[1-\frac{3}{20n}(1-6r^2+3r^4)+\cdots\right].$$

In order to have the probability that the sum of the inclinations will be contained between $-l$ and l, it is necessary to divide the preceding function by the number of all the possible combinations, and this number is $(2i)^n$. We have therefore, for this probability,

$$\begin{aligned}\sqrt{\frac{6}{\pi}}\int dr\, e^{-\frac{3}{2}r^2}&\left[1-\frac{3}{20n}(1-6r^2+3r^4)+\cdots\right]\\ &=\sqrt{\frac{6}{\pi}}\left[\int dr\, e^{-\frac{3}{2}r^2}-\frac{3r}{20n}(1-r^2)e^{-\frac{3}{2}r^2}\right].\end{aligned}$$

But we have

$$2i = h, \qquad \frac{l}{i} = r\sqrt{n};$$

the limits of the integral are therefore $-\frac{h}{2} r\sqrt{n}$ and $+\frac{h}{2}\sqrt{n}$; consequently the probability that the mean inclination of the orbits will be contained within the limits $\frac{1}{2}h - \frac{rh}{2\sqrt{n}}$ and $\frac{1}{2}h + \frac{rh}{2\sqrt{n}}$, will be expressed by the preceding integral.

If we make $\frac{3}{2}r^2 = s^2$, this integral becomes

$$\frac{2}{\sqrt{\pi}} \int ds\, e^{-s^2} \left[1 - \frac{3}{20n}(1 - 4s^2 + \frac{4}{3}s^4) + \cdots \right]$$

or

$$(a^{iv}) \qquad \frac{2}{\sqrt{\pi}} \left[\int ds\, e^{-s^2} - \frac{1}{20n} e^{-s^2} (3s - 2s^3) + \cdots \right].$$

When the value of s to its limit is quite great, then $\int ds\, e^{-s^2}$ approaches $\frac{1}{2}\sqrt{\pi}$, in a way to differ from it less than any given magnitude, if we increase indefinitely the number n; moreover, the terms following $-\frac{1}{20n} e^{-s^2}(3s - 2s^3)$ become then entirely insensibles. We can therefore, by the increase of n, tighten at the same time the limits $\pm \frac{h}{2\sqrt{n}}$ and to increase at the same time the probability that the mean inclination of the orbits will fall between the limits $\frac{1}{2}h \pm \frac{rh}{2\sqrt{n}}$, in a way that the difference from certitude to this probability and the interval contained between these limits are less than every assignable magnitude.

When s is quite small, we have, by a convergent series,

$$\int ds\, e^{-s^2} = s - \frac{1}{1.2}\frac{s^3}{3} + \frac{1}{1.2.3}\frac{s^5}{5} - \cdots$$

This series can be employed when s does not surpass $\frac{3}{2}$; but when it surpasses it, we can make use of the continued fraction that I have given in Book X of the *Mécanique céleste*,

$$\int ds\, e^{-s^2} = \frac{1}{2}\sqrt{\pi} - \frac{e^{-s^2}}{2s} \left\{ \cfrac{1}{1 + \cfrac{q}{1 + \cfrac{2q}{1 + \cfrac{3q}{1 + \cfrac{4q}{1 + \ddots}}}}} \right\}$$

q being equal to $\frac{1}{2s^2}$. The continued fraction $\frac{1}{1+\frac{q}{1+\ddots}}$ is reduced, according as we stop it at the first, at the second, … terms in the following fractions, alternatively greater and smaller than the continued fraction:

$$\frac{1}{1}, \quad \frac{1}{1+q}, \quad \frac{1+2q}{1+3q}, \quad \frac{1+5q}{1+6q+3q^2}, \quad \frac{1+9q+8q^2}{1+10q+15q^2}, \quad \ldots$$

The numerators of these fractions are deduced the one from the others, by observing that the numerator of the i^{th} fraction is equal to the numerator of the $(i-1)^{\text{st}}$ fraction, plus to the numerator of the $(i-2)^{\text{nd}}$ multiplied by $(i-1)q$. The denominators are deduced the one from the others in the same manner.

IV.

We can now apply our formulas to the observed comets, by making use of the givens of article II. We have, after these givens,

$$n = 97, \qquad h = 100^{\text{G}},$$

$$\frac{rh}{2\sqrt{n}} = 1^{\text{G}}.87763,$$

this which gives

$$s = \frac{1^{\text{G}}.87763}{100^{\text{G}}}\, 2\sqrt{97}\sqrt{\frac{3}{2}} = 0.452731.$$

We can here make use of the expression of the integral $\int ds\, e^{-s^2}$ in series, and then we have

$$\frac{2}{\sqrt{\pi}}\int ds\, e^{-s^2} = 0.4941.$$

The probability that the mean inclination must be contained within the limits $50^{\text{G}} \pm 1^{\text{G}}.87663$ is, by formula (a^{iv}), equal to 0.4933, or $\frac{1}{2}$ very nearly; the probability that this inclination must be below is therefore $\frac{1}{4}$, and the probability that it must be above is $\frac{1}{4}$. All these probabilities are too little different from $\frac{1}{2}$ in order that the observed result throws out the hypothesis of an equal facility of the inclinations of the orbits, and so to indicate the existence of a primitive cause which has influence on these inclinations, a cause that we can not refrain from admitting in the inclinations of these planets.

The same thing holds with respect to the sense of the movement. The probability that, on ninety-seven comets, forty-five at most are retrogrades, is the sum of the forty-six first terms of the binomial $(p+q)^{97}$, by making $p = q = \frac{1}{2}$; but the sum of the forty-eight first terms is the half of the binomial or $\frac{1}{2}$; whence it is easy to conclude that the sought probability is

$$\frac{1}{2} - \frac{97.96\ldots 50}{1.2.3\ldots 48.2^{97}}\left(1+\frac{48}{50}+\frac{48.47}{50.51}\right).$$

Now we have

$$\frac{97.96\ldots 50}{1.2.3\ldots 48.2^{97}} = \frac{1.2.3\ldots 97}{(1.2.3\ldots 49)^2}\,\frac{49}{2^{97}};$$

moreover, we have generally, when s is a great number,

$$1.2.3\ldots s = s^{s+\frac{1}{2}}\,e^{-s}\sqrt{2\pi}\left(1+\frac{1}{12s}+\cdots\right),$$

this which gives

$$\frac{1.2.3\ldots 97}{(1.2.3\ldots 49)^2}\,\frac{49}{2^{97}} = \frac{\left(\frac{48.5}{49}\right)^{98}\frac{1165}{1164}e}{\sqrt{\pi\cdot 48.5}\left(\frac{589}{588}\right)^2}.$$

We find thus the sought probability equal to 0.2713, a fraction much too great in order that it can indicate a cause which has favored, in the origin, the direct movements. Thus the cause which has determined the sense of the movements of rotation and of revolution of the planets and of the satellites do not appear to have influenced on the movement of the comets.

V.

If we neglect the terms of order $\frac{1}{n}$, the integral $\frac{2}{\sqrt{\pi}}\int ds\,e^{-s^2}$ or $\frac{2}{\sqrt{\pi}}\sqrt{\frac{3}{2}}\int dr\;e^{-\frac{3}{2}r^2}$ expresses the probability that the sum of the inclinations of the orbits will be contained within the limits $\frac{h}{2} - \frac{rh}{2\sqrt{n}}$ and $\frac{h}{2} + \frac{rh}{2\sqrt{n}}$; but this same probability is, by article II, equal to

$$\begin{aligned}\frac{1}{1.2.3\ldots n.2^n}\Big[(n+r\sqrt{n})^n - n(n+r\sqrt{n}-2)^n \\ + \frac{n(n-1)}{1.2}(n+r\sqrt{n}-4)^n - \cdots \\ - (n-r\sqrt{n})^n + n(n-r\sqrt{n}-2)^n \\ - \cdots\Big];\end{aligned}$$

this function is therefore equal to the preceding integral. Now we have, without the exclusion of the negative quantities elevated to the power n in the first member, the following equation:

$$\begin{aligned}&(n+r\sqrt{n})^n - n(n+r\sqrt{n}-2)^n + \frac{n(n-1)}{1.2}(n+r\sqrt{n}-4)^n - \cdots \\ &\quad = (n+r\sqrt{n})^n - n(n+r\sqrt{n}-2)^n + \cdots \\ &\quad + (n-r\sqrt{n})^n - n(n-r\sqrt{n}-2)^n + \cdots\end{aligned}$$

The first member is, as we know, equal to $1.2.3\ldots n.2^n;$ the second expression of the probability becomes thus, by eliminating $(n-r\sqrt{n})^n - n(n-r\sqrt{n}-2)^n + \cdots$ by means of its value given by the preceding equation,

$$\frac{1}{1.2.3\ldots n.2^{n-1}}\Big[(n+r\sqrt{n})^n - n(n+r\sqrt{n}-2)^n + \cdots - 1.2.3\ldots n.2^{n-1}\Big];$$

be equating it to the integral which expresses the same probability, we will have this remarkable equation

$$(b)\left\{\begin{aligned}\frac{1}{1.2.3\ldots n.2^n}\Big[(n+r\sqrt{n})^n - n(n+r\sqrt{n}-2)^n \\ + \frac{n(n-1)}{1.2}(n+r\sqrt{n}-4)^n - \cdots - 1.2.3\ldots n.2^{n-1} \\ = \sqrt{\frac{3}{2\pi}}\int dr\, e^{-\frac{3}{2}r^2}\end{aligned}\right.$$

If, instead of eliminating $(n-r\sqrt{n})^n - n(n-r\sqrt{n}-2)^n + \cdots,$ we eliminated $(n+r\sqrt{n})^n - n(n+r\sqrt{n}-2)^n + \cdots,$ we would have an equation which would coincide with the preceding, by making r negative in it; thus this equation holds, r being positive or negative, the integral must begin with r, and the series of differences must stop when the quantity elevated to the power n becomes negative.

Equation (*b*) differentiated with respect to r gives

$$\frac{\sqrt{n}}{1.2.3\ldots(n-1).2^n}[(n+r\sqrt{n})^{n-1}-n(n+r\sqrt{n}-2)^{n-1}+\cdots]=\sqrt{\frac{3}{2\pi}}e^{-\frac{3}{2}r^2};$$

by differentiating again, we will have

$$\frac{n}{1.2.3\ldots(n-2).2^n}[(n+r\sqrt{n})^{n-2}-n(n+r\sqrt{n}-2)^{n-2}+\cdots]=-3r\sqrt{\frac{3}{2\pi}}dr\,e^{-\frac{3}{2}r^2}.$$

By continuing to differentiate thus, we will have in a very approximate manner the values of the successive differentials of the first member of equation (*b*), provided that the number of these differentiations are very small relative to the number n. All these equations hold, r being positive or negative; and when r is null, they become

$$\begin{aligned}
&\frac{\sqrt{n}}{1.2.3\ldots(n-1)2^n}\left[n^{n-1}-n(n-2)^{n-1}+\frac{n(n-1)}{1.2}(n-4)^{n-1}-\cdots\right]=\sqrt{\frac{3}{2\pi}},\\
&\frac{n}{1.2.3\ldots(n-2)2^n}\left[n^{n-2}-n(n-2)^{n-2}+\cdots\right]=0,\\
&\frac{n\sqrt{n}}{1.2.3\ldots(n-3)2^n}\left[n^{n-3}-n(n-2)^{n-3}+\cdots\right]=-3\sqrt{\frac{3}{2\pi}},\\
&\frac{n^2}{1.2.3\ldots(n-2)2^n}\left[n^{n-4}-n(n-2)^{n-4}+\cdots\right]=0,\\
&\ldots
\end{aligned}$$

The second members of these equations are zero, when the exponent of the power is of the form $n-2s$, this which is easy to see besides, by observing that

$$n^{n-2s}-n(n-2)^{n-2s}+\cdots$$

is the half of the series $n^{n-2s}-n(n-2)^{n-2s}+\cdots$, without the exclusion of the negative quantities elevated to the power $n-2s$, a series which, being the n^{th} finite difference of a power less than n, is null.

We can, by integrating successively equation (*b*), obtain some analogous theorems on the differences of the powers superior to n; thus we have by a first integration

$$(b')\left\{\begin{aligned}&\frac{1}{1.2.3\ldots(n+1)\sqrt{n}.2^n}[(n+r\sqrt{n})^{n+1}-n(n+r\sqrt{n}-2)^{n+1}+\cdots-N_n]\\&\qquad\qquad=\frac{1}{2}r+\sqrt{\frac{3}{2\pi}}\int\int dr^2\,e^{-\frac{3}{2}r^2},\end{aligned}\right.$$

the integrals beginning with r, and N_n being equal to

$$n^{n+1}-n(n-2)^{n+1}+\cdots$$

In order to determine this function, we will observe that we have

$$\begin{aligned}&n^{n+1}-n(n-2)^{n+1}+\cdots\\&\quad=n\Big[n^n-n(n-2)^n+\frac{n(n-1)}{1.2}(n-4)^n-\cdots\Big]\\&\quad+2n[(n-1+r'\sqrt{n-1})^n-(n-1)(n-1+r'\sqrt{n-1}-2)^n+\cdots]\end{aligned}$$

by making $r'\sqrt{n-1}=-1$. We have next

$$n^n-n(n-2)^n+\cdots=1.2.3\ldots n.2^{n-1},$$

because the first member of this equation is the half of the series of the differences, without the exclusion of the negative quantities elevated to the power n. Moreover, if we change, in equation (b'), n into $n-1$, r into r', and if we suppose in it next $r'=-\frac{1}{\sqrt{n-1}}$, we will have very nearly

$$\begin{aligned}&(n-1+r'\sqrt{n-1})^n-(n-1)(n-1+r'\sqrt{n-1}-2)^n+\cdots\\&=N_{n-1}+1.2.3\ldots n\sqrt{n-1}.2^{n-1}\Big[-\frac{1}{2\sqrt{n-1}}+\frac{1}{2(n-1)}\sqrt{\frac{3}{2\pi}}\Big]\end{aligned}$$

We will have therefore

$$N_n=2nN_{n-1}+1.2.3\ldots n.2^{n-1}\sqrt{\frac{3}{2\pi}}\,\frac{n}{\sqrt{n-1}};$$

if we make

$$N_n=1.2.3\ldots n.2^n\delta_n,$$

we will have

$$\delta_n - \delta_{n-1} = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\,\frac{n}{\sqrt{n-1}},$$

this which gives very nearly, by integrating,

$$\delta_n = \sqrt{\frac{3}{2\pi}}\,\frac{1}{3}(n+1)\sqrt{n};$$

it is easy to see that we can neglect here the arbitrary constant. Therefore

$$N_n = 1.2.3\ldots(n+1)2^n\sqrt{n}\,\frac{1}{3}\sqrt{\frac{3}{2\pi}},$$

hence

$$\begin{aligned}\frac{1}{1.2.3\ldots(n+1)2^n\sqrt{n}}&[(n+r\sqrt{n})^{n+1} - n(n+r\sqrt{n}-2)^{n+1}+\cdots]\\ &= \frac{1}{3}\sqrt{\frac{3}{2\pi}} + \frac{1}{2}r + \sqrt{\frac{3}{2\pi}}\int\!\!\int dr^2\, e^{-\frac{3}{2}r^2}.\end{aligned}$$

By integrating anew, we have

$$\begin{aligned}\frac{1}{1.2.3\ldots(n+2)2^n n}&[(n+r\sqrt{n})^{n+2} - n(n+r\sqrt{n}-2)^{n+2}+\cdots\\ &\qquad - n^{n+2} + n(n-2)^{n+2} - \cdots]\\ &= \sqrt{\frac{3}{2\pi}}\,\frac{r}{3} + \frac{1}{4}r^2 + \sqrt{\frac{3}{2\pi}}\int\!\!\int\!\!\int dr^3\, e^{-\frac{3}{2}r^2}.\end{aligned}$$

all the integrals must begin with r. But we have

$$n^{n+2} - n(n-2)^{n+2} + \cdots = 1.2.3\ldots(n+2)2^{n-1}\frac{n}{6}.$$

In effect, we have, as we know,

$$\Delta^n u = \left(e^{\alpha\frac{du}{dx}} - 1\right)^n$$

by applying to the characteristic d the exponents of the powers of $\frac{du}{dx}$, in the development of the second member of this equation, and α being the variation of x. If we make $u = x^{n+2}$, we will have, without exclusion of the powers of the negative quantities,

$$(x+2n)^{n+2} - n(x+2n-\alpha)^{n+2} + \cdots$$
$$= 1.2.3\ldots(n+2)\alpha^n\left(\frac{x^2}{2} + \frac{\alpha n x}{2} + \frac{\alpha^2 n^2}{8} + \frac{\alpha^2 n}{3.8}\right),$$

this which gives without this exclusion, and making $x = -n$ and $\alpha = 2$,

$$n^{n+2} - n(n-2)^{n+2} + \cdots = 1.2.3\ldots(n+2)2^n\frac{n}{6},$$

and, with the exclusion of the powers of the negative quantities,

$$n^{n+2} - n(n-2)^{n+2} + \cdots = 1.2.3\ldots(n+2)2^{n-1}\frac{n}{6};$$

we have therefore

$$\frac{1}{1.2.3\ldots(n+2)2^n n}[(n+r\sqrt{n})^{n+2} - n(n+r\sqrt{n}-2)^{n+2} + \cdots]$$
$$= \frac{1}{12} + \sqrt{\frac{3}{2\pi}}\,\frac{r}{3} + \frac{1}{4}r^2 + \sqrt{\frac{3}{2\pi}}\int\!\!\int\!\!\int dr^3\, e^{-\frac{3}{2}r^2},$$

and thus in sequence.

VI.

The problem which we have resolved in article I, relatively to the inclinations, is the same as the one in which we proposed to determine the probability that the mean error of a number n of observations will be contained within some given limits, by supposing that the errors of each observation can equally be extended within the interval h. We are going to consider now the general case in which the facilities of the errors follow any law.

We divide the interval h, into an infinite number of parts $i+i'$, the negative errors can be extended from zero to $-i$, and the positive errors from zero to i'. For each point of the interval h, we elevate some ordinates which express the facilities of the corresponding errors; we name q the number of the parts contained from the ordinate relative to the error zero to the ordinate of the center of gravity of the area of the curve formed by these ordinates. This put, we represent by $\phi\left(\frac{s}{i+i'}\right)$ the probability of the error s for each observation, and we consider the function

$$\phi\left(\frac{-i}{i+i'}\right)e^{-i\varpi\sqrt{-1}} + \phi\left(\frac{-(i-1)}{i+i'}\right)e^{-(i-1)\varpi\sqrt{-1}} + \cdots$$
$$+\,\phi\left(\frac{0}{i+i'}\right) + \cdots + \phi\left(\frac{i'-1}{i+i'}\right)e^{(i'-1)\varpi\sqrt{-1}} + \phi\left(\frac{i'}{i+i'}\right)e^{i'\varpi\sqrt{-1}}.$$

By raising this function to the power n, the coefficient of $e^{r\varpi\sqrt{-1}}$ in the development of this power will be the probability that the sum of the errors of n observations will be r, whence it follows that, by multiplying the preceding function by $e^{-q\varpi\sqrt{-1}}$ and elevating the product to the power n, the coefficient of $e^{r\varpi\sqrt{-1}}$ in the development of this product will be the probability that the sum of the errors will be $r+nq$. This product is

(o) $$\left[\sum\phi\left(\frac{r}{i+i'}\right)e^{(r-q)\varpi\sqrt{-1}}\right]^n.$$

the sign $\sum$ must extend from $r=-i$ to $r=i'$. If we make

$$\frac{r}{i+i'} = \frac{x}{h}, \quad \frac{q}{i+i'} = \frac{q'}{h}, \quad \frac{1}{i+i'} = \frac{dx}{h}$$

the function (o) becomes, by reducing the exponentials into series,

$$\frac{(i+i')^n}{h^n}\Big[\int\phi\Big(\frac{x}{h}\Big)dx + (i+i')\varpi\sqrt{-1}\int\frac{x-q'}{h}\phi\Big(\frac{x}{h}\Big)dx$$
$$-\frac{(i+i')^2}{1.2}\varpi^2\int\frac{(x-q')^2}{h^2}\phi\Big(\frac{x}{h}\Big)dx + \cdots\Big]^n;$$

x is the abscissa of which the ordinate is $\phi\big(\frac{x}{h}\big)$, the origin of the abscissas corresponding to the ordinate relative to the error zero; q' is the abscissa corresponding to the ordinate of the center of gravity of the area of the curve; the integrals must be taken from $x=\frac{-ih}{i+i'}$ to $x=\frac{i'h}{i+i'}$. We have, by the nature of the center of gravity of the curve,

$$\int\frac{x-q'}{h}\phi\Big(\frac{x}{h}\Big)dx = 0;$$

by making

$$k = \int \phi\Big(\frac{x}{h}\Big)dx, \qquad k' = \int \frac{(x-q')^2}{h^2}\phi\Big(\frac{x}{h}\Big)dx, \qquad \ldots,$$

the preceding function becomes thus

$$\frac{(i+i')^n}{h^n}k^n\left[1 - \frac{k'}{2k}(i+i')^2\varpi^2 + \cdots\right]^n.$$

If, conformably to the analysis of article IV, we multiply the function (o) by $2\cos l\varpi$, the term independent of ϖ in the product will express the probability that the sum of the errors will be either $nq - l$ or $nq + l$; by multiplying this product by $d\varpi$, and integrating from ϖ null to $\varpi = \pi$, the integral divided by π will express this probability which will be thus, by rejecting the odd powers of ϖ, which are multiplied by $\sqrt{-1}$ and which result from the development of the sine of ϖ and from its multiples in the function (o),

$$(o') \qquad \frac{(i+i')^n k^n}{h^n}\frac{2}{\pi}\int \cos l\varpi\left[1 - \frac{k'}{2k}(i+i')^2\varpi^2 + \cdots\right]^n d\varpi.$$

Let presently

$$(i+i')\varpi = t;$$

we will have

$$\log\left[1 - \frac{k'}{2k}(i+i')^2\varpi^2 + \cdots\right]^n = n\log\left(1 - \frac{k'}{2k}t^2 + \cdots\right) = n\left(-\frac{k'}{2k}t^2 + \cdots\right),$$

this which gives for $\left[1 - \frac{k'}{2k}(i+i')^2\varpi^2 + \cdots\right]^n$ an expression of this form

$$e^{-\frac{k'}{2k}nt^2}(1 + Ant^4 + \cdots).$$

The function (o') will become therefore

$$(o'') \qquad \frac{(i+i')^n k^n}{h^n}\frac{2}{\pi}\frac{1}{i+i'}\int \cos\frac{lt}{i+i'}dt\, e^{-\frac{k'}{2k}nt^2}(1 + Ant^4 + \cdots).$$

The error of each observation must necessarily fall with the interval h, we have

$$\frac{(i+i')k}{h} = 1.$$

Let $\frac{l}{i+i'} = r\sqrt{n}$; the preceding expression will become, by having regard only to its first term,

$$\frac{2}{\pi(i+i')}\int \cos rt\sqrt{n}\, e^{-\frac{k'}{2k}nt^2}\, dt,$$

this which, by integrating from t null to t infinity, becomes, by the analysis of article III,

$$\frac{2}{(i+i')\sqrt{\pi}}\sqrt{\frac{k}{2k'n}}\, e^{-\frac{k}{2k'}r^2}.$$

If we multiply this function by dl, by integrating it we will have the probability that the sum of the errors will be contained within the limits $nq \pm l$ or $nq \pm (i+i')r\sqrt{n};$ now we have

$$dl = (i+i')dr\sqrt{n};$$

this probability will be therefore

$$\frac{2}{\sqrt{\pi}}\sqrt{\frac{k}{2k'}}\int e^{-\frac{k}{2k'}r^2}dr.$$

$i+i'$ being equal to h, and q' can be substituted for q, the preceding limits will become

$$nq' \pm hr\sqrt{n},$$

and those of the mean of the errors will be

$$q' \pm \frac{rh}{\sqrt{n}}.$$

In the case that we have considered in article III, q' is null; $k = h$, $k' = \frac{1}{12}h$; the preceding expression becomes, by making $r = \frac{1}{2}r'$ in it,

$$\frac{2}{\sqrt{\pi}}\sqrt{\frac{3}{2}}\int e^{-\frac{3}{2}r'^2}dr',$$

and the limits of the mean of the errors is $\pm\frac{\frac{1}{2}r'h}{\sqrt{n}}$: this which is conformed to the article cited.

In general, q' is null when the curve of the facilities of errors is symmetric on each side of the ordinate corresponding to the error zero. If the law of the facilities is represented by $A(\frac{1}{4}h^2 - x^2),$ we will have

$$k = \frac{A}{6}h^2, \qquad 2k' = \frac{A}{60}h^2$$

and, consequently,

$$\frac{k}{2k'} = 10;$$

thus the probability that the mean error of the observations will be contained within the limits $\pm \frac{rh}{\sqrt{n}}$ will be

$$\frac{2}{\sqrt{\pi}}\sqrt{10}\int e^{-10r^2}dr.$$

By applying in this case the method of article I, we will have the expression of the same probability by a series of a very great number of terms, analogous to that of the finite differences, by which we have determined the probability in the case of an equal facility of errors. But this new series, which we have given in the *Mémoires* cited from *l'Académie des Sciences* for the year 1778, page 249,[6] is too complicated to offer for the sake of its comparison with the preceding expression of the probability of the results which can interest the geometers.

In the case where the errors can be extended to infinity, the preceding analysis gives yet the probability that the mean error of a very great number of observations will be closed up within the given limits. In order to see how we can now apply this analysis, we suppose that $e^{-\frac{x}{p}}$ is the expression of the facility of errors, the exponent of e must always be negative and the same as regards the equal errors positives and negatives. By supposing the errors positives, we will have

$$\int e^{-\frac{x}{p}}dx = p\left(1 - e^{-\frac{h}{2p}}\right),$$

by taking the integral from x null to $x = \frac{1}{2}h$. In order to have the entire value of k, it is necessary to double this quantity, because the negative errors give a quantity equal to the preceding; by supposing therefore h great enough in order

[6] *Mémoire sur les probabilités*, §IX.

that $e^{-\frac{h}{2p}}$ disappears beyond unity, this which holds exactly in the case h infinite, we will have very nearly

$$k = 2p.$$

We will find in the same manner, by taking the integral $\int \frac{x^2 dx}{h^2} e^{-\frac{x}{p}}$,

$$k' = \frac{4p^2}{h^2};$$

thus

$$\frac{k}{2k'} = \frac{h^2}{4p^2}.$$

The probability that the mean error will be contained within the limits $\pm \frac{rh}{\sqrt{n}}$ will be therefore

$$\frac{2}{\sqrt{\pi}} \frac{h}{2p} \int e^{-\frac{h^2}{4p^2} r^2} dr.$$

Let $rh = r'p$, the limits become $\pm \frac{r'p}{\sqrt{n}}$, and the probability that the mean error will be contained within the limits becomes

$$\frac{1}{\sqrt{\pi}} \int e^{-\frac{1}{4} r'^2} dr';$$

then the consideration of h supposed infinite disappears.

VII.

The manner which has led us to equation (*b*) of article V leaves room to improve a direct method in order to arrive to it; its research is the object of the following analysis.

We designate by $\phi(r, n)$ the second member of this equation which it is the question to determine; by differentiating it with respect to r, it will give

$$\frac{1}{1.2.3\ldots(n-1)2^n} [(n + r\sqrt{n})^{n-1} - n(n + r\sqrt{n} - 2)^{n-1} + \cdots] = \frac{1}{\sqrt{n}} \phi'(r, n),$$

$\phi'(r, n)$, $\phi''(r, n)$, $\ldots$ designating the successive differences of $\phi(r, n)$, divided by the corresponding powers of dr; but we have

$$
\begin{aligned}
&(n+r\sqrt{n})^{n-1} - n(n+r\sqrt{n}-2)^{n-1} + \frac{n(n-1)}{1.2}(n+r\sqrt{n}-4)^{n-1} - \cdots \\
&\quad = (n-1+r'\sqrt{n-1})^{n-1} - (n-1)(n-1+r'\sqrt{n-1}-2)^{n-1} + \cdots \\
&\qquad - (n-1+r''\sqrt{n-1})^{n-1} + (n-1)(n-1+r''\sqrt{n-1}-2)^{n-1} + \cdots
\end{aligned}
$$

by making

$$
r'\sqrt{n-1} = r\sqrt{n}+1, \qquad r''\sqrt{n-1} = r\sqrt{n}-1.
$$

Equation (*b*) gives, by changing n into $n-1$ in it,

$$
\begin{aligned}
&\frac{1}{1.2.3\ldots(n-1)2^{n-1}}[(n-1+r'\sqrt{n-1})^{n-1} - (n-1)(n-1+r'\sqrt{n-1}-2)^{n-1} + \cdots \\
&\qquad - (n-1+r''\sqrt{n-1})^{n-1} + (n-1)(n-1+r''\sqrt{n-1}-2)^{n-1} - \cdots] \\
&= \phi(r', n-1) - \phi(r'', n-1);
\end{aligned}
$$

we have therefore this equation in the finite and infinitely small partial differences

$$
(p) \qquad \phi(r', n-1) - \phi(r'', n-1) = \frac{2}{\sqrt{n}}\phi'(r,n).
$$

We can obtain a second equation in this manner; we have

$$
\begin{aligned}
&(n+r\sqrt{n})^{n} - n(n+r\sqrt{n}-2)^{n} + \frac{n(n-1)}{1.2}(n+r\sqrt{n}-4)^{n} - \cdots \\
&\quad = (n+r\sqrt{n})[(n+r\sqrt{n})^{n-1} - n(n+r\sqrt{n}-2)^{n-1} + \cdots] \\
&\qquad + 2n[(n+r\sqrt{n}-2)^{n-1} - (n-1)(n+r\sqrt{n}-4)^{n-1} + \cdots].
\end{aligned}
$$

Equation (*b*) gives, by differentiating it with respect to r,

$$
\begin{aligned}
&\frac{n+r\sqrt{n}}{1.2.3\ldots n.2^{n}}[(n+r\sqrt{n})^{n-1} - n(n+r\sqrt{n}-2)^{n-1} + \cdots] \\
&\qquad = \left(\frac{1}{\sqrt{n}} + \frac{r}{n}\right)\phi'(r,n),
\end{aligned}
$$

and the same equation gives, by changing in it as above n into $n-1$ and r into r'',

$$\frac{2n}{1.2.3\ldots n.2^{n-1}}[(n-1+r''\sqrt{n-1})^{n-1}-(n-1)(n-1+r''\sqrt{n-1}-2)^{n-1}+\cdots]$$
$$=\phi(r'',n-1)+\frac{1}{2};$$

therefore

$$\frac{1}{1.2.3\ldots n.2^{n}}[(n+r\sqrt{n})^{n}-n(n+r\sqrt{n}-2)^{n}+\cdots]$$
$$=\left(\frac{1}{\sqrt{n}}+\frac{r}{n}\right)\phi'(r,n)+\phi(r'',n-1)+\frac{1}{2};$$

substituting this value into equation (*b*), we will have

$$\left(\frac{1}{\sqrt{n}}+\frac{r}{n}\right)\phi'(r,n)+\phi(r'',n-1)=\phi(r,n). \tag{q}$$

This equation, combined with equation (*p*), gives

$$\left(-\frac{1}{\sqrt{n}}+\frac{r}{n}\right)\phi'(r,n)+\phi(r',n-1)=\phi(r,n). \tag{q'}$$

We have

$$r'=r\left(1+\frac{1}{2n}+\frac{3}{8n^2}+\cdots\right)+\frac{1}{\sqrt{n}}\left(1+\frac{1}{2n}+\frac{3}{8n^2}+\cdots\right),$$
$$r''=r\left(1+\frac{1}{2n}+\frac{3}{8n^2}+\cdots\right)-\frac{1}{\sqrt{n}}\left(1+\frac{1}{2n}+\frac{3}{8n^2}+\cdots\right).$$

By substituting these values into equations (*q*) and (*q*′) and by developing in series the functions $\phi(r',n-1)$ and $\phi(r'',n-1)$, we see that these two equations differ only in this that the terms affected of $\frac{1}{\sqrt{n}}$ have contrary signs; we can therefore equate separately to zero the terms of the development of equation (*q*) which have at no point $\sqrt{n}$ for divisor; and then we have an equation of this form

$$
(r) \qquad \begin{cases} \dfrac{1}{2n}[3r\phi'(r,n-1)+\phi''(r,n-1)] \\ \quad = \phi(r,n)-\phi(r,n-1)-\dfrac{r}{n}[\phi'(r,n)-\phi'(r,n-1)] \\ \qquad + \dfrac{M}{n^2}+\dfrac{M'}{n^3}+\cdots, \end{cases}
$$

M, M', $\ldots$ being some rational and entire functions of r, multiplied by the differentials of $\phi(r,n-1)$, and that it is easy to form. We find thus

$$
\begin{aligned} M = -\frac{3}{8}\phi'(r,n-1) - \frac{4+r^2}{8}\phi''(r,n-1) \\ -\frac{r}{4}\phi'''(r,n-1)-\frac{1}{24}\phi^{iv}(r,n-1). \end{aligned}
$$

Equation (p) gives, by integrating it and designating by $\phi_{\prime}(r,n)$ the integral $\int dr\ \phi(r,n)$, beginning with r, and observing that $dr' = dr'' = \frac{dr\sqrt{n}}{\sqrt{n-1}}$,

$$
\phi_{\prime}(r',n-1)-\phi_{\prime}(r'',n-1) = \frac{2}{\sqrt{n-1}}\phi(r,n).
$$

By substituting for r' and r'' their preceding values and developing in series the functions $\phi_{\prime}(r',n-1)$ and $\phi_{\prime}(r'',n-1)$, we have an equation of this form

$$
(r') \qquad \begin{cases} \dfrac{1}{\sqrt{n-1}}[\phi(r,n)-\phi(r,n-1)] \\ \quad = \dfrac{1}{6n\sqrt{n-1}}[3r\phi'(r,n-1)+\phi''(r,n-1)] \\ \qquad + \dfrac{N}{n^2\sqrt{n-1}}+\dfrac{N'}{n^3\sqrt{n-1}}+\cdots, \end{cases}
$$

N, N', $\ldots$ being some functions of the same nature as M, M', $\ldots$ and which it is easy to form in the same manner; we find thus

$$
\begin{aligned} N = \frac{3r}{8}\phi'(r,n-1)+\frac{4+3r^2}{24}\phi''(r,n-1) \\ +\frac{r}{12}\phi'''(r,n-1)+\frac{1}{120}\phi^{iv}(r,n-1). \end{aligned}
$$

If we substitute into equation (r), instead of $\phi(r,n)-\phi(r,n-1)$, its value given by equation (r'), we will have

$$(s)\qquad \begin{cases} \dfrac{1}{3n}[3r\phi'(r,n-1)+\phi''(r,n-1)] \\ \quad = -\dfrac{r}{6n^2}\,\dfrac{d}{dr}[3r\phi(r,n-1)+\phi''(r,n-1)] \\ \qquad + \dfrac{M+N}{n^2}+\dfrac{M'+N'}{n^3}+\cdots-\dfrac{r}{n^3}\,\dfrac{dN}{dr}-\dfrac{r}{n^4}\,\dfrac{dN'}{dr'}-\cdots, \end{cases}$$

In order to integrate this equation, we suppose

$$\phi(r,n-1)=\Psi(r)+\frac{\Pi(r)}{n}+\frac{\Gamma(r)}{n^2}+\cdots;$$

by substituting this expression into the preceding equation, and comparing the coefficients of the descending powers of n, we will have the following equations

$$0=3r\Psi'(r)+\Psi''(r),$$
$$3r\Pi'(r)+\Pi''(r)=-\frac{1}{2}r\,\frac{d}{dr}[3r\Psi'(r)+\Psi''(r)]+3(\overline{M}+\overline{N}),$$

$\overline{M}$ and $\overline{N}$ being that which M and N become when we change $\phi(r,n-1)$ in it into $\Psi(r)$. Be continuing thus we will have the equations necessary to determine $\Gamma(r)$ and the following functions.

The first equation gives, by integrating it,

$$\Psi'(r)=Ae^{-\frac{3}{2}r^2},$$

A being an arbitrary constant. In order to integrate the second, we must observe that the preceding expressions of M and of N give

$$\overline{M}=\frac{3Ar}{4}(1-r^2)e^{-\frac{3}{2}r^2},\qquad \overline{N}=-\frac{3Ar}{20}(1-r^2)e^{-\frac{3}{2}r^2}.$$

The equation in $\Pi'(r)$ becomes thus

$$3r\Pi'(r)+\Pi''(r)=\frac{36A}{20}(r-r^3)e^{-\frac{3}{2}r^2};$$

by multiplying it by $e^{\frac{3}{2}r^2}$ and integrating, we will have

$$\Pi'(r) = Be^{-\frac{3}{2}r^2} + \frac{3A}{20}(6r^2 - 3r^4)e^{-\frac{3}{2}r^2},$$

B being a second arbitrary. We will have in the same manner $\Gamma'(r)$, $\ldots,$ and we will obtain thus $\phi(r, n-1)$.

In order to determine the arbitraries A, B, $\ldots,$ we will observe that, if we integrate $\int dr\, \phi'(r, n-1)$ from r nul to $r = \sqrt{n}$, that which returns to take it to r infinity, because we can neglect the terms multiplied by the exponential $e^{-\frac{3}{2}n}$, on account of the magnitude supposed in n, we will have for this integral a quantity that we will designate by $L\sqrt{\frac{3}{2n}}, L$ being a linear function of A, $\frac{B}{n}$, $\ldots;$ but, when $r = \sqrt{n}$, the first member of equation (*b*) becomes, whatever be n, equal to $\frac{1}{2}$; we have therefore

$$L\sqrt{\frac{3}{2n}} = \frac{1}{2}.$$

By equating to zero in this equation the coefficients of the successive powers of $\frac{1}{n}$, we will have as many equations which will determine the arbitraries A, B, $\ldots;$ thus $\phi'(r, n-1)$ being, by that which precedes, equal to

$$\left(A + \frac{B}{n} + \cdots\right)e^{-\frac{3}{2}r^2} + \frac{3A}{20n}(6r^2 - 3r^4)e^{-\frac{3}{2}r^2} + \cdots,$$

we have, by integrating from r null to r infinity,

$$\int dr\, \phi'(r, n-1) = \sqrt{\frac{\pi}{6}}\left(A + \frac{B}{n} + \frac{3A}{20n} + \cdots\right);$$

equating this quantity to $\frac{1}{2}$ and comparing the powers of $\frac{1}{n}$, we have

$$A = \sqrt{\frac{3}{2\pi}}, \qquad B = -\frac{3A}{20}, \qquad \cdots,$$

that which gives

$$\phi'(r, n-1) = \sqrt{\frac{3}{2\pi}}e^{-\frac{3}{2}r^2}\left[1 - \frac{3}{20n}(1 - 6r^2 + 3r^4) + \cdots\right].$$

By changing n into $n+1$ and neglecting the quantities of order $\frac{1}{n^2}$, we will have the expression of $\phi'(r, n)$ which results from articles III and V; for we see, by article V, that $\phi(r,n)$ must be a half of the probability that we have determined in article IV, and of which the half is equal to the integral of dr multiplied by this expression of $\phi'(r,n)$.

VIII.

We can reduce equations (q) and (q') to a single equation in the infinitely small and finite differences. In effect, if in equation (q) we increase r by $\frac{2}{\sqrt{n}}$; then r'' is changed into r', and we have

$$\left(\frac{1}{\sqrt{n}}+\frac{r+\frac{2}{\sqrt{n}}}{n}\right)\phi'\left(r+\frac{2}{\sqrt{n}},n\right)+\phi(r',n-1)=\phi\left(r+\frac{2}{\sqrt{n}},n\right).$$

By subtracting from this equation the equation (q'), member by member, we have

$$\begin{aligned}\frac{1}{\sqrt{n}}\left[\phi'\left(r+\frac{2}{\sqrt{n}},n\right)+\phi(r',n)\right]+\frac{r+\frac{2}{\sqrt{n}}}{n}\phi'\left(r+\frac{2}{\sqrt{n}},n\right)-\frac{r}{n}\phi'(r,n)\\ =\phi\left(r+\frac{2}{\sqrt{n}},n\right)-\phi(r,n).\end{aligned}$$

Let $s=r\sqrt{n}$, and we designate $\phi(r,n)$ by $\Psi(s)$, that which gives

$$dr\,\phi'(r,n)=ds\,\Psi'(s)$$

and, consequently,

$$\phi'(r,n)=\sqrt{n}\,\Psi'(s);$$

the preceding equation will become

$$\Psi'(s+2)+\Psi'(s)=\Psi(s+2)-\Psi(s)-\frac{s+2}{n}\Psi'(s+2)+\frac{s}{n}\Psi'(s).$$

By differentiating, we will have

$$(x)\qquad\left\{\begin{aligned}\Psi''(s+2)+\Psi''(s)=[\Psi'(s+2)-\Psi'(s)]\frac{n-1}{n}\\ -\frac{s+2}{n}\Psi''(s+2)+\frac{s}{n}\Psi''(s).\end{aligned}\right.$$

This equation is susceptible to the general method which I have presented in *Mémoires de l'Académie des Sciences* for the year 1782, page 44.[7] I make therefore, conforming to this method, and by employing the cosine in the place of the exponentials,

$$\Psi'(s) = \int \cos st\, \Pi(t)\, dt.$$

The question is to determine the function $\Pi(t)$ and the limits of the integral. For this, we will substitute this integral, in place of $\Psi'(s)$, into equation (*x*), and we will make the coefficients $s+2$ and s of this equation disappear by means of integration by parts; we will have thus

$$(y) \qquad \begin{cases} 0 = \dfrac{t}{n}\sin(s+1)t\sin t\,\Pi(t) \\ \qquad + \displaystyle\int \sin(s+1)t\left[(t\cos t - \sin t)\Pi(t) - \frac{t\sin t}{n}\Pi'(t)\right]dt. \end{cases}$$

Following the method cited, we determine $\Pi(t)$ by equating the function under the integral sign to zero, that which gives

$$0 = (t\cos t - \sin t)\Pi(t) - \frac{t\sin t}{n}\Pi'(t),$$

whence we deduce, by integrating,

$$\Pi(t) = A\left(\frac{\sin t}{t}\right)^n$$

and, consequently,

$$\Psi'(s) = A\int \cos st\left(\frac{\sin t}{t}\right)^n dt = A\int \cos rt\sqrt{n}\left(\frac{\sin t}{t}\right)^n dt,$$

A being an arbitrary constant. We will have next, by the same method, the limits of this last integral by equating to zero the part outside of the $\int$ sign in equation (*y*); now, this part is null when t is null and when t is infinity, because $\Pi(t)$ becomes null then. We can therefore take $t = 0$ and $t = \infty$ for these limits. This expression of $\Psi'(s)$ is of the same form as that which we have found in article

[7] *Mémoire sur les approximations des formules qui sont fonctions de très grands nombres,* § XV.

IV, for the probability that the sum of the inclination of the orbits of n comets will be $\frac{nh}{2} \pm \frac{rh\sqrt{n}}{2}$; and, by treating it by the method of the article cited, we will arrive, in order to determine $\phi(r,n)$, to the same formulas which we just gave.

IX.

We can extend the preceding researches to the differences of fractional powers. For this, we consider the function

$$(n-i)(n-i-1)(n-i-2)\cdots(f-i)2^{n-i}\big[(n+r\sqrt{n})^{n-i} - n(n+r\sqrt{n}-i)^{n-i} \\ + \frac{n.n-1}{1.2}(n+r\sqrt{n}-4)^{n-i} + \cdots\big],$$

i being any whole or fractional number very small relatively to n, and f being the number immediately superior to i. By designating this function by $\phi(r,n)$, we will have first, by following the preceding analysis, equation (*p*). We will have next, in place of equation (*q*), this one

$$\frac{n}{n-i}\left(\frac{1}{\sqrt{n}} + \frac{r}{n}\right)\phi'(r,n) + \frac{n}{n-i}\phi(r'', n-1) = \phi(r,n).$$

By combining these two equations and reducing to series, as above, we will have, by neglecting the powers superior to $\frac{1}{n}$,

$$0 = 3r\phi'(r,n-1) + \phi''(r,n-1) + 3i\phi(r,n-1),$$

and, by changing $n-1$ into n,

(*u*) $$0 = 3r\phi'(r,n) + \phi''(r,n) + 3i\phi(r,n).$$

We satisfy this equation when i is a whole number by making

$$\phi(r,n) = A\frac{d^{i-1}e^{-\frac{3}{2}r^2}}{dr^{i-1}},$$

A being an arbitrary constant and the differential characteristic d must be changed into the integral sign $\int$, if $i-1$ is negative, and then we obtain the preceding results; but, if i is fractional, the integration of equation (*u*) presents more difficulties. We can obtain it then by definite integrals.

We consider the case of $i = \frac{1}{2}$; we will have for the integral of equation (*u*)

$$\phi(r,n) = \int \frac{1}{\sqrt{x}} e^{-\frac{x^2}{6}} (a \cos rx + b \sin rx) dx,$$

a and b being two arbitrary constants, and the integral being taken from x null to x infinity. In effect, if, conformably to the method exhibited on pages 49 and the following of *Mémoires de l'Académie des Sciences* for the year 1782,[8] we make

$$\phi(r,n) = \int \cos rx\, \Psi(x) dx;$$

by substituting this value into the differential equation (*u*), and making the coefficient r vanish in this equation by means of integration by parts, we will have

$$0 = 3x \cos rx\, \Psi(x) + \int \cos rx \left[\left(\frac{3}{2} - x^2 \right) dx \Psi(x) - 3d.x\Psi(x) \right].$$

Following the method cited, we determine $\Psi(x)$ by equating to zero the part under the $\int$ sign, and we have

$$0 = \left(\frac{3}{2} - x^2 \right) dx \Psi(x) - 3d.x\Psi(x),$$

an equation which, integrated, gives

$$\Psi(x) = \frac{a}{\sqrt{x}} e^{-\frac{1}{6}x^2}.$$

We determine next the limits of the integral by equating to zero the part $3x$ cos $rx\, \Psi(x)$ under the integral sign. This part becomes

$$3a\sqrt{x} \cos rx\, e^{-\frac{1}{6}x^2},$$

and it is null with x $\langle$null$\rangle$ and when x is infinity. Thus the integral

$$\int \frac{dx}{\sqrt{x}} \cos rx\, e^{-\frac{1}{6}x^2}$$

[8] *Mémoire sur les approximations des formules qui sont fonctions de très grands nombres*, §XVII.

must be taken within these limits. If, in place of the integral

$$\int \cos rx \, \Psi(x) dx,$$

we had considered this one

$$\int \sin rx \, \Psi(x) dx,$$

we would have found for $\Psi(x)$

$$\frac{b}{\sqrt{x}} e^{-\frac{1}{6}x^2}.$$

The reunion of these two integrals is therefore the complete integral of equation (*u*).

In order to determine the constants a and b, we observe that, if we make $r = \sqrt{n}$ and $x = \frac{x'}{\sqrt{n}}$, the integral $\int \frac{1}{\sqrt{x}} e^{-\frac{x^2}{6}} (a \cos rx + b \sin rx) dx$ becomes

$$\frac{1}{n^{\frac{1}{4}}} \int \frac{dx'}{\sqrt{x'}} (a \cos x' + b \sin x') e^{-\frac{x'^2}{6n}}.$$

When n is a very great number, we can suppose the factor $e^{-\frac{x'^2}{6n}}$ equal to unity, in every extent of the integral taken from x' null to x' infinity, because then this factor begins to deviate sensibly from unity only when x' is of the order $\sqrt{n}$, and the integral, taken from a value of this order for x' to x' infinity, can be neglected relatively to the entire integral. Now we have, as I have shown in Book VIII of the *Journal de l'École Polytechnique*, page 248,[9]

$$\int \frac{dx' \cos x'}{\sqrt{x'}} = \int \frac{dx' \sin x'}{\sqrt{x'}} = \sqrt{\frac{1}{2}\pi}.$$

The preceding integral is reduced therefore to

$$\frac{1}{n^{\frac{1}{4}}} \sqrt{2\pi} \, \frac{a+b}{2};$$

this is the expression of $\phi(r, n)$ when we make $r = \sqrt{n}$ in it. Then we have

[9] *Mémoire sur divers points d'analyse.* XV Cahier, T. VIII; 1809.

$$\phi(r,n)=\frac{2^n}{1.3.5\ldots(2n-1)}\left[n^{n-\frac{1}{2}}-n(n-1)^{n-\frac{1}{2}}+\frac{n(n-1)}{1.2}(n-2)^{n-\frac{1}{2}}-\cdots\right].$$

The formula (μ'') on page 82[10] of the *Mémoires de l'Académie des Sciences* for the year 1782 gives, by having regard only to its first term,

$$n^{n-\frac{1}{2}}-n(n-1)^{n-\frac{1}{2}}\cdots=\frac{1}{2^n}\,1.3.5\ldots(2n-1)\sqrt{\frac{2}{n}};$$

we have therefore

$$\frac{a+b}{2}=\frac{1}{n^{\frac{1}{4}}\sqrt{\pi}}.$$

If we make next in $\phi(r,n)$, $r=-\sqrt{n}$, this function becomes null; we have consequently

$$0=\int\frac{dx}{\sqrt{x}}e^{-\frac{x^2}{6n}}(a\cos x\sqrt{n}+b\sin x\sqrt{n}).$$

or

$$0=\int\frac{dx'}{\sqrt{x'}}(a\cos x'+b\sin x'),$$

that which gives

$$a=b.$$

Therefore

$$a=b=\frac{1}{n^{\frac{1}{4}}\sqrt{\pi}}$$

and, consequently,

$$\phi(r,n)=\frac{1}{n^{\frac{1}{4}}\sqrt{\pi}}\int\frac{dx}{\sqrt{x}}e^{-\frac{x^2}{6n}}(\cos rx+\sin rx)$$

10 *Mémoire sur les approximations des formules qui sont fonctions de très grands nombres*, §XXVIII.

or

$$\phi(r,n) = \frac{1}{6n^{\frac{1}{4}}\sqrt{\pi}} \int \frac{dx(9+2x^2)}{x^{\frac{3}{2}}} \sin rx.e^{-\frac{x^2}{6}}, \text{([11])}$$

the integrals being taken from x null to x infinity.

The same analysis leads us to determine generally $\phi(r,n),$ whatever be the number i. By supposing it less than unity, we will satisfy the differential equation (u) in $\phi(r,n)$ by the assumption of

$$\phi(r,n) = \int \frac{e^{-\frac{x^2}{6}}}{x^{1-i}} (a \cos rx + b \sin rx) dx,$$

a and b being some constants that we will determine thus.

By supposing $r = \sqrt{n}$, we will have

$$\phi(r,n) = \frac{\Delta^n s^{n-1}}{(1-i)(2-i)\cdots(n-i)},$$

s increasing from unity and being null at the origin. Formula (μ'') of page 82[12] of the *Mémoires de l'Académie des Sciences* for the year 1782 gives, by considering in it only its first term,

$$\Delta^n s^{n-1} = (1-i)(2-i)\cdots(n-i)\frac{2^i}{n^i};$$

we have therefore, in the case of $r = \sqrt{n}$,

$$\phi(r,n) = \frac{2^i}{n^i}.$$

If we make next, in the preceding expression of $\phi(r,n),$ $r = \sqrt{n}$ and $x = \frac{x'}{\sqrt{n}}$, it becomes

[11] This formula is false, because, for $r = 0$, the second member is made void without that it be likewise of the first. The exact formula is

$$\phi(r,n) = \frac{1}{6n^{\frac{1}{4}}\sqrt{\pi}} \int \sin rx.e^{-\frac{x^2}{6}} \left(\frac{2x^2+3}{r} + 6x\right) \frac{dx}{x^{\frac{3}{2}}}. \text{ (\textit{Note of editor.})}$$

[12] *Mémoire sur les approximations des formules qui sont fonctions de très grands nombres*, §XXVIII.

$$\frac{1}{n^{\frac{i}{2}}}\int \frac{}{x'^{(1-i)}}e^{-\frac{x'^2}{6n}}(a\cos x' + b\sin x')dx';$$

now we have, by the formulas of the Volume cited of the *Journal de l'École Polytechnique*, page 250,[13]

$$\int \frac{1}{x'^{(1-i)}}\cos x' dx' = \frac{k}{i}\cos\frac{i\pi}{2},$$
$$\int \frac{1}{x'^{(1-i)}}\sin x' dx' = \frac{k}{i}\sin\frac{i\pi}{2},$$

k being the integral $\int dt\, e^{-t^{\frac{1}{i}}}$ taken from t null to t infinity; by taking therefore for unity the factor $e^{-\frac{x'^2}{6n}}$, as for it we can when n is a very great number, the expression of $\phi(r, n)$ becomes

$$\frac{k}{in^{\frac{1}{2}}}\left(a\cos\frac{i\pi}{2} + b\sin\frac{i\pi}{2}\right).$$

By equating it to $\frac{2^i}{n^i}$, we will have

$$a\cos\frac{i\pi}{2} + b\sin\frac{i\pi}{2} = \frac{i.2^i}{kn^{\frac{i}{2}}}.$$

By making next $r = -\sqrt{n}$ in $\phi(r, n)$, it is reduced to zero; but then, in its expression as definite integral, $\sin rx$ is changed into $-\sin x\sqrt{n}$. Thence it is easy to conclude that we have

$$0 = a\cos\frac{i\pi}{2} - b\sin\frac{i\pi}{2},$$

consequently

$$a = \frac{i.2^i}{2kn^{\frac{1}{2}}\cos\frac{i\pi}{2}}, \qquad b = \frac{i.2^i}{2kn^{\frac{i}{2}}\sin\frac{i\pi}{2}}.$$

We have therefore

[13] *Mémoire sur divers points d'analyse.* XV Cahier, T. VIII; 1809.

$$\phi(r,n) = \frac{i.2^i}{kn^{\frac{i}{2}}\sin i\pi}\int\left(\sin\frac{i\pi}{2}\cos rx + \cos\frac{i\pi}{2}\sin rx\right)\frac{e^{-\frac{x^2}{6}}dx}{x^{1-i}}$$

or

$$\phi(r,n) = \frac{i.2^i}{kn^{\frac{i}{2}}\sin i\pi}\int\frac{e^{-\frac{x^2}{6}}\sin\left(rx+\frac{i\pi}{2}\right)dx}{x^{1-i}}.$$

X.

We can obtain quite simply all the results which precede by the following analysis.

We consider generally the integral $\int\frac{e^{-sx}dx}{x^{n-i+1}}$ taken from x null to x infinity, $n-i$ being equal to $i'+\frac{1}{2f}$, i' expressing a whole number positive or zero. By integrating by parts, from $x=\alpha$ to x infinity, we have

$$\begin{aligned}\frac{(-1)^{i'}e^{-s\alpha}}{\frac{1}{2f}\left(\frac{1}{2f}+1\right)\left(\frac{1}{2f}+2\right)\cdots(n-i)\alpha^{\frac{1}{2f}}}&\left[s^{i'}-\frac{1}{2f}\frac{s^{i'-1}}{\alpha}+\frac{1}{2f}\left(\frac{1}{2f}+1\right)\frac{s^{i'-2}}{\alpha^2}+\cdots\right.\\ &\left.+(-1)^{i'}\frac{1}{2f}\left(\frac{1}{2f}+1\right)\cdots(n-i)\frac{1}{\alpha^{i'}}\right]\\ &+\frac{(-1)^{i'+1}s^{i'+1}\int\frac{dx\,e^{-sx}}{x^{\frac{1}{2f}}}}{\frac{1}{2f}\left(\frac{1}{2f}+1\right)\cdots(n-i)}\end{aligned}$$

We have generally

$$\Delta^n\frac{e^{-s\alpha}}{\alpha^{n-c}} = 0,$$

when we suppose α infinitely small; because, if we develop $e^{-s\alpha}$ into a series ordered with respect to the powers of $s\alpha$, all the powers of s inferior to n become null in the function $\Delta^n\frac{e^{-s\alpha}}{\alpha^{n-c}}$, and all the powers equal to n or superior are null by the supposition of α infinitely small. It follows thence that $\Delta^n\int\frac{e^{-sx}dx}{x^{n-i+1}}$ is equal to

$$\frac{(-1)^{i'+1}\Delta^n\left(s^{i'+1}\int\frac{e^{-sx}dx}{x^{\frac{1}{2f}}}\right)}{\frac{1}{2f}\left(\frac{1}{2f}+1\right)\cdots(n-i)},$$

the integral being taken from x null to x infinity; now we have

$$\Delta^n\int\frac{e^{-sx}dx}{x^{n-i+1}}=\int\frac{e^{-sx}(e^{-x}-1)^n dx}{x^{n-i+1}};$$

by making next $x=\frac{x'}{s}$, we have

$$\int\frac{e^{-sx}dx}{x^{\frac{1}{2f}}}=s^{\frac{1}{2f}-1}\int\frac{e^{-x'}dx'}{x'^{\frac{1}{2f}}},$$

the two integrals being taken from x and x' nulls to their values infinities; we have therefore

$$\int\frac{e^{-sx}(e^{-x}-1)^n dx}{x^{n-i+1}}=\frac{(-1)^{i'+1}\Delta^n s^{n-i}\int\frac{e^{-x'}dx'}{x'^{\frac{1}{2f}}}}{\frac{1}{2f}\left(\frac{1}{2f}+1\right)\cdots(n-i)},$$

this which gives

$$\Delta^n s^{n-i}=\frac{1}{2f}\left(\frac{1}{2f}+1\right)\cdots(n-i)(-1)^{i'+1}\frac{\int\frac{e^{-sx}(e^{-x}-1)^n dx}{x^{n-i+1}}}{\int\frac{e^{-x'}dx'}{x'^{\frac{1}{2f}}}},$$

an equation which is the same as formula (μ''') of *Mémoires de l'Académie des Sciences*, for the year 1782,[14] as it is easy to convince ourselves.

We suppose $i'=n-1$ and f a positive whole number. If we make $s=-\frac{n}{2}-\frac{r\sqrt{n}}{2}$, the integral $\int\frac{e^{-sx}(e^{-x}-1)^n dx}{x^{n-i+1}}$ will become

$$\int\frac{e^{\frac{1}{2}rx\sqrt{n}}dx}{x^{\frac{1}{2f}}}\left(\frac{e^{-\frac{x}{2}}-e^{\frac{x}{2}}}{x}\right)^n.$$

We make $x=2x'\sqrt{-1}$, and then this last integral is transformed into the following

[14] *Mémoire sur les approximations des formules qui sont fonctions de très grands nombres*, §XXIX.

$$\frac{(-1)^n 2\sqrt{-1}}{(2\sqrt{-1})^{\frac{1}{2f}}}\int \frac{1}{x'^{\frac{1}{2f}}}(\cos rx'\sqrt{n}+\sqrt{-1}\sin rx'\sqrt{n})\left(\frac{\sin x'}{x'}\right)^n dx';$$

we have therefore

$$(x)\quad \left\{\begin{aligned}\Delta^n s^{n-i} &= \frac{1}{2f}\left(\frac{1}{2f}+1\right)\cdots(n-i)\frac{2\sqrt{-1}}{(2\sqrt{-1})^{\frac{1}{2f}}}\\ &\quad\times\frac{\int \frac{1}{x'^{\frac{1}{2f}}}(\cos rx'\sqrt{n}+\sqrt{-1}\sin rx'\sqrt{n})\left(\frac{\sin x'}{x'}\right)^n dx'}{\int \frac{e^{-x'}dx'}{x'^{\frac{1}{2f}}}},\end{aligned}\right.$$

the integrals being taken from x' null to $x' = \pm\infty$.

We suppose first f infinity; we have generally $\frac{1}{k^{2f}} = 1$ by neglecting the terms of order $\frac{1}{2f}$; because if we make $\frac{1}{k^{2f}} = 1+q$, by taking the logarithms, we will have

$$\frac{1}{2f}\log k = \log(1+q),$$

that which gives

$$q = \frac{1}{2f}\log k.$$

This put, the preceding equation becomes

$$(z)\quad \left\{\begin{aligned}&2^{n-1+\frac{1}{2f}}\Delta^n s^{n-1+\frac{1}{2f}}\\ &\quad = \frac{1.2.3\ldots(n-1)}{2f}2^n\int(\sqrt{-1}\cos rx'\sqrt{n}-\sin rx'\sqrt{n})\left(\frac{\sin x'}{x'}\right)^n dx';\end{aligned}\right.$$

now we have, with the exclusion of the powers of the negative quantities,

$$\begin{aligned}2^{n-1+\frac{1}{2f}}\Delta^n s^{n-1+\frac{1}{2f}} &= 1^{\frac{1}{2f}}\left[(n+r\sqrt{n})^{n-1+\frac{1}{2f}} - n(n+r\sqrt{n}-2)^{n-1+\frac{1}{2f}}+\cdots\right]\\ &\quad -(-1)^{\frac{1}{2f}}\left[(n+r\sqrt{n})^{n-1+\frac{1}{2f}} - n(n+r\sqrt{n}-2)^{n-1+\frac{1}{2f}}+\cdots\right].\end{aligned}$$

$1^{\frac{1}{2f}}$ is susceptible of $2f$ values of which a single one is real and equal to unity. We obtain these values by observing that

$$1 = \cos 2l\pi + \sqrt{-1}\sin 2l\pi,$$

and that thus

$$1^{\frac{1}{2f}} = (\cos 2l\pi + \sqrt{-1}\sin 2l\pi)^{\frac{1}{2f}} = \cos\frac{2l\pi}{2f} + \sqrt{-1}\sin\frac{2l\pi}{2f},$$

l being a positive whole number which can be extended from $l = 1$ to $l = 2f$. In order to have the real value of $1^{\frac{1}{2f}}$, it is necessary to give to l its greatest value $2f$. Then the imaginary part of the preceding expression of $\Delta^n s^{n-1+\frac{1}{2f}}$ is produced by the parts affected of $(-1)^{\frac{1}{2f}}$. This last quantity has likewise $2f$ values represented by $\cos\frac{(2l-1)\pi}{2f} + \sqrt{-1}\sin\frac{(2l-\pi)}{2f}$, l can again be extended from $l = 1$ to $l = 2f$. But, having chosen $l = 2f$ in order to have $1^{\frac{1}{2f}}$, we must likewise choose this value of l in order to determine $(-1)^{\frac{1}{2f}}$, and then the imaginary part of $(-1)^{\frac{1}{2f}}$ becomes $\sqrt{-1}\sin\frac{(4f-1)\pi}{2f}$; and, in the case of f infinity, it becomes $-\sqrt{-1}\,\frac{\pi}{2f}$; this which gives, by neglecting the terms of order $\frac{1}{f^2}$,

$$\frac{\pi\sqrt{-1}}{2f}\left[(n + r\sqrt{n})^{n-1} - n(n + r\sqrt{n} - 2)^{n-1} + \cdots\right],$$

for the imaginary part of the preceding expression of

$$2^{n-1+\frac{1}{2f}}\Delta^n s^{n-1+\frac{1}{2f}}.$$

By equating to the imaginary part of the expression given by equation (z), we will have

$$\frac{(n + r\sqrt{n})^{n-1} - n(n + r\sqrt{n} - 2)^{n-1} + \cdots}{1.2.3\ldots(n-1)2^n} = \frac{1}{\pi}\int \cos rx'\sqrt{n}\left(\frac{\sin x'}{x'}\right)dx'.$$

The second member of this equation being integrated by the method of article III, will have the same results as those above.

We suppose now, in equation (x), $f = 1$, we will have, by changing in it x to $-x''$ in the numerator of the second member, and observing that the integral $\int \frac{1}{\sqrt{x'}}e^{-x'}dx'$ of the numerator is equal to $\sqrt{\pi}$,

$$\Delta^n s^{n-\frac{1}{2}} = \frac{1.3.5\ldots(2n-1)}{2^{n-1}\sqrt{2\pi}}(-1)^{\frac{3}{4}}\int\frac{1}{\sqrt{x''}}(\cos rx''\sqrt{n}-\sqrt{-1}\sin rx''\sqrt{n})\left(\frac{\sin x''}{x''}\right)^n dx''.$$

Here the integrals must be taken from x'' null to x'' infinity. We have, by excluding the powers of the negative quantities,

$$\begin{aligned}2^{n-\frac{1}{2}}\Delta^n s^{n-\frac{1}{2}} = {} & (n-r\sqrt{n})^{n-\frac{1}{2}} - n(n-r\sqrt{n}-2)^{n-\frac{1}{2}} + \cdots \\ & -\sqrt{-1}\Big[(n+r\sqrt{n})^{n-\frac{1}{2}} - n(n+r\sqrt{n}-2)^{n-\frac{1}{2}} + \cdots\Big].\end{aligned}$$

By substituting this value into the preceding equation and taking $\frac{1-\sqrt{-1}}{\sqrt{2}}$ in the place of $(-1)^{\frac{3}{4}}$, we will have, by comparing the real quantities to the reals and the imaginaries to the imaginaries, the double equation

$$\begin{aligned}&\frac{(n\pm r\sqrt{n})^{n-\frac{1}{2}} - n(n\pm r\sqrt{n}-2)^{n-\frac{1}{2}} + \cdots}{1.3.5\ldots(2n-1)} \\ &= \frac{1}{\sqrt{2\pi}}\int\frac{1}{\sqrt{x''}}(\cos rx''\sqrt{n}\pm\sin rx''\sqrt{n})\left(\frac{\sin x''}{x''}\right)^n dx''.\end{aligned}$$

If we reduce into series $\frac{\sin x''}{x''}$, and if we make $x''=\frac{t}{\sqrt{n}}$, we will have

$$1-\frac{t^2}{6n}+\cdots;$$

we could therefore substitute $e^{-\frac{t^2}{6}}$ in the place of $\left(\frac{\sin x''}{x''}\right)^n$, and then the second member of the preceding equation becomes

$$\frac{1}{n^{\frac{1}{4}}\sqrt{2\pi}}\int\frac{1}{\sqrt{t}}(\cos rt\pm\sin rt)\,e^{\frac{-t^2}{6}}\,dt,$$

that which coincides with the results of the preceding article.

By generalizing this analysis, we will arrive easily to this rigorous expression, i being less than unity and the powers of the negative quantities being excluded,

$$
\begin{aligned}
&\frac{1}{(1-i)(2-i)\cdots(n-i)2^{n-i}}\Big[(n+r\sqrt{n})^{n-i} - n(n+r\sqrt{n}-2)^{n-i}\\
&\qquad\qquad\qquad\qquad\qquad + \frac{n(n-1)(n-2)}{1.2}(n+r\sqrt{n}-4)^{n-i} - \cdots\Big]\\
&\quad = \frac{i.2^i}{k\sin i\pi}\int \frac{1}{x^{1-i}}\sin\left(rx\sqrt{n}+\frac{i\pi}{2}\right)\left(\frac{\sin x}{x}\right)^n dx,
\end{aligned}
$$

the integral being taken from x null to x infinity, and k being the integral $\int e^{-x^{\frac{1}{i}}}dx$ taken within the same limits. We will have, by some successive differentiations, the values relative to i greater than unity.