RECHERCHES
SUR
L'INTEGRATION DES ÉQUATIONS DIFFÉRENTIELLES
AUX DIFFÉRENCES FINIES
ET SUR
LEUR USAGE DANS LA THÉORIE DES HASARDS

M. de la Place, Professor at the École Royale militaire

Mémoires de l'Académie Royale des Sciences de Paris (Savants étrangers),
1773, T. VII (1776) pp. 37-163.[1]

I.

The first researches that one has made on the summation of arithmetic progressions and on geometric progressions contained the germ of the integral Calculus in finite differences in one and two variables; here is how: an arithmetic progression is a sequence of terms which increase equally, and it was necessary to find the sum according to this condition; it is clear that each term of the sequence is the finite difference of the sum of the preceding terms, to that same sum augmented by this term; one proposed therefore to find this sum according to the nature of its finite difference; thus by whatever manner that one is arrived there, one has veritably integrated a quantity in the finite differences. The geometers who have come next have pushed further these researches; they have determined the sum of the squares and of the superior and entire powers of the natural numbers; they have arrived there first by some indirect methods: they did not perceive that that which they sought returned to finding a quantity of which the finite difference was known; but as soon as they had made this reflection, they have resolved directly, not only the cases already known, but many others more extended. In general, $\phi(x)$ representing any function whatsoever of the variable x, of which the finite difference is supposed constant, they have proposed to find a quantity of which the finite difference is equal to that function,

[1] Read to the Academy 10 February 1773. OC 8 pp. 69-174.

and this is the object of the integral Calculus in the finite differences in a single variable.

Similarly, the research of the general term of a geometric progression returns to finding the x^{th} term of a sequence[2] such that each term is to the one which precedes it in constant ratio. Let y_{x-1} be the $(x-1)^{\text{st}}$ term and y_x be the x^{th} term: the law of the sequence requires that one have $y_x = py_{x-1}$, whatever be x, p being constant. Now it is clear that, in whatever manner that one is arrived to find y_x, one has veritably integrated the equation in the finite differences $y_x = py_{x-1}$. Next, one has generalized this research by proposing to find the general term of the sequences such that each of their terms is equal to many of the preceding multiplied by some constants any whatsoever; these sequences have been named for this *récurrentes*. One has arrived first to find their general term by some indirect ways, although quite ingenious; one did not perceive that this returned to integrating a linear equation in finite differences; but, when one had made this reflection, one tried to apply to these equations the methods known for the linear equations in the infinitely small differences, with the modifications that the assumption of finite differences requires, and one resolved in this manner some cases much more extended than those which were already.

Mr. Moivre is, I believe, the first who had determined the general term of the recurrent sequences; but Mr. de Lagrange is the first who is aware that this research depends on the integration of a linear equation in finite differences, and who had applied the good method of undetermined coefficients of Mr. d'Alembert (*see* Vol. I of the *Mémoires de Turin*). I myself have proposed next to deepen this interesting calculus, in a Memoir printed in Volume IV of those of Turin;[3] and next, having had occasion to reflect further there, I have made on this new researches of which I will render account shortly. I must observe here that Mr. the marquis de Condorcet has given excellent things on this matter, in his *Traité du Calcul intégral*, and in the *Mémoires de l'Académie*.

It was until then only a question of equations in ordinary finite differences and of the sequences which depend on them; but the solution of many problems on the chances has led me to a new kind of sequence which I have named *récurro-récurrentes*, and of which I believe to have given first the theory and indicated

---

[2] *Translator's note*: The word suite is used to refer to both a sequence and a series. It is rendered according to its usage.

[3] Recherches sur le calcul intégral aux differences infiniment petites, & aux différences finies. *Mélanges de philosophie et de mathématiques de la Société royale de Turin, pour les années 1766-1769 (Miscellanea Taurensia IV),* 273-345, 1771.

the usage in the Science of probabilities (see T. VI of *Savants étranges*.[4]) The equations on which these sequences depend are nearly, in the finite differences, that which the equations in the partial differences are in the infinitely small differences; that which I have given on these equations is only a trial: in deepening them, I have seen that they were quite important in the Theory of chances, and that they gave a method to treat them much more generally that one had done yet: this is that which engages me to consider them anew; but, the new researches that I have made on this object supposing those that I have already given, I am going to begin again here all this matter.

II.

One can imagine thus the equations in finite differences; I imagine the sequence

$$y_1, y_2, y_3, y_4, y_5, \dots, y_x$$

formed following a law such as one has constantly

$$\text{(A)} \qquad X_x = M_x y_x + N_x \Delta y_x + P_x \Delta^2 y_x + \ldots + S_x \Delta^n y_x;$$

the numbers 1, 2, 3, $\dots$, x, placed at the base of y, indicating the rank which y occupies in the sequence, or, this which returns to the same, the index of the series; the quantities X_x, M_x, N_x, $\dots$ are some functions any whatsoever of the variable x, of which the difference is supposed constant and equal to unity. The characteristic Δ serves to express the finite difference of the quantity before which it is placed, as in the infinitesimal Analysis the letter d expresses the infinitely small difference of the quantities. This put, the preceding equation is an equation in finite differences, which can generally represent the equations of this kind, where the variable y_x and its differences are under a linear form.

Although I have supposed the constant difference of x equal to unity, this diminishes nothing from the generality of the preceding equation (A); because, if this difference, instead of being 1, is equal to q, one will make $\frac{x}{q} = x'$, and y_x being a function of x will become a function of qx'; I name $y_{x'}$ this last function. Now one has, by hypothesis,

$$\begin{aligned}\Delta y_x = y_{x+q} - y_x &= f(x+q) - f(x) \\ &= f[q(x'+a)] - f(qx') = y_{x'+1} - y_{x'} = \Delta y_{x'},\end{aligned}$$

[4] Mémoire sur les suites récurro-récurrentes et sur leur usages dans la théorie des hasards, *Mémoires de l'Académie Royale des Sciences de Paris (Savants étranges)* 6, 1774, p. 353-371.

the constant difference of x' being 1. Similarly,

$$\Delta^2 y_x = y_{x+2q} - 2y_{x+q} + y_x = y_{x'+2} - 2y_{x'+1} + y_{x'} = \Delta^2 y_{x'},$$

and thus of the remaining. Equation (A) will be therefore transformed into the following

$$X_{x'} = M_{x'} y_{x'} + N_{x'} \Delta y_{x'} + \ldots + S_{x'} \Delta^n y_{x'}, \tag{A}$$

in which the difference of x' is equal to unity.

One can form easily other differential equations, in which y_x and its differences would enter in any manner whatsoever; but those which are contained in equation (A) are the only ones which it is truly interesting to consider.

Before researching to integrate them, I am going to recall here a principle quite useful in the analysis of the infinitely small differences, and which applies equally and with the same advantage to finite differences; here is in what it consists:

Each function of x which, containing n arbitrary irreducible constants, satisfying as y_x in a differential equation of order n, between x and y_x, is the complete expression of y_x.

By *irreducible constants*, I intend that they are such that two or many can not be reduced to one alone; it follows thence that, if a function containing n irreducible arbitrary constants satisfy as y_x in a differential equation of order $n - 1$, this equation is surely identical; because, if it was not, the most general function of x which was able to satisfy as y_x would contain only $n - 1$ irreducible arbitrary constants.

For the convenience of the calculus, I will suppose that the quantities noted in this manner, ${}^1H$, ${}^2H$, $\ldots$, or ${}^1M$, ${}^2M$, $\ldots$, express somedifferent quantities and which can have not relation among themselves; but these here, H_1, H_2, H_3, $\ldots$ H_x or M_1, M_2, M_3, $\ldots$, M_x represent the different terms of a sequence formed according to one law any whatsoever, the numbers 1, 2, 3, $\ldots$, x designating the rank of the H or of the M in the sequence. This put, since one has

$$\begin{aligned}
&\Delta y_x = y_{x+1} - y_x,\\
&\Delta y^2 y_x = y_{x+2} - 2y_{x+1} + y_x,\\
&\Delta^3 y_x = y_{x+2} - 3y_{x+2} + 3y_{x+1} - y_x,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}$$

I am able to give to equation (A) this form

$$
\begin{aligned}
X_x = \ & + y_x(M_x - N_x + P_x - \dots) \\
& + y_{x+1}(N_x - 2P_x + \dots) \\
& + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + y_{x+n}S_x.
\end{aligned}
$$

whence it results that each linear equation in finite differences can be generally represented by this here

$$
\text{(B)} \qquad y_x = H_x y_{x-1} + {}^1H_x y_{x-2} + {}^2H_x y_{x-3} + \cdots + {}^{n-1}H_x y_{x-n} + X_x;
$$

the equation

$$
y_x = H_x y_{x-1} + X_x
$$

is of the first order, this here

$$
y_x = H_x y_{x-1} + {}^1H_x y_{x-2} + X_x
$$

is of the second order, and thus in sequence.

As in the series I will have need of characteristics in order to designate the finite difference of the quantities, their finite integrals, the product of all the terms of a sequence, I will serve myself for this with the following.

The characteristic Δ placed before a quantity will designate for it, as above, the finite difference: thus ΔH_x will express the finite difference of H_x; the characteristic Σ placed before a quantity will designate for it the finite integral: thus H_x will signify the finite integral of H_x; finally the characteristic ∇ will designate the product of all the terms of a sequence: thus ∇H_x will represent the product $H_1 H_2 H_3 \dots H_x$ of all the terms of the sequence $H_1,\ H_2,\ H_3,\ \dots,\ H_x$.

III.

PROBLEM I. — *The differential equation of the first order*

$$
y_x = H_x y_{x-1} + X_x
$$

being given, one proposes to integrate it.

I make in this equation $y_x = u_x \nabla H_x$; it becomes

$$
u_x \nabla H_x = H_x u_{x-1} \nabla H_{x-1} + X_x;
$$

but one has

$$H_x \nabla H_{x-1} = \nabla H_x,$$

hence

$$u_x = u_{x-1} + \frac{X_x}{\nabla H_x} \qquad \text{or} \qquad \Delta u_{x-1} = \frac{X_x}{\nabla H_x};$$

and, as this equation holds whatever be x, one will have

$$\Delta u_x = \frac{X_{x+1}}{\nabla H_{x+1}},$$

hence, by integrating,

$$u_x = A + \sum \frac{X_{x+1}}{\nabla H_{x+1}},$$

A being an arbitrary constant. One has therefore

$$y_x = \nabla H_x \left(A + \sum \frac{X_{x+1}}{\nabla H_{x+1}} \right).$$

If H_x was constant and equal to p, one would have

$$\nabla H_x = p^x \qquad \text{and} \qquad y_x = p^x \left(A + \sum \frac{X_{x+1}}{p^{x+1}} \right).$$

IV.

PROBLEM II. — *The differentio-differential equation*

(B) $$y_x = H_x y_{x-1} + {}^1H_x y_{x-2} + {}^2H_x y_{x-3} + \ldots + {}^{n-1}H_x y_{x-n} + X_x$$

being given, one proposes to integrate it.

I make

(C) $$y_x = \alpha_x y_{x-1} + T_x,$$

α_x and T_x being two new variables, and I conclude from it the following equations:

$$\begin{aligned}
y_{x-1} &= \alpha_{x-1} y_{x-2} + T_{x-1},\\
y_{x-2} &= \alpha_{x-2} y_{x-3} + T_{x-2},\\
y_{x-3} &= \alpha_{x-3} y_{x-4} + T_{x-3},\\
\ldots&\ldots\ldots\ldots\ldots\ldots,\\
y_{x-n+1} &= \alpha_{x-n+1} y_{x-n} + T_{x-n+1};
\end{aligned}$$

I multiply the first of these equations by $-{}^1\beta$, the second by $-{}^2\beta$, the third by $-{}^3\beta$, $\ldots$ and I add them with equation (C): this which gives me

$$\begin{aligned}
y_x = (\alpha_x + {}^1\beta) y_{x-1} &+ (-{}^1\beta\alpha_{x-1} + {}^2\beta) y_{x-2}\\
&+ (-{}^2\beta\alpha_{x-2} + {}^3\beta) y_{x-3} + \ldots - {}^{n-1}\beta\alpha_{x-n+1} y_{x-n}\\
&+ T_x - {}^1\beta T_{x-1} - {}^2\beta T_{x-2} - \ldots - {}^{n-1}\beta T_{x-n+1}.
\end{aligned}$$

By comparing this equation with equation (B), one will have

1°

$$T_x = {}^1\beta T_{x-1} + {}^2\beta T_{x-2} + \ldots + {}^{n-1}\beta T_{x-n+1} + X_x;$$

2° The following equations:

$$\begin{aligned}
{}^1\beta + \alpha_x &= H_x,\\
{}^2\beta - {}^1\beta\alpha_{x-1} &= {}^1H_x,\\
{}^3\beta - {}^2\beta\alpha_{x-2} &= {}^2H_x\\
\ldots\ldots&\ldots\ldots,\\
-{}^{n-1}\beta\alpha_{x-n+1} &= {}^{n-1}H_x.
\end{aligned}$$

Thence one will conclude

$$\begin{aligned}
{}^1\beta &= H_x - \alpha_x,\\
{}^2\beta &= {}^1H_x + \alpha_{x-1} H_x - \alpha_x\alpha_{x-1},\\
{}^3\beta &= {}^2H_x + \alpha_{x-2}{}^1H_x + \alpha_{x-1}\alpha_{x-2} H_x - \alpha_x\alpha_{x-1}\alpha_{x-2},\\
\ldots&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
{}^{n-1}\beta &= {}^{n-2}H_x + \alpha_{x-n+2}{}^{n-3}H_x + \alpha_{x-n+3}\alpha_{x-n+2}{}^{n-4}H_x + \ldots\\
&\qquad - \alpha_x\alpha_{x-1}\ldots\alpha_{x-n+2} = -\frac{{}^{n-1}H_x}{\alpha_{x-n+1}},
\end{aligned}$$

because of the equation

$$-{}^{n-1}\beta\alpha_{x-n+1} = {}^{n-1}H_x;$$

one will have therefore, in order to resolve the problem, the following two equations:

$$\text{(D)}\qquad \begin{cases} T_x = (H_x - \alpha_x)T_{x-1} + ({}^1H_x + \alpha_{x-1}H_x - \alpha_x\alpha_{x-1})T_{x-2} + \dots \\ \qquad - \dfrac{{}^{n-1}H_x}{\alpha_{x-n+1}}T_{x-n+1} + X_x, \end{cases}$$

$$\text{(E)}\qquad 0 = t - \frac{H_x}{\alpha_x} - \frac{{}^1H_x}{\alpha_x\alpha_{x-1}} - \frac{{}^2H_x}{\alpha_x\alpha_{x-1}\alpha_{x-2}} - \dots - \frac{{}^{n-1}H_x}{\alpha_x\dots\alpha_{x-n+1}}.$$

Equations (D) and (E) are of a degree inferior to the proposed, and equation (D) is of the same form; now it is not necessary to integrate generally these equations in order to integrate equation (B) of the problem; it suffices to know for α_x a quantity which satisfies equation (E). I name δ_x this value; one will substitute it into equation (D), which I name (D′) after this substitution, and one will seek the complete integral of equation (D′); next, by means of the equation $y_x = \delta_x y_{x-1} + T_x$, one will conclude, by integrating by problem I,

$$y_x = \nabla\delta_x\left(A + \sum\frac{T_{x+1}}{\nabla\delta_{x+1}}\right),$$

A being an arbitrary constant.

This equation is the complete integral of equation (B), because, equation (D′) being necessarily of order $n-1$, the complete expression of T_x contains $n-1$ irreducible arbitrary constants; hence, $\nabla\delta_x\left(A + \sum\frac{T_{x+1}}{\nabla\delta_{x+1}}\right)$ contains n arbitrary constants. These constants are moreover irreducibles, because $\nabla\delta_x\sum\frac{T_{x+1}}{\nabla\delta_{x+1}}$ contains in it $n-1$ irreducibles, and none of them is reducible with the constant A.

The preceding expression of y_x can serve to make known the integral of equation (B) of the problem; because, since equation (D′) is linear, one can suppose that the expression of T_x has this form

$$T_x = \nabla\lambda_x\left({}^1A + \sum\frac{{}^1T_{x+1}}{\nabla\lambda_{x+1}}\right),$$

${}^1T_x$ depending on the integration of a linear equation of order $n-2$; one has therefore

$$y_x = \nabla\delta_x\left[A + {}^1A\sum\frac{\nabla\lambda_{x+1}}{\nabla\delta_{x+1}} + \sum\frac{\sum\frac{{}^1T_{x+1}}{\nabla\lambda_{x+1}}}{\nabla\delta_{x+1}}\right];$$

by continuing to reason thus, one will see that the expression of y_x is of this form

$$y_x = A\nabla\delta_x + {}^1A\nabla^1\delta_x + {}^2A\nabla^2\delta_x + \ldots + {}^{n-1}A\nabla^{n-1}\delta_x + L_x,$$

A, ${}^1A$, ${}^2A, \ldots$ being arbitrary.

If one supposes $X_x = 0$ in equation (B), it is easy to see, by the sequence of operations that I just indicated, that L_x will be null; thus, in this case

$$y_x = A\nabla\delta_x + {}^1A\nabla^1\delta_x + \ldots + {}^{n-1}A\nabla^{n-1}\delta_x,$$

δ_x satisfying under the assumption for α_x in equation (E); ${}^1\delta_x$, ${}^2\delta_x$, $\ldots$ will satisfy similarly; because, since the equation $y_x = A\nabla^1\delta_x$, for example, satisfies equation (B) by supposing $X = 0$, one will have

$$\nabla^1\delta_x = H_x\nabla^1\delta_{x-1} + {}^1H_x\nabla^1\delta_{x-2} + \ldots,$$

hence

$$0 = 1 - \frac{H_x}{{}^1\delta_x} - \frac{{}^1H_x}{{}^1\delta_x\,{}^1\delta_{x-1}} - \cdots$$

V.

I suppose, in equations (D′) and (B), $X_x = 0$; I will have the two expressions following from y_x:

(1) $$y_x = \nabla\delta_x\left(A + \sum\frac{T_{x+1}}{\nabla\delta_{x+1}}\right),$$

(2) $$y_x = A\nabla\delta_x + {}^1A\nabla^1\delta_x + {}^2A\nabla^2\delta_x + \ldots + {}^{n-1}A\nabla^{n-1}\delta_x.$$

These two expressions, different in appearance, must really coincide; I suppose therefore that the complete integral of equation (D′) is

$$T_x = {}^1AR_x + {}^2A\,{}^1R_x + \ldots + {}^{n-1}A\,{}^{n-2}R_x;$$

by substituting this value of T_x into equation (1), one will have

$$y_x = \nabla\delta_x\left(A + {}^1A\frac{R_{x+1}}{\nabla\delta_{x+1}} + {}^2A\frac{{}^1R_{x+1}}{\nabla\delta_{x+1}} + \ldots + {}^{n-1}A\frac{{}^{n-2}R_{x+1}}{\nabla\delta_{x+1}}\right).$$

By comparing this last equation with equation (2), one will have

$$\nabla\delta_x\sum\frac{R_{x+1}}{\nabla\delta_{x+1}} = \nabla^1\delta_x,$$

$$\nabla\delta_x\sum\frac{{}^1R_{x+1}}{\nabla\delta_{x+1}} = \nabla^2\delta_x,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Therefore

$$R_x = \nabla\delta_x\,\Delta\frac{\nabla^1\delta_{x-1}}{\nabla\delta_{x-1}},$$

$${}^1R_x = \nabla\delta_x\,\Delta\frac{\nabla^2\delta_{x-1}}{\nabla\delta_{x-1}},$$

$${}^2R_x = \nabla\delta_x\Delta\frac{\nabla^3\delta_{x-1}}{\nabla\delta_{x-1}},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Therefore, if I know how to resolve equation (B) by supposing $X = 0$, I will know how to resolve equation (D′) by supposing similarly $X_x = 0$. Let therefore u_x, ${}^1u_x$, ${}^2u_x, \ldots$ be the particular values of y_x in equation (B), so that its complete integral is

$$y_x = Au_x + {}^1A\,{}^1u_x + {}^2A\,{}^2u_x + \ldots + {}^{n-1}A\,{}^{n-1}u_x,$$

one will have

$$u_x = \nabla \delta_x, \qquad {}^1u_x = \nabla {}^1\delta_x, \qquad \dots,$$

and the complete integral of equation (D′), by supposing $X_x = 0$ in it, will be

$$T_x = {}^1Au_x \Delta \frac{{}^1u_{x-1}}{u_{x-1}} + {}^2Au_x \Delta \frac{{}^2u_{x-1}}{u_{x-1}} + \dots + {}^{n-1}Au_x \Delta \frac{{}^{n-1}u_{x-1}}{u_{x-1}}.$$

Presently, if I know how to integrate equation (D′) by supposing X_x anything, I will be able, under the same assumption, to integrate equation (B), since one has, by that which precedes,

$$y_x = u_x \left(A + \sum \frac{T_{x+1}}{u_{x+1}} \right);$$

therefore the difficulty to integrate the equation

(B) $$y_x = H_x y_{x-1} + {}^1H_x y_{x-2} + \dots + {}^{n-1}H_x y_{x-n} + X_x,$$

when one knows how to integrate this one

(*b*) $$y_x = H_x y_{x-1} + {}^1H_x y_{x-2} + \dots + {}^{n-1}H_x y_{x-n},$$

is reduced to integrate the equation

(D′) $$T_x = (H - \delta_x) T_{x-1} + \dots - \frac{{}^{n-1}H_x}{\delta_{x-n+1}} T_{x-n+1} + X_x,$$

which is of degree $n-1$, and when one knows how to integrate by supposing $X_x = 0$; one will make similarly the integration of (D′) to depend on the integration of an equation of degree $n-2$, and thus in sequence; whence there results that the equation

$$y_x = H_x y_{x-1} + {}^1H_x y_{x-2} + \dots + {}^{n-1}H_x y_{x-n} + X_x$$

is integrable in the same cases as this one

$$y_x = H_x y_{x-1} + \dots + {}^{n-1}H_x y_{x-n}.$$

VI.

The process which I just indicated in order to restore the integral of equation (B) to that of equation (*b*) can serve to demonstrate the liaison which these two integrals have between them; but it would be quite painful to employ it to integrate equation (B). It would be therefore very useful to have immediately the general expression of y_x in equation (B), when one has that of equation (*b*).

It take for this equation

$$y_x = u_x \left(A + \sum \frac{T_{x+1}}{u_{x+1}} \right),$$

T_x being supposed to be the complete expression of T_x in equation (D′). Now, this equation (D′) being of the same form as equation (B), if one names $\overset{1}{u}_x$, ${}^1\overset{1}{u}_x$, ${}^2\overset{1}{u}_x, \dots$ the particular integrals of T_x in equation (D′), when one supposes $X_x = 0$ there, one will have, in the same manner and whatever be X_x,

$$T_x = \overset{1}{u}_x \left({}^1A + \sum \frac{{}^1T_{x+1}}{\overset{1}{u}_{x+1}} \right),$$

${}^1T_x$ being the complete expression of ${}^1T_x$ in an equation of order $n-2,$ which I name (D″) and which results from (D′) in the same manner as this one results from equation (B); one will have similarly

$${}^1T_x = \overset{2}{u}_x \left({}^2A + \sum \frac{{}^2T_{x+1}}{\overset{2}{u}_{x+1}} \right),$$

and thus in sequence until one arrives to the equation of the first order

$${}^{n-2}T_x = S_x\, {}^{n-2}T_{x-1} + X_x,$$

of which the integral is

$${}^{n-2}T_x = \overset{n-1}{u}_x \left({}^{n-1}A + \sum \frac{X_{x+1}}{\overset{n-1}{u}_{x+1}} \right).$$

If one substitutes presently into the expression of y_x the value of T_x into ${}^1T_x$, that of ${}^1T_x$ into ${}^2T_x$, etc., one will have

$$\text{(K)}\quad y_x = u_x\left\{A + \sum\frac{\overset{1}{u}_{x+1}}{u_{x+1}}\left({}^1A + \sum\frac{\overset{2}{u}_{x+1}}{\overset{1}{u}_{x+1}}\left[{}^2A\ldots + \sum\frac{\overset{n-1}{u}_{x+n-1}}{\overset{n-2}{u}_{x+n-1}}\left({}^{n-1}A + \sum\frac{X_{x+n}}{\overset{n-1}{u}_{x+n}}\right)\cdots\right]\right)\right\}.$$

It is necessary presently to determine $\overset{1}{u}_x$, $\overset{2}{u}_x, \ldots$; now one has, by the previous Article,

$$\overset{1}{u}_x = R_x = u_x\Delta\frac{{}^1u_{x-1}}{u_{x-1}},$$

similarly

$$\begin{aligned} {}^1\overset{1}{u}_x &= u_x\Delta\frac{{}^2u_{x-1}}{u_{x-1}}, \\ {}^2\overset{1}{u}_x &= u_x\Delta\frac{{}^3u_{x-1}}{u_{x-1}}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots; \end{aligned}$$

one will have likewise

$$\begin{aligned} \overset{2}{u}_x &= \overset{1}{u}_x\Delta\frac{{}^1\overset{1}{u}_{x-1}}{\overset{1}{u}_{x-1}}, \\ {}^1\overset{2}{u}_x &= \overset{1}{u}_x\Delta\frac{{}^2\overset{1}{u}_{x-1}}{\overset{1}{u}_{x-1}}, \\ {}^2\overset{2}{u}_x &= \overset{1}{u}_x\Delta\frac{{}^3\overset{1}{u}_{x-1}}{\overset{1}{u}_{x-1}}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots; \end{aligned}$$

formula (K) will become

$$\text{(O)}\quad y_x = u_x\left\{A + \sum\Delta\frac{{}^1u_x}{u_x}\left({}^1A + \sum\Delta\frac{{}^1\overset{1}{u}_{x+1}}{\overset{1}{u}_{x+1}}\left[{}^2A\ldots + \sum\Delta\frac{{}^1\overset{n-2}{u}_{x+n-2}}{\overset{n-2}{u}_{x+n-2}}\left({}^{n-1}A + \sum\frac{X_{x+n}}{\overset{n-1}{u}_{x+n}}\right)\cdots\right]\right)\right\};$$

if one knows only the number $n-1$ of particular integrals of y_x, in the equation

$$y_x = H_x y_{x-1} + {}^1H_x y_{x-2} + \ldots + {}^{n-1}H_x y_{x-n},$$

the integration will be of difficulty no longer; I suppose that this is the integral ${}^{n-1}u_x$ which is unknown; since one knows u_x, ${}^1u_x, \dots,$ ${}^{n-2}u_x$, one will know $\overset{1}{u}_x$, $\overset{2}{u}_x$,... until $\overset{n-1}{u_x}$ exclusively. In order to determine $\overset{n-1}{u_x}$, it is necessary to integrate the equation

$$ {}^{n-2}T_x = S_x \, {}^{n-2}T_{x-1} + X_x, $$

by supposing $X_x = 0$, this which will be easy by Problem I if one knows S_x. In order to find it, I observe that, in equation (D′), the coefficient of T_{x-1} is

$$ H_x - \delta_x = H_x - \frac{u_x}{u_{x-1}}, $$

because of

$$ \delta_x = \frac{u_x}{u_{x-1}}. $$

Similarly the one of ${}^1T_{x-1}$, in equation (D″), is

$$ H_x - \frac{u_x}{u_{x-1}} - \frac{\overset{1}{u}_x}{\overset{1}{u}_{x-1}}, $$

and thus in sequence; hence,

$$ S_x = H_x - \frac{u_x}{u_{x-1}} - \frac{\overset{1}{u}_x}{\overset{1}{u}_{x-1}} - \dots - \frac{\overset{n-2}{u}_x}{\overset{n-2}{u}_{x-1}}. $$

If, instead of knowing the integral of the equation

$$ y_x = H_x y_{x-1} + \dots + {}^{n-1}H_x y_{x-n}, $$

one knows a number n or $n-1$ of values for α_x, in equation (E), the preceding formulas will serve equally, because δ_x, ${}^1\delta_x, \dots$ being these values, one has

$$ u_x = \nabla\delta_x, \qquad {}^1u_x = \nabla\, {}^1\delta_x, \qquad \dots $$

VII.

Formula (O) has not at all yet the total degree of simplicity that the complete integral of y_x can have, because one has seen (Art. IV) that this integral has the following form

$$y_x = Au_x + {}^1A^1u_x + \ldots + {}^{n-1}A^{n-1}u_x + L_x;$$

it is necessary therefore to restore equation (O) to this form; for this, I divide equation (O) by u_x, and I conclude from it, by differentiating it,

$$\Delta\frac{y_{x-1}}{u_{x-1}} = \Delta\frac{{}^1u_{x-1}}{u_{x-1}}\left\{{}^1A + \sum\Delta\frac{{}^1\overset{1}{u}_x}{\overset{1}{u}_x}\left[{}^2A\ldots + \sum\Delta\frac{{}^1\overset{n-2}{u}_{x+n-3}}{\overset{n-2}{u}_{x+n-3}}\left({}^{n-1}A + \sum\frac{X_{x+n-1}}{\overset{n-1}{u}_{x+n-1}}\right)\ldots\right]\right\},$$

whence one will conclude, by dividing by $\Delta\frac{{}^1u_{x-1}}{u_{x-1}}$ and differentiating,

$$\Delta\frac{\Delta\frac{y_{x-2}}{u_{x-2}}}{\Delta\frac{{}^1u_{x-2}}{u_{x-2}}} = \Delta\frac{{}^1\overset{1}{u}_{x-1}}{\overset{1}{u}_{x-1}}[{}^2A + \ldots].$$

One will have therefore, by continuing to differentiate thus, an equation of this form

$${}^{n-1}A + \sum\frac{X_{x-1}}{\overset{n-1}{u}_{x-1}} = \gamma_x y_x + {}^1\gamma_x y_{x-1} + {}^2\gamma_x y_{x-2} + \ldots + {}^{n-1}\gamma_x y_{x-n+1},$$

$\gamma_x,\ {}^1\gamma_x, \ldots$ being some functions of $u_x,\ {}^1u_x, \ldots$ and of their finite differences. I observe now that, in order to form the values of $\overset{1}{u}_x,\ \overset{2}{u}_x,\ \overset{3}{u}_x, \ldots,$ I have considered (preceding Article) the quantities $u_x,\ {}^1u_x,\ {}^2u_x, \ldots$ in this order

$$u_x,\ {}^1u_x,\ {}^2u_x, \ldots, {}^{n-1}u_x;$$

but if, instead of that, I had considered them in the following order

$${}^1u_x,\ u_x,\ {}^2u_x, \ldots, {}^{n-1}u_x,$$

I would arrive to the following equation

$${}^{n-1}A + \sum\frac{X_{x+1}}{\left(\overset{n-1}{u}_{x+1}\right)} = (\gamma_x)y_x + ({}^1\gamma_x)y_{x-1} + \ldots + ({}^{n-1}\gamma_x)y_{x-n+1},$$

$\left(\overset{n-1}{u}_x\right), (\gamma_x), \ldots$ being that which $\overset{n-1}{u}_x, \gamma_x, \ldots$ become when one changes u_x into ${}^1u_x$, and ${}^1u_x$ into u_x. If I had supposed $X_{x+1} = 0$, I would have arrived to the two equations

$$
\begin{aligned}
{}^{n-1}A &= \gamma_x y_x + {}^{1}\gamma_x y_{x-1} + \ldots + {}^{n-1}\gamma_x y_{x-n+1},\\
{}^{n-1}A &= (\gamma_x) y_x + ({}^{1}\gamma_x) y_{x-1} + \ldots + ({}^{n-1}\gamma_x) y_{x-n+1},
\end{aligned}
$$

in which the constant ${}^{n-1}A$ is visibly the same, since I have supposed, in order to form the one and the other equation, that the complete value of y_x is

$$y_x = Au_x + {}^{1}A\,{}^{1}u_x + \ldots + {}^{n-1}A\,{}^{n-1}u_x.$$

One will have therefore, by comparing these two equations,

$$
\begin{aligned}
&\gamma_x y_x + {}^{1}\gamma_x y_{x-1} + \ldots + {}^{n-1}\gamma_x y_{x-n+1}\\
&\quad = (\gamma_x) y_x + ({}^{1}\gamma_x) y_{x-1} + \ldots + ({}^{n-1}\gamma_x) y_{x-n+1},
\end{aligned}
$$

an equation which must be an identity; because, if it were not, this equation being differential of order $n-1$ would have however for the complete integral

$$y_x = Au_x + \ldots + {}^{n-1}A\,{}^{n-1}u_x,$$

an equation which contains n arbitrary constants, this which would be absurd (Art. II).

One has therefore

$${}^{n-1}A + \sum \frac{X_{x+1}}{\binom{n-1}{u_{x+1}}} = {}^{n-1}A + \sum \frac{X_{x+1}}{\overset{n-1}{u_{x+1}}},$$

hence

$$\binom{n-1}{u_{x+1}} = \overset{n-1}{u_{x+1}}.$$

Thus the expression of $\overset{n-1}{u_x}$ remains always the same, whether one changes u_x into ${}^{1}u_x$, and ${}^{1}u_x$ into u_x; one will be assured in the same manner that if in $\overset{n-1}{u_x}$ one changes u_x into ${}^{2}u_x$, and ${}^{2}u_x$ into u_x; or ${}^{1}u_x$ into ${}^{2}u_x$, and ${}^{2}u_x$ into ${}^{1}u_x$, and generally ${}^{k}u_x$ into ${}^{i}u_x$, and ${}^{i}u_x$ into ${}^{k}u_x$, k and i being less than $n-1$, the expression $\overset{n-1}{u_x}$ will always remain the same, and that thus, whatever order that one gives to the quantities u_x, ${}^{1}u_x$, ${}^{2}u_x, \ldots$ in order to form $\overset{n-1}{u_x}$, this expression will remain always the same, provided that ${}^{n-1}u_x$ is considered as the last of these quantities.

I make $\overset{n-1}{u}_{x+1} = {}^{n-1}z_{x+1}$; next, instead of considering ${}^{n-1}u_x$ as the last of the quantities u_x, ${}^1u_x, \dots$ I suppose actually that ${}^{n-2}u_x$ is this last; let ${}^{n-2}z_{x+1}$ be that which becomes then ${}^{n-1}z_{x+1}$, that is to say when one changes ${}^{n-2}u_x$ into ${}^{n-1}u_x$, and ${}^{n-1}u_x$ into ${}^{n-2}u_x$. One will have, by a process similar to the preceding,

$$ {}^{n-2}A + \sum \frac{X_{x+1}}{{}^{n-2}z_{x+1}} = \underline{\gamma}_x y_x + {}^1\underline{\gamma}_x y_{x-1} + \dots + {}^{n-1}\underline{\gamma}_x y_{x-n+1}, $$

$\underline{\gamma}$, ${}^1\underline{\gamma}_x$, $\dots$ being that which γ_x, ${}^1\gamma_x, \dots$ become when one changes ${}^{n-1}u_x$ into ${}^{n-2}u_x$ and ${}^{n-2}u_x$ into ${}^{n-1}u_x$; one will have similarly

$$ {}^{n-3}A + \sum \frac{X_{x+1}}{{}^{n-3}z_{x+1}} = \underline{\underline{\gamma}}_x y_x + {}^1\underline{\underline{\gamma}}_x y_{x-1} + \dots + {}^{n-1}\underline{\underline{\gamma}}_x y_{x-n+1}, $$

${}^{n-3}z_{x+1}$, $\underline{\underline{\gamma}}_x$, ${}^1\underline{\underline{\gamma}}_x$ being that which ${}^{n-1}z_{x+1}$, γ_x, ${}^1\gamma_x, \dots$ become when one changes ${}^{n-1}u_x$ into ${}^{n-3}u_x$ and ${}^{n-2}u_x$ into ${}^{n-1}u_x$. This set, by disposing in the following order all the equations that one can form thus

$$ (>) \quad \begin{cases} {}^{n-1}A + \sum \dfrac{X_{x+1}}{{}^{n-1}z_{x+1}} = \gamma_x y_x + {}^1\gamma_x y_{x-1} + {}^2\gamma_x y_{x-2} + \dots + {}^{n-1}\gamma_x y_{x-n+1}, \\ {}^{n-2}A + \sum \dfrac{X_{x+1}}{{}^{n-2}z_{x+1}} = \underline{\gamma}_x y_x + {}^1\underline{\gamma}_x y_{x-1} + {}^2\underline{\gamma}_x y_{x-2} + \dots + {}^{n-1}\underline{\gamma}_x y_{x-n+1}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ A + \sum \dfrac{X_{x+1}}{z_{x+1}} = \dfrac{\gamma_x}{n-1} y_x + \dfrac{{}^1\gamma_x}{n-1} y_{x-1} + \dfrac{{}^2\gamma_x}{n-1} y_{x-2} + \dots + \dfrac{{}^{n-1}\gamma_x}{n-1} y_{x-n+1}, \end{cases} $$

and adding them altogether, after having multiplied the first by ${}^{n-1}u_x$, the second by ${}^{n-2}u_x$, etc., finally the last by u_x, one will have an equation of this form

$$\lambda_x y_x + \ldots + {}^{n-1}\lambda_x y_{x-n+1} = u_x\left(A + \sum \frac{X_{x+1}}{z_{x+1}}\right)$$
$$+ {}^1u_x\left({}^1A + \sum \frac{X_{x+1}}{{}^1z_{x+1}}\right)$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ {}^{n-1}u_x\left({}^{n-1}A + \sum \frac{X_{x+1}}{{}^{n-1}z_{x+1}}\right),$$

this which gives, by making $X_{x+1} = 0$,

$$\lambda_x y_x + {}^1\lambda_x y_{x-1} + \ldots + {}^{n-1}\lambda_x y_{x-n+1} = Au_x + {}^1A^1u_x + \ldots + {}^{n-1}A^{n-1}u_x;$$

but one has in this case

$$y_x = Au_x + {}^1A^1u_x + \ldots,$$

hence

$$y_x = \lambda_x y_x + {}^1\lambda_x y_{x-1} + \ldots + {}^{n-1}\lambda_x y_{x-n+1}.$$

Now this equation must be an identity, because otherwise, although of order $n-1$, its integral would contain the n arbitrary constants which the complete expression of y_x contains; one has therefore for the complete integral of equation (B) of Problem II, whatever be X_x,

$$y_x = u_x\left(A + \sum \frac{X_{x+1}}{z_{x+1}}\right)$$
$$+ {}^1u_x\left({}^1A + \sum \frac{X_{x+1}}{{}^1z_{x+1}}\right)$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ {}^{n-1}u_x\left({}^{n-1}A + \sum \frac{X_{x+1}}{{}^{n-1}z_{x+1}}\right),$$

Thence results this quite simple rule, in order to have the complete integral of the equation

$$y_x = H_x y_{x-1} + {}^1H_x y_{x-2} + \ldots + {}^{n-1}H_x y_{x-n} + X_x,$$

when one knows how to integrate this here

$$y_x = H_x y_{x-1} + {}^1H_x y_{x-2} + \ldots + {}^{n-1}H_x y_{x-n}.$$

Let

$$y_x = Au_x + {}^1A\,{}^1u_x + {}^2A\,{}^2u_x + \ldots + {}^{n-1}A\,{}^{n-1}u_x$$

be the integral of this last, and let one make

$$\begin{array}{lll}
\overset{1}{u}_x = u_x \Delta \dfrac{{}^1u_{x-1}}{u_{x-1}}, & \overset{2}{u}_x = \overset{1}{u}_x \Delta \dfrac{{}^1\overset{1}{u}_{x-1}}{\overset{1}{u}_{x-1}}, & \overset{3}{u}_x = \overset{1}{u}_x \Delta \dfrac{{}^1\overset{2}{u}_{x-1}}{\overset{2}{u}_{x-1}}, \\
{}^1\overset{1}{u}_x = u_x \Delta \dfrac{{}^2u_{x-1}}{u_{x-1}}, & {}^1\overset{2}{u}_x = \overset{1}{u}_x \Delta \dfrac{{}^2\overset{1}{u}_{x-1}}{\overset{1}{u}_{x-1}}, & \ldots\ldots\ldots\ldots\ldots\ldots, \\
{}^2\overset{1}{u}_x = u_x \Delta \dfrac{{}^3u_{x-1}}{u_{x-1}}, & {}^2\overset{2}{u}_x = \overset{1}{u}_x \Delta \dfrac{{}^3\overset{1}{u}_{x-1}}{\overset{1}{u}_{x-1}}, & \\
\ldots\ldots\ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots\ldots\ldots, &
\end{array}$$

until one arrives to form ${}^n\overset{-1}{u}_x{}^1$, or ${}^n\overset{-1}{u}_x{}^1 = {}^{n-1}z_x$. If, in the expression of ${}^{n-1}z_x$, one changes ${}^{n-1}u_x$ into ${}^{n-2}u_x$ and ${}^{n-2}u_x$ into ${}^{n-1}u_x$, one will form ${}^{n-2}z_x$; if, in the same expression of ${}^{n-1}z_x$, one changes ${}^{n-1}u_x$ into ${}^{n-3}u_x$, and reciprocally ${}^{n-3}u_x$ into ${}^{n-1}u_x$, one will form ${}^{n-3}z_x$, and thus in sequence; the complete integral of equation

(B) $$y_x = H_x y_{x-1} + {}^1H_x y_{x-2} + \cdots + {}^{n-1}H_x y_{x-n} + X_x$$

will be

$$
\text{(H)}\qquad
\begin{cases}
y_x = u_x\left(A + \sum \dfrac{X_{x+1}}{z_{x+1}}\right) \\
\qquad + {}^1u_x\left({}^1A + \sum \dfrac{X_{x+1}}{{}^1z_{x+1}}\right) \\
\qquad + \dots\dots\dots\dots\dots\dots\dots \\
\qquad + {}^{n-1}u_x\left({}^{n-1}A + \sum \dfrac{X_{x+1}}{{}^{n-1}z_{x+1}}\right).
\end{cases}
$$

VIII.

I take now the equations (>) of the preceding Article; they give

$$
\begin{cases}
{}^{n-1}A + \sum \dfrac{X_{x+2}}{{}^{n-1}z_{x+2}} = \gamma_{x+1}y_{x+1} + \dots + {}^{n-1}\gamma_{x+1}y_{x-n+2}, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\
A + \sum \dfrac{X_{x+2}}{z_{x+2}} = \dfrac{\gamma_{x+1}}{n-1}y_{x+1} + \dots + \dfrac{{}^{n-1}\gamma_{x+1}}{n-1}y_{x-n+2};
\end{cases}
$$

if one multiplies the first by ${}^{n-1}u_x$, the second by ${}^{n-2}u_x$, $\dots$, one will have, by adding them together, an equation of this form

$$\lambda_x y_{x+1} + {}^1\lambda_x y_{x+2} + \dots + {}^{n-1}\lambda_x y_{x-n+2} = Au_x + {}^1A\,{}^1u_x + \dots + {}^{n-1}A\,{}^{n-1}u_x;$$

therefore

$$\lambda_x y_{x+1} + {}^1\lambda_x y_{x+2} + \dots + {}^{n-1}\lambda_x y_{x-n+2} = y_x,$$

an equation which must be an identity; hence,

$$
\begin{aligned}
y_x = u_x&\left(A + \sum \frac{X_{x+2}}{z_{x+2}}\right) \\
&+ {}^1u_x\left({}^1A + \sum \frac{X_{x+2}}{{}^1z_{x+2}}\right) \\
&+ \dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

One will find similarly

$$y_x = u_x\left(A + \sum \frac{X_{x+3}}{z_{x+3}}\right) + {}^1u_x\left({}^1A + \sum \frac{X_{x+3}}{{}^1z_{x+3}}\right) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and thus in sequence until one arrives to this last equation inclusively,

$$y_x = u_x\left(A + \sum \frac{X_{x+n}}{z_{x+n}}\right) + {}^1u_x\left({}^1A + \sum \frac{X_{x+n}}{{}^1z_{x+n}}\right) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

All these equations being the complete integral of equation (B) are identically the same; in comparing them together, one will form the following equations

$$\frac{u_x}{z_{x+1}} + \frac{{}^1u_x}{{}^1z_{x+1}} + \ldots + \frac{{}^{n-1}u_x}{{}^{n-1}z_{x+1}} = 0.$$

$$\frac{u_x}{z_{x+2}} + \frac{{}^1u_x}{{}^1z_{x+2}} + \ldots + \frac{{}^{n-1}u_x}{{}^{n-1}z_{x+2}} = 0.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\frac{u_x}{z_{x+n-1}} + \frac{{}^1u_x}{{}^1z_{x+n-1}} + \ldots + \frac{{}^{n-1}u_x}{{}^{n-1}z_{x+n-1}} = 0.$$

IX.

The integration of equation (B) of Problem II being reduced to the integration of this same equation when $X_x = 0,$ there is no longer a question to resolve the problem but to integrate this here, but this appears very difficult in general; thus I will limit myself to the particular cases. Here is of it a quite expanded, in which the integration succeeds, and which embraces all the cases already known; it is the one in which one has

$$(\mathrm{B}') \quad y_x = C\phi_x y_{x-1} + {}^1C\phi_x\phi_{x-1}y_{x-2} + \ldots + {}^{n-1}C\phi_x\phi_{x-1}\ldots\phi_{x-n+1}y_{x-n}.$$

If $\phi_x = 1$, one will have the equation of the recurrent sequences.

Equation (E) of Article IV becomes in this case

$$(\mathrm{E}') \qquad 0 = 1 - \frac{C\phi_x}{\alpha_x} - \frac{{}^1C\phi_x\phi_{x-1}}{\alpha_x\alpha_{x-1}} - \ldots - \frac{{}^{n-1}C\phi_x\phi_{x-1}\ldots\phi_{x-n+1}}{\alpha_x\ldots\alpha_{x-n+1}}.$$

Now (Art. IV), it suffices in order to integrate equation (B′) to know a number n of values for α_x in equation (E′). Let therefore $\alpha_x = a\phi_x,\ a$ being constant, and equation (E′) will give

$$(h) \qquad a^n = Ca^{n-1} + {}^1Ca^{n-2} + {}^2Ca^{n-3} + \ldots + {}^{n-1}C;$$

whence one will have a number n of values for a, and consequently for α_x, since $\alpha_x = a\phi_x$.

Let p, ${}^1p$, ${}^2p$, $\ldots$, ${}^{n-1}p$ be the different values of a in equation (*h*). One will have (Art. IV)

$$\delta_x = p\phi_x, \quad {}^1\delta_x = {}^1p\phi_x, \quad {}^2\delta_x = {}^2p\phi_x, \quad \ldots$$

Now one has (Art. V)

$$\begin{aligned} u_x &= \nabla\delta_x = \phi_1\phi_2\phi_3\ldots\phi_x p^x, \\ {}^1u_x &= \nabla{}^1\delta_x = \phi_1\phi_2\phi_3\ldots\phi_x {}^1p^x, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

The complete integral of equation (B′) is therefore

$$y_x = \phi_1\phi_2\phi_3\ldots\phi_x(Ap^x + {}^1A\,{}^1p^x + \ldots + {}^{n-1}A\,{}^{n-1}p^x).$$

One will determine the arbitrary constants A, ${}^1A$, ${}^2A, \ldots$ by means of n values of y_x, under as many particular assumptions for x. Let

$$y_1 = M, \quad y_2 = {}^1M, \quad \ldots, \quad y_n = {}^{n-1}M;$$

and one will have

$$\frac{M}{\phi_1} = Ap + {}^1A^1p + {}^2A^2p + \ldots + {}^{n-1}A^{n-1}p,$$
$$\frac{{}^1M}{\phi_1\phi_2} = Ap^2 + {}^1A^1p^2 + {}^2A^2p^2 + \ldots + {}^{n-1}A^{n-1}p^2,$$
$$\frac{{}^2M}{\phi_1\phi_2\phi_3} = Ap^3 + {}^1A^1p^3 + {}^2A^2p^3 + \ldots + {}^{n-1}A^{n-1}p^3,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$\frac{{}^{n-1}M}{\phi_1\phi_2\ldots\phi_n} = Ap^n + {}^1A^1p^n + {}^2A^2p^n + \ldots + {}^{n-1}A^{n-1}p^n,$$

In order to resolve these equations, one can make use of the ordinary methods of elimination: but here is one of them which appears to me simpler.

I multiply the first equation by ${}^{n-1}p$, and I subtract it from the second; I multiply similarly the second by ${}^{n-1}p$, and I subtract it from the third, and thus in sequence, this which produces the following equations:

$$\frac{{}^1M}{\phi_1\phi_2} - \frac{M}{\phi_1}{}^{n-1}p = Ap(p - {}^{n-1}p) + {}^1A^1p({}^1p - {}^{n-1}p) + \ldots + {}^{n-2}A^{n-2}p({}^{n-2}p - {}^{n-1}p),$$
$$\frac{{}^2M}{\phi_1\phi_2\phi_3} - \frac{{}^1M}{\phi_1\phi_2}{}^{n-1}p = Ap^2(p - {}^{n-1}p) + {}^1A^1p^2({}^1p - {}^{n-1}p) + \ldots + {}^{n-2}A^{n-2}p^2({}^{n-2}p - {}^{n-1}p),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$\frac{{}^{n-1}M}{\phi_1\ldots\phi_n} - \frac{{}^{n-2}M}{\phi_1\ldots\phi_{n-1}}{}^{n-1}p = Ap^{n-1}(p - {}^{n-1}p) + \ldots + {}^{n-2}A^{n-2}p^{n-1}({}^{n-2}p - {}^{n-1}p),$$

I multiply again the first of these equations by ${}^{n-2}p$, and I subtract it from the second; I multiply similarly the second by ${}^{n-2}p$, and I subtract it from the third, this which gives

$$\frac{{}^2M}{\phi_1\phi_2\phi_3} - \frac{{}^1M}{\phi_1\phi_2}({}^{n-1}p - {}^{n-2}p) + \frac{M}{\phi_1}{}^{n-1}p^{n-2}p$$
$$= Ap(p - {}^{n-1}p)(p - {}^{n-2}p)$$
$$+ {}^1A^1p({}^1p - {}^{n-1}p)({}^1p - {}^{n-2}p)$$
$$+ \dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$+ {}^{n-3}A^{n-3}p({}^{n-3}p - {}^{n-1}p)({}^{n-3}p - {}^{n-2}p),$$
$$\frac{{}^3M}{\phi_1\phi_2\phi_3\phi_4} - \frac{{}^2M}{\phi_1\phi_2\phi_3}({}^{n-1}p - {}^{n-2}p) + \frac{{}^1M}{\phi_1\phi_2}{}^{n-1}p^{n-3}p$$
$$= Ap^2(p - {}^{n-1}p)(p - {}^{n-2}p)$$
$$+ \dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$+ {}^{n-3}A^{n-3}p^2({}^{n-3}p - {}^{n-1}p)({}^{n-3}p - {}^{n-2}p),$$
$$+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots;$$

by operating on these last equations, as on the previous, one will have

$$\frac{{}^3M}{\phi_1\phi_2\phi_3\phi_4} - \frac{{}^2M}{\phi_1\phi_2\phi_3}({}^{n-1}p - {}^{n-2}p + {}^{n-3}p)$$
$$+ \frac{{}^1M}{\phi_1\phi_2}[({}^{n-2}p + {}^{n-1}p){}^{n-3}p + {}^{n-1}p^{n-2}p] - \frac{M}{\phi_1}{}^{n-1}p^{n-2}p^{n-3}p$$
$$= Ap(p - {}^{n-1}p)(p - {}^{n-2}p)(p - {}^{n-3}p) + \dots,$$

and thus in sequence.

Thence it is easy to conclude that, if one names:

f the sum of the quantities ${}^1p$, ${}^2p$, ${}^3p$, $\dots$, ${}^{n-1}p$,
h the sum of their products two by two,
i the sum of their products three by three,
q the sum of their products four by four, etc.,
${}^1f$ the sum of the quantities p, ${}^2p$, ${}^3p$, $\dots$, ${}^{n-1}p$,
${}^1h$ the sum of their products two by two,
${}^1i$ the sum of their products three by three, etc.,

and thus in sequence, one will have

$$A = \frac{{}^{n-1}M - \phi_n f^{n-2}M + \phi_n\phi_{n-1}h^{n-3}M - \phi_n\phi_{n-1}\phi_{n-2}i^{n-4}M + \ldots}{\phi_1\phi_2\phi_3\ldots\phi_n p(p - {}^1p)(p - {}^2p)(p - {}^3p)\ldots},$$
$${}^1A = \frac{{}^{n-1}M - \phi_n {}^1f^{n-2}M + \phi_n\phi_{n-1}{}^1h^{n-3}M - \ldots}{\phi_1\phi_2\phi_3\ldots\phi_n {}^1p({}^1p - p)({}^1p - {}^2p)({}^1p - {}^3p)\ldots}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

One can determine in a quite simple manner the quantities f, h, i, q, ${}^1f$, ${}^1h$, ${}^1i$, ${}^1q$, $\ldots$; I take for this the equation

$$(h) \qquad a^n - Ca^{n-1} - {}^1C^{n-2} - \ldots - {}^{n-1}C = 0;$$

I divide it by $a - p$, and the resulting equation will be

$$a^{n-1} - fa^{n-2} - ha^{n-3} - ia^{n-4} + qa^{n-5} + \ldots = 0.$$

I multiply this result by $a - p$, and I will have the following equation

$$a^n - (p + f)a^{n-1} + (pf + h)a^{n-2} - (ph + i)a^{n-3} + \ldots = 0;$$

I compare it with equation (*h*), and I conclude from it

$$\begin{aligned} f &= +C - p, \\ h &= -{}^1C - pf, \\ i &= +{}^2C - ph, \\ &\ldots\ldots\ldots\ldots, \end{aligned}$$

and, consequently,

$$\begin{aligned} {}^1f &= +C - {}^1p, \\ {}^1h &= -{}^1C - {}^1p.{}^1f, \\ &\ldots\ldots\ldots\ldots, \end{aligned}$$

I suppose until here that all the roots of equation (*h*) are unequal, but it can happen that one or many of these roots are equal among themselves; here is in this case the method that it is necessary to follow.

I suppose that one has $p = {}^1p$; one will make ${}^1p = p + dp$, and the equation

$$y_x = \phi_1\phi_2\phi_3\ldots\phi_x(Ap^x + {}^1A{}^1p^x + {}^2A{}^2p^x + \ldots + {}^{n-1}A{}^{n-1}p^x)$$

will give, by reducing $(p+dp)^x$ into series,

$$y_x = \phi_1\phi_2\ldots\phi_x\left\{p^x\left[A + {}^1A\left(1 + \frac{xdp}{p} + \frac{x(x-1)}{1.2}\frac{dp^2}{p^2} + \ldots\right)\right] + {}^2A^2p^x + \ldots\right\}.$$

Let

$$A + {}^1A = B \quad \text{and} \quad {}^1A\frac{dp}{p} = D,$$

B and D being some arbitrary and finite constants; ${}^1A$ will be therefore infinitely great of order $\frac{1}{dp}$; ${}^1A\frac{dp^2}{p^2}$, ${}^1A\frac{dp^3}{p^3}, \ldots$ will be infinitely small. Hence

$$y_x = \phi_1\phi_2\ldots\phi_x[p^x(B + Dx) + {}^2A^2p^x + {}^3A^3p^x + \ldots].$$

If, moreover, one has $p = {}^2p$, one will make ${}^2p = p + dp$ in this expression of y_x, and one will have

$$y_x = \phi_1\phi_2\ldots\phi_x\left\{p^x\left[B + {}^2A + \left(D + {}^2A\frac{dp}{p}\right)x + {}^2A\frac{dp^2}{p^2}\frac{x(x-1)}{1.2} + \ldots\right] + {}^3A^3p^x + \ldots\right\}.$$

Let

$${}^2A + B = {}^1B, \qquad D + {}^2A\frac{dp}{p} = {}^1D \qquad \text{and} \qquad {}^2\mathrm{A}\frac{dp^2}{p^2} = {}^1E,$$

${}^1B$, ${}^1D$ and ${}^1E$ being some arbitrary and finite constants; one will have

$$y_x = \phi_1\phi_2\ldots\phi_x\left\{p^x\left[{}^1B + {}^1Dx + {}^1E\frac{x(x-1)}{1.2} + \ldots\right] + {}^3A^3p^x + \ldots\right\};$$

if moreover one had $p = {}^3p$, one would have

$$y_x = \phi_1\phi_2\ldots\phi_x\left\{p^x\left[{}^2B + {}^2Dx + {}^2E\frac{x(x-1)}{1.2} + {}^2F\frac{x(x-1)(x-2)}{1.2.3}\right] + {}^4A^4p^x + \ldots\right\},$$

and thus in sequence; one would determine the arbitrary constants, at least of n particular values of y_x.

If equation (*h*) has two imaginary roots p and ${}^1p$, one will make

$$p = a + b\sqrt{-1} \qquad \text{and} \qquad {}^1p = a - b\sqrt{-1}.$$

Let

$$\frac{a}{\sqrt{aa+bb}} = \cos q \qquad \text{and} \qquad \frac{b}{\sqrt{aa+bb}} = \sin q;$$

one will have

$$\begin{aligned} Ap^x + {}^1A\,{}^1p^x &= (aa+bb)^{\frac{x}{2}}[A(\cos q + \sqrt{-1}\sin q)^x + {}^1A(\cos q - \sqrt{-1}\sin q)^x] \\ &= (aa+bb)^{\frac{x}{2}}[(A + {}^1A)\cos qx + (A - {}^1A)\sqrt{-1}\sin qx)^x] \end{aligned}$$

because

$$(\cos q \pm \sqrt{-1}\sin q)^x = \cos qx \pm \sqrt{-1}\sin qx.$$

Let

$$A + {}^1A = B \quad \text{and} \quad (A - {}^1A)\sqrt{-1} = {}^1B,$$

B and ${}^1B$ being reals; one will have

$$Ap^x + {}^1A\,{}^1p^x = (aa+bb)^{\frac{x}{2}}(B\cos qx + {}^1B\sin qx);$$

one will have therefore then

$$y^x = \phi_1\phi_2\ldots\phi_x\big[(aa+bb)^{\frac{x}{2}}(B\cos qx + {}^1B\sin qx) + {}^2A\,{}^2p^x + \ldots\big];$$

it will be the same process if there were a greater number of imaginaries.

If one supposes, in the preceding calculations, $\phi_x = 1$, one will have the case of the recurrent sequences. Thence results this theorem:

If one names Y_x the general term of a recurrent sequence, such as one has

$$Y_x = CY_{x-1} + {}^1CY_{x-2} + \ldots + {}^{n-1}CY_{x-n},$$

the general term of a sequence such as one has

$$y_x = C\phi_x y_{x-1} + {}^1C\phi_x\phi_{x-1}y_{x-2} + \ldots + {}^{n-1}C\phi_x\ldots\phi_{x-n+1}y_{x-n},$$

and in which the arbitrary constants which arrive by integrating are the same as in the preceding, will be

$$y_x = \phi_1\phi_2\ldots\phi_x Y_x.$$

This is it of which it is easy to be assured besides; because, if one substitutes this value of y_x into the equation

$$y_x = C\phi_x y_{x-1} + \dots,$$

one will have

$$\phi_1\phi_2\dots\phi_x Y_x = C\phi_1\phi_2\dots\phi_x Y_{x-1} + \dots,$$

hence

$$Y_x = CY_{x-1} + {}^1CY_{x-2} + \dots,$$

an equation which holds by assumption.

X.

When one has, by the preceding article, the integral of the equation

$$y_x = C\phi_x y_{x-1} + {}^1C\phi_x\phi_{x-1}y_{x-2} + \dots + {}^{n-1}C\phi_x\dots\phi_{x-n+1}y_{x-n} + X_x,$$

by supposing $X_x = 0$, it is easy to conclude this same integral, X_x being anything. For this, I observe that, since, X_x being null, one has

$$y_x = \phi_1\phi_2\dots\phi_x(Ap^x + {}^1A\,{}^1p^x + \dots {}^{n-1}A\,{}^{n-1}p^x),$$

one will have, by Article V,

$$\begin{aligned} u_x &= \phi_1\phi_2\phi_3\dots\phi_x p^x, \\ {}^1u_x &= \phi_1\phi_2\phi_3\dots\phi_x\,{}^1p^x, \\ {}^2u_x &= \phi_1\phi_2\phi_3\dots\phi_x\,{}^2p^x, \\ &\dots\dots\dots\dots\dots\dots\dots, \end{aligned}$$

whence one will conclude, by Article VII,

$$\overset{1}{u}_x = \phi_1\phi_2\ldots\phi_x p^x \Delta \frac{{}^1p^{x-1}}{p^{x-1}} = \phi_1\phi_2\ldots\phi_x({}^1p - p){}^1p^{x-1},$$
$$ {}^1\overset{1}{u}_x = \phi_1\phi_2\ldots\phi_x({}^2p - p){}^2p^{x-1},$$
$$ {}^2\overset{1}{u}_x = \phi_1\phi_2\ldots\phi_x({}^3p - p){}^3p^{x-1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$\overset{2}{u}_x = \phi_1\phi_2\ldots\phi_x({}^2p - p)({}^2p - {}^1p){}^2p^{x-2},$$
$$ {}^1\overset{2}{u}_x = \phi_1\phi_2\ldots\phi_x({}^3p - p)({}^3p - {}^1p){}^3p^{x-2},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$\overset{3}{u}_x = \phi_1\phi_2\ldots\phi_x({}^3p - p)({}^3p - {}^1p)({}^3p - {}^2p){}^3p^{x-3},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and thus in sequence, hence

$$\overset{n-1}{u}_{x+1} = {}^{n-1}z_{x+1} = \phi_1\phi_2\ldots\phi_{x+1}({}^{n-1}p - p)({}^{n-1}p - {}^1p)({}^{n-1}p - {}^2p)\ldots{}^{n-1}p^{x-n+2};$$

similarly

$$ {}^{n-2}z_{x+1} = \phi_1\phi_2\ldots\phi_{x+1}({}^{n-2}p - p)({}^{n-2}p - {}^1p)\ldots{}^{n-2}p^{x-n+2},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots;$$

whence one will conclude, by substituting these values into formula (H) of article VII and making $X_x = \phi_1\phi_2\ldots\phi_x{}^1X_x$ for abridgment,

$$\begin{aligned} y_x = {} & \frac{\phi_1\phi_2\ldots\phi_x}{(p-{}^1p)(p-{}^2p)(p-{}^3p)\ldots}p^{x+n-1}\left(G + \sum\frac{{}^1X_{x+1}}{p^{x+1}}\right) \\ & + \frac{\phi_1\phi_2\ldots\phi_x}{({}^1p-p)({}^1p-{}^2p)\ldots}{}^1p^{x+n-1}\left({}^1G + \sum\frac{{}^1X_{x+1}}{{}^1p^{x+1}}\right) \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

If $p = {}^1p$, one will make ${}^1p = p + dp$. Let $K = \frac{1}{(p-{}^2p)(p-{}^3p)\ldots}$, and one will have

$$y_x = \phi_1\phi_2\ldots\phi_x p^{x+n-1}\left\{B + Dx - \frac{K}{p}\sum\frac{{}^1X_{x+1}}{p^{x+1}}(x+1) + \left[\frac{dK}{dp} + \frac{K}{p}(x+n-1)\right]\sum\frac{{}^1X_{x+1}}{p^{x+1}}\right\}$$
$$+ \frac{\phi_1\phi_2\ldots\phi_x}{({}^2p-p)^2({}^2p-{}^3p)\ldots}{}^2p^{x+n-1}\left({}^2G + \sum\frac{{}^1X_{x+1}}{{}^2p^{x+1}}\right),$$

B and D being two arbitrary constants.

If, moreover, one has $p = {}^2p$, one will have, in this last expression of y_x, ${}^2p = p + dp$, and thus in sequence.

One can therefore integrate generally all the differential equations contained in the following formula

$$y_x = C\phi_x y_{x-1} + {}^1C\phi_x\phi_{x-1}y_{x-2} + \ldots + \mathrm{X}_x;$$

whence it results that, if one designates by θ_x any function whatsoever of x, the following equation

$$\theta_x y_x = C\theta_{x-1}\phi_x y_{x-1} + {}^1C\theta_{x-2}\phi_x\phi_{x-1}y_{x-2} + \ldots + \mathrm{X}_x$$

is generally integrable, since by making $\theta_x y_x = t_x$ this equation is of the same form as the preceding.

XI.

Here is now another kind of linear differential equations, of which the order depends on the variable x; let, for example,

$$\begin{aligned} y_x = {} & a_{x-1}y_{x-1} + b_{x-2}y_{x-2} + f_{x-3}y_{x-3} + X_x \\ & + a_{x-4}y_{x-4} + b_{x-5}y_{x-5} + f_{x-6}y_{x-6} \\ & + a_{x-7}y_{x-7} + b_{x-8}y_{x-8} + \ldots \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + a_3y_3 + b_2y_2 + f_1y_1. \end{aligned}$$

It is easy to bring these equations back to the form of equation (B) of problem II, because one has

$$\begin{aligned} y_{x-3} = {} & a_{x-4}y_{x-4} + b_{x-5}y_{x-5} + f_{x-6}y_{x-6} + X_{x-3} \\ & + a_{x-7}y_{x-7} + b_{x-8}y_{x-8} + \ldots \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + a_3y_3 + b_2y_2 + f_1y_1. \end{aligned}$$

If one subtracts this last equation from the preceding, one will have

$$y_x = a_{x-1}y_{x-1} + b_{x-2}y_{x-2} + (f_{x-3}+1)y_{x-3} + X_x - X_{x-3},$$

an equation contained in equation (B).

XII.

Presently here is a quite extended use of the integral Calculus in the finite differences, in order to determine directly the general expression of the quantities subject to a certain law which serves to form them, an expression that until here it seems to me that one has always sought to draw by way of induction, a method not only indirect, but which, moreover, must be often at fault.

In order to make myself better understood, I take the following example:

Let x be the sine of an angle z and u its cosine; one has generally, as one knows,

$$\sin nz = 2u\sin(n-1)z - \sin(n-2)z,$$

whence one draws

$$\begin{aligned}
&\sin z = x,\\
&\sin 2z = x(2u),\\
&\sin 3z = x(4u^2-1),\\
&\sin 4z = x(8u^3-4u),\\
&\sin 5z = x(16u^4-12u^2+1),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

It is necessary now to determine the general expression of $\sin nz$.

One can arrive by way of induction, by continuing further these expressions and seeking to discover the law of the different coefficients of the powers of u; but it will happen, if it is not in this example, at least in an infinity of others, that this law will be very complicated and very difficult to grasp: it matters consequently to have a general and sure method in order to find it in all the possible cases.

Let, for this, the differential equation be

$$(\nabla)\qquad y_x = \begin{cases} y_n = y_{n-1}(a_n u + b_n) \\ \qquad + y_{n-2}({}^1a_n u^2 + {}^1b_n u + {}^1c_n) \\ \qquad + y_{n-3}({}^2a_n u^3 + {}^2b_n u^2 + {}^2c_n u + {}^2f_n) \\ \qquad + \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

I suppose that one has

$$\begin{aligned} &y_1 = \alpha u + \beta, \\ &y_2 = \delta u^2 + \gamma u + \Omega, \\ &y_3 = \varpi u^3 + \pi u^2 + \theta u + \sigma, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Here is how I conclude the general expression of y_n.

I make

$$y_n = A_n u^n + B_n u^{n-1} + C_n u^{n-2} + \ldots,$$

hence,

$$\begin{aligned} y_{n-1} &= A_{n-1}u^{n-1} + B_{n-1}u^{n-2} + C_{n-1}u^{n-3} + \ldots, \\ y_{n-2} &= A_{n-2}u^{n-2} + B_{n-2}u^{n-3} + C_{n-2}u^{n-4} + \ldots, \end{aligned}$$

and thus in sequence; if one substitutes these values of $y_{n-1}, y_{n-2}, \ldots$ into equation (∇), one will have

$$\begin{aligned} y_n = u^n&(a_n A_{n-1} + {}^1a_n A_{n-2} + {}^2a_n A_{n-3} + \ldots \\ &+ u^{n-1}(a_n B_{n-1} + {}^1a_n B_{n-2} + {}^2a_n B_{n-3} + \ldots \\ &\qquad + b_n A_{n-1} + {}^1b_n A_{n-2} + {}^2b_n A_{n-3} + \ldots) \\ &+ u^{n-2}(a_n C_{n-1} + {}^1a_n C_{n-2} + {}^2a_n C_{n-3} + \ldots \\ &\qquad + b_n B_{n-1} + {}^1b_n B_{n-2} + {}^2b_n B_{n-3} + \ldots \\ &\qquad + {}^1c_n A_{n-2} + {}^2c_n A_{n-3} + {}^2c_n A_{n-4} + \ldots) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

By comparing this expression of y_n with the preceding, one will have the following equations

$$A_n = a_n A_{n-1} + {}^1a_n A_{n-2} + {}^2a_n A_{n-3} + \dots,$$
$$B_n = a_n B_{n-1} + {}^1a_n B_{n-2} + {}^2a_n B_{n-3} + \dots$$
$$+ b_n A_{n-1} + {}^1b_n A_{n-2} + {}^2b_n A_{n-3} + \dots,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

by means of which one will determine, by the preceding methods, A_n, B_n, $\dots$, and one will have thus the general expression of y_n.

I suppose that one wishes to have the general expression of $\sin nz$; it is easy to see, by that which precedes, that it will have this form

$$\sin nz = x(A_n u^{n-1} + B_n u^{n-3} + C_n u^{n-5} + D_n u^{n-7} + \dots);$$

therefore

$$\sin(n-1)z = x(A_{n-1}u^{n-2} + B_{n-1}u^{n-4} + C_{n-1}u^{n-6} + \dots)$$
$$\sin(n-2)z = x(A_{n-2}u^{n-3} + B_{n-2}u^{n-5} + C_{n-2}u^{n-7} + \dots).$$

If one substitutes these values of $\sin(n-1)z$ and $\sin(n-2)z$ into the equation

$$\sin nz = 2u\sin(n-1)z - \sin(n-2)z,$$

one will have

$$\sin nz = x(2A_{n-1}u^{n-1} + 2B_{n-1}u^{n-3} + 2C_{n-1}u^{n-5} + \dots - A_{n-2}u^{n-3} - B_{n-2}u^{n-5} - \dots)$$

and, if one compares this expression with the preceding, one will have

$$\text{(A)} \qquad \begin{cases} A_n = 2A_{n-1}, \\ B_n = 2B_{n-1} - A_{n-2}, \\ C_n = 2C_{n-1} - B_{n-2}, \\ \dots\dots\dots\dots\dots\dots \end{cases}$$

By means of these equations one will determine A_n, B_n, C_n, $\dots$, but one must make here an observation in which it is necessary to pay attention to all the researches which depend on the integral Calculus in the finite differences; this which renders its use very delicate. This observation consists in this that the preceding equations (A) begin to exist not at all immediately, that is to say when n has one same value in these equations. In order to demonstrate, I observe that the fundamental equation

$$\sin nz = 2u\sin(n-1)z - \sin(n-2)z,$$

by means of which I have concluded $\sin 2z$, $\sin 3z$, $\sin 4z$, $\ldots$, suppose known the first two sines $\sin 0z$ and $\sin 1z$; it can therefore begin to take place only when $n = 2$; hence also, equations (A) can begin to exist only when $n = 2$. The first of these equations begin to exist when $n = 2$, in which case one has $A_2 = 2A_1$; thus, the smallest index of A_n, that is to say the least value that n can have in this expression, is unity; the second equation can therefore begin to take place only when $n = 3$, in which case one has $B_3 = 2B_2 - A_1$; hence, the least index of B_n is 2; the third equation can therefore begin to take place only when $n = 4$, in which case one has $C_4 = 2C_3 - B_2$; hence, the smallest index of C_n is 3, and thus in sequence. This put:

If one integrates the first equation, one will have

$$A_n = 2^n H,$$

H being arbitrary; now, putting $n = 1$, $A_n = 1$, whence $H = \frac{1}{2}$, one has $A_n = 2^{n-1}$, hence $A_{n-2} = 2^{n-3}$. If one substitutes this value of A_{n-2} into the second equation and if next one integrates it; one will have

$$B_n = -2^{n-3}(n + H);$$

since the differential equation in B_n commences to exist when $n = 3$, the arbitrary constant H must be determined by the value of B_n, when $n = 2$; now, u not being able to have a negative exponent in the expression of $\sin nz$, it follows that $B_2 = 0$, hence $H = -2$; therefore

$$B_n = -2^{n-3}(n-2) \quad \text{and} \quad B_{n-2} = -2^{n-5}(n-4).$$

If one substitutes this value of B_{n-2} into the third equation, and if next one integrates it, one will have

$$C_n = 2^{n-5}\left(\frac{n^2 - 7n}{2} + H\right)$$

now, putting $n = 3$, $C_n = 0$, whence $H = 6$, one has $C_n = 2^{n-5}\frac{(n-3)(n-4)}{1.2}$, and thus to infinity. Therefore

$$\sin nz = x\Big[2^{n-1}u^{n-1} - \frac{n-2}{1}2^{n-3}u^{n-3} + \frac{(n-3)(n-4)}{1.2}2^{n-5}u^{n-5} - \frac{(n-4)(n-5)(n-6)}{1.2.3}2^{n-7}u^{n-7} + \dots\Big].$$

Let next $z = \text{angle} \sin x$; one will have, by differentiating,

$$\frac{dz}{dx} = \frac{1}{\sqrt{1-x^2}},$$

and I wish to have the general expression of $\frac{d^nz}{dx^n}$, dx being supposed constant. For this, let $u = \frac{1}{\sqrt{1-x^2}}$; one will have

$$\frac{du}{dx} = \frac{x}{(1-x^2)^{\frac{3}{2}}},$$
$$\frac{d^2u}{dx^2} = \frac{2x^2+1}{(1-x^2)^{\frac{5}{2}}},$$
$$\frac{d^3u}{dx^3} = \frac{6x^3+9x}{(1-x^2)^{\frac{7}{2}}},$$
$$\dots\dots\dots\dots\dots\dots$$

It is easy to see, by considering the law of these expressions of du, d^2u, $\dots$, that the general expression of $\frac{d^nu}{dx^n}$ has the following form

$$\frac{d^nu}{dx^n} = \frac{A_nx^n + B_nx^{n-2} + C_nx^{n-4} + D_nx^{n-6} + \dots}{(1-x^2)^{n+\frac{1}{2}}};$$

by differentiating this expression, one has

$$\frac{d^{n+1}u}{dx^{n+1}} = \frac{\begin{array}{c|c|c|c} (n+1)A_nx^{n+1} + (n+3)B_n & x^{n-1} + (n+5)C_n & x^{n-3} + (n+7)D_n & x^{n-5} + \dots \\ + nA_n & + (n-2)B_n & + (n-4)C_n & + \dots \end{array}}{(1-x^2)^{n+\frac{3}{2}}}$$

but one has

$$\frac{d^{n+1}u}{dx^{n+1}} = \frac{A_{n+1}x^{n+1} + B_{n+1}x^{n-1} + C_{n+1}x^{n-3} + D_{n+1}x^{n-5} + \dots}{(1-x^2)^{n+\frac{3}{2}}};$$

by comparing these two expressions of $\frac{d^{n+1}u}{dx^{n+1}}$, one will have the following equations:

$$\begin{aligned}
&A_{n+1} = (n+1)A_n,\\
&B_{n+1} = (n+3)B_n + nA_n,\\
&C_{n+1} = (n+5)C_n + (n-2)B_n,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

All these equations begin to exist immediately and when $n = 1$; this put, the first gives

$$A_n = 1.2.3\dots n;$$

the second gives

$$B_n = 1.2.3\dots n(n+1)(n+2)\left[H + \sum \frac{n}{(n+1)(n+2)(n+3)}\right],$$

or

$$B_n = 1.2.3\dots n(n+1)(n+2)\left[Q + \frac{1}{2}\frac{1}{(n+1)(n+2)} - \frac{1}{n+2}\right].$$

One will determine the constant Q by this condition that B_n is zero when $n = 1$; one has therefore $Q = \frac{1}{2.2}$. Therefore

$$B_n = 1.2.3\dots n\,\frac{1}{2}\,\frac{n(n-1)}{1.2}.$$

The third equation gives, by integrating and adding the appropriate constants,

$$C_n = 1.2.3\dots n\,\frac{1.3}{2.4}\,\frac{n(n-1)(n-2)(n-3)}{1.2.3.4};$$

one will find similarly

$$D_n = 1.2.3\dots n\,\frac{1.3.5}{2.4.6}\,\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1.2.3.4.5.6},$$

and thus in sequence. Hence

$$\frac{d^n z}{dx^n} = \frac{1.2.3\ldots(n-1)}{(1-x^2)^{n-\frac{1}{2}}}\Big[x^{n-1} + \frac{1}{2}\frac{(n-1)(n-2)}{1.2}x^{n-3}$$
$$+ \frac{1.3}{2.4}\frac{(n-1)(n-2)(n-3)(n-4)}{1.2.3.4}x^{n-5}$$
$$+ \frac{1.3.5}{2.4.6}\frac{(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}{1.2.3.4.5.6}x^{n-7}$$
$$+ \frac{1.3.5.7}{2.4.6.8}\frac{(n-1)(n-2)\ldots(n-8)}{1.2.3\ldots 8}x^{n-9}$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Big].$$

I have supposed, in the two preceding examples, the law of the exponents known, because it was very easy to perceive; but, if it happened that it was complicated, this which must be extremely rare, one will be able to determine it by the preceding method.

XIII.

Here is yet a remarkable usage of the integral Calculus in the finite differences in order to determine the nature of the functions according to some given conditions, this which is often useful, principally in the Calculus of partial differences.[5]

One proposes to find a function of x such that by making successively $x = \phi(x)$ and $x = \psi(x)$, one has

$$(\sigma) \qquad f[\phi(x)] = H_x f[\psi(x)] + X_x,$$

$\phi(x)$, $\psi(x)$, H_x being some given functions of x.

[5] I had found this method at the end of 1772, on the occasion of some problems which Mr. Monge, skillful professor of Mathematics at the schools of the Genoese at Mézières, proposed to me; I did part of it for him then; at the same time, I sent it to Mr. de la Grange, and I have presented it to the Academy in the month of February 1773. Since this time, Mr. the marquis de Condorcet has had printed in the Volume of the Academy for the year 1771 a quite beautiful memoir on this object; but the route which I have differs from his in this that he does not propose, as I do it, to restore the question to the differential equations of which the difference is constant and equal to unity. *Translator's note*: On 10 March and 17 March 1773, as reported in the Procès-Verbaux of the Paris Academy, Laplace read the paper "Recherches sur l'integration des differentielles aux différences finies et sur leur application à l'analyse des hasards."

For this let

$$u_z = \psi(x) \quad \text{and} \quad u_{z+1} = \phi(x).$$

From the first of these equations, I conclude

$$x = \Gamma(u_z) \quad \text{and} \quad \phi(x) = H(u_z),$$

$\Gamma(u_z)$ and $H(u_z)$ representing some known functions of u_z; hence,

$$u_{z+1} = H(u_z),$$

a differential equation of which the constant difference is equal to unity, and which one can integrate in many cases.

The integral of this equation will give u_z as function of z, and the equation $x = \Gamma(u_z)$ will give x as function of z. Substituting this value of x in H_x and X_x, the quantities will becomes some functions of z, which I designate by L_z and Z_z. Moreover, one has

$$f[\phi(x)] = f(u_{z+1}) \quad \text{and} \quad f[\psi(x)] = f(u_z);$$

equation (σ) will become therefore, by supposing $f(u_z) = y_z$,

$$y_{z+1} = L_z y_z + Z_z,$$

an equation integrable by Problem I.

One must observe here, consistent with a remark due to Mr. Euler, that the constants which come by integrating the finite differential equations of which the variable is z, and of which the constant difference is unity, can be supposed some functions any whatsoever of $\sin 2\pi z$ and $\cos 2\pi z$, π expressing the ratio of the circumference to the diameter.

Presently, if one puts back into the expression of y_z instead of z its value in x, one will have $f[\psi(x)]$, and, if one changes $\psi(x)$ into x, one will have the function of x, which satisfies the Problem. The following examples clarify this method:

The question is to find a function of x such that by changing successively x into x^q and into mx, one has

$$f(x^q) = f(mx) + p,$$

m and p being constants.

I make $u_z = mx$, and $u_{z+1} = x^q$; hence,

$$u_{z+1} = \left(\frac{u_z}{m}\right)^q.$$

In order to integrate this equation, I make $u_1 = a$; therefore $u_2 = \frac{a^q}{m^q}$, $u_3 = \frac{a^{q^2}}{m^{q^2+q}}, \ldots$ Let $u_z = \frac{a^{g_z}}{m^{f_z}}$; therefore

$$u_{z+1} = \frac{a^{qg_z}}{m^{qf_z+q}} = \frac{a^{g_{z+1}}}{m^{f_{z+1}}}.$$

Therefore

$$g_{z+1} = qg_z,$$

this which gives

$$g_z = Aq^z.$$

Now, putting $z = 2$, $g_z = q$, whence $A = \frac{1}{q}$, one has $g_z = q^{z-1}$. Moreover, one has $f_{z+1} = qf_z + q$. Therefore $f_z = Aq^z + \frac{q}{1-q}$. Now, putting $z = 2$, $f_z = q$; therefore $A = \frac{1}{q-1}$ and $f_z = \frac{1}{q-1}(q^z - q)$; therefore

$$u_z = \frac{a^{q^{z-1}}}{m^{\frac{1}{q-1}(q^z-q)}}.$$

This expression of u_z is complete, since a is arbitrary; now the equation

$$f(x^q) = f(mx) + p$$

will become

$$y_{z+1} = y_z + p.$$

Therefore

$$y_z = C + pz = f(mx).$$

It is necessary presently to have the value of z in x; now, since one has $u_z = mx$, one will have

$$mx = \frac{a^{q^{z-1}}}{m^{\frac{1}{q-1}(q^z-q)}},$$

whence one draws[6]

$$lmx = q^z \frac{la}{q} - \frac{1}{q-1}(q^z - q)\, lm$$

or

$$q^z \left(\frac{la}{q} - \frac{lm}{q-1} \right) = l \frac{mx}{m^{\frac{q}{q-1}}};$$

let $\frac{la}{q} - \frac{lm}{q-1} = K,$ and one will find

$$z = \frac{ll \frac{mx}{m^{\frac{q}{q-1}}}}{lq} - \frac{lK}{lq},$$

hence

$$y_z = A + p \frac{ll \frac{mx}{m^{\frac{q}{q-1}}}}{lq},$$

A being an arbitrary constant which can be any function whatsoever of $\sin 2\pi z$ and $\cos 2\pi z$. Let $\Gamma(\sin 2\pi z, \cos 2\pi z)$ be this function; by substituting instead of z its value, one will have

$$A = \Gamma\left(\sin 2\pi \frac{ll \frac{mx}{m^{\frac{q}{q-1}}}}{lq}, \cos 2\pi \frac{ll \frac{mx}{m^{\frac{q}{q-1}}}}{lq} \right).$$

Therefore

$$y_z = f(mx) = \Gamma\left(\sin 2\pi \frac{ll \frac{mx}{m^{\frac{q}{q-1}}}}{lq}, \cos 2\pi \frac{ll \frac{mx}{m^{\frac{q}{q-1}}}}{lq} \right) + p \frac{ll \frac{mx}{m^{\frac{q}{q-1}}}}{lq};$$

thus the function of x demanded is

$$f(x) = \Gamma\left(\sin 2\pi \frac{ll \frac{x}{m^{\frac{q}{q-1}}}}{lq}, \cos 2\pi \frac{ll \frac{x}{m^{\frac{q}{q-1}}}}{lq} \right) + p \frac{ll \frac{x}{m^{\frac{q}{q-1}}}}{lq}.$$

[6] *Translator's note*: Laplace uses l to denote the natural logarithm. It appears as l in this document.

It is a question again to find $f(x)$ such that

$$[f(x)]^2 = f(2x) + 2.$$

One could first think that it is impossible to satisfy this equation, at least to suppose $f(x)$ equal to a constant; this is indeed that which some able geometers have believed (*see* the second Volume of the *Mémoires de Turin*, p. 320); but one is going to see there are an infinity of other ways to satisfy it.

Let

$$u_z = x \qquad \text{and} \qquad u_{z+1} = 2x;$$

therefore

$$u_{z+1} = 2u_z \qquad \text{and} \qquad u_z = A2^z = x.$$

Moreover, one has

$$f(2x) = f(u_{z+1}), \qquad \text{which I designate by} \qquad t_{z+1},$$

and

$$f(x) = f(u_z) = t_z;$$

and one will have

$$t_{z+1} = t_z^2 - 2.$$

In order to integrate this equation, I suppose $t_1 = a + \frac{1}{a}$, therefore

$$t_2 = a^2 + \frac{1}{a^2}, \quad t_3 = a^4 + \frac{1}{a^4}, \quad \ldots,$$

and generally

$$t_z = a^{2^{z-1}} + \frac{1}{a^{2^{z-1}}},$$

a complete expression of t_x, since a is arbitrary; now one has $2^{z-1} = \frac{x}{2A}$, therefore

$$t_z = a^{\frac{x}{2A}} + a^{-\frac{x}{2A}}, \quad \text{or} \quad t_z = b^x + b^{-x},$$

b being an arbitrary constant; now this constant can be supposed any function whatsoever of $\sin 2\pi z$ and $\cos 2\pi z$, and since $z = H + \frac{lx}{l2}$, H being any constant whatsoever, one will have

$$b = f(\sin 2\pi \frac{lx}{l\,2}, \cos 2\pi \frac{lx}{l\,2}),$$

hence the function of x demanded is

$$\left[f(\sin 2\pi \frac{lx}{l\,2}, \cos 2\pi \frac{lx}{l\,2})\right]^x + \left[f(\sin 2\pi \frac{lx}{l\,2}, \cos 2\pi \frac{lx}{l\,2})\right]^{-x}$$

It is a question again to find $f(x - y\sqrt{-1}),$ *such that one has*

$$f(x + y\sqrt{-1}) - f(x - y\sqrt{-1}) = 2M\sqrt{-1}.$$

By supposing $y = g + hx$, one will have

$$f[g\sqrt{-1} + x(1 + h\sqrt{-1})] - f[x(1 - h\sqrt{-1}) - g\sqrt{-1}] = 2M\sqrt{-1}.$$

Let

$$x(1 + h\sqrt{-1}) + g\sqrt{-1} = u_{z+1},$$
$$x(1 - h\sqrt{-1}) - g\sqrt{-1} = u_{z;}$$

one will have therefore

$$x = \frac{u_z + g\sqrt{-1}}{1 - h\sqrt{-1}};$$

therefore

$$u_{z+1} = \frac{1 + h\sqrt{-1}}{1 - h\sqrt{-1}} u_z + \frac{2g\sqrt{-1}}{1 - h\sqrt{-1}},$$

an equation of which the integral is

$$u_z = A\left(\frac{1 + h\sqrt{-1}}{1 - h\sqrt{-1}}\right)^2 - \frac{g}{h} = x(1 - h\sqrt{-1}) - g\sqrt{-1};$$

hence,

$$zl\frac{1+h\sqrt{-1}}{1-h\sqrt{-1}} = l(g+hx)+K.$$

Now, if one names $\varpi\pi$ the angle of which the tangent is h, and π the ratio of the semi-circumference to the radius, one will have

$$l\frac{1+h\sqrt{-1}}{1-h\sqrt{-1}} = 2\sqrt{-1}\varpi\pi;$$

therefore

$$z = \frac{l(g+hx)}{2\sqrt{-1}\varpi\pi} + K'.$$

Now one has

$$f(u_{z+1}) - f(u_z) = 2M\sqrt{-1};$$

and, by representing $f(u_z)$ by t_z,

$$t_{z+1} = t_z + 2M\sqrt{-1},$$

therefore

$$t_z = H + 2Mz\sqrt{-1};$$

substituting instead of z its value, one will have

$$t_z = M\frac{l(g+hx)}{\varpi\pi} + L,$$

L being an arbitrary constant, which can be any function whatsoever of $\sin 2\pi z$ and $\cos 2\pi z$, or of $\sin\frac{l(g+hx)}{\varpi\sqrt{-1}}$ and of $\cos\frac{l(g+hx)}{\varpi\sqrt{-1}}$, and consequently of $e^{\frac{l(g+hx)}{\varpi}}$; now, $e^{l(g+hx)} = g+hx;$ therefore L can be a function of $(g+hx)^{\frac{1}{\varpi}};$ hence

$$f(x-y\sqrt{-1}) = M\frac{l(g+hx)}{\varpi\pi} + \Gamma\left[(g+hx)\frac{1}{\varpi}\right].$$

XIV.

On the equations in finite differences, when one has many equations among many variables.

I suppose that one has the following two equations among the three variables y_x, ${}^1y_x$ and x

$$y_x + A_x y_{x-1} = B_x\, {}^1y_x + C_x\, {}^1y_{x-1}, \tag{1}$$

$$y_x + {}^1A_x y_{x-1} = {}^1B_x\, {}^1y_x + {}^1C_x\, {}^1y_{x-1}. \tag{2}$$

The simplest way to integrate them is to reduce them by elimination to two other equations, the one between y_x and x, the other between ${}^1y_x$ and x; for this, I multiply the first by ${}^1C_x$, the second by C_x, and I subtract the one from the other; this which gives

$$({}^1C_x - C_x)y_x + ({}^1C_x A_x - C_x\, {}^1A_x)y_{x-1} = ({}^1C_x B_x - C_x\, {}^1B_x)\, {}^1y_x,$$

hence

$$\left\{ \begin{aligned} &({}^1C_{x-1} - C_{x-1})y_{x-1} + ({}^1C_{x-1} A_{x-1} - C_{x-1}\, {}^1A_{x-1})y_{x-1} \\ &\qquad = ({}^1C_{x-1} B_{x-1} - C_{x-1}\, {}^1B_{x-1})\, {}^1y_{x-1}. \end{aligned} \right. \tag{3}$$

I multiply equation (1) by α, equation(2) by ${}^1\alpha$, and I add them with equation (3), this which gives

$$\begin{aligned} &(\alpha + {}^1\alpha)y_x + (\alpha A_x + {}^1\alpha {}^1A_x + {}^1C_{x-1} - C_{x-1})y_{x-1} + ({}^1C_{x-1} A_{x-1} - C_{x-1}\, {}^1A_{x-1})y_{x-2} \\ &\quad = (\alpha B + {}^1\alpha {}^1B)\, {}^1y_x + (\alpha C_x + {}^1\alpha {}^1C_x + {}^1C_{x-1} B_{x-1} - C_{x-1}\, {}^1B_{x-1})\, {}^1y_{x-1}; \end{aligned}$$

I make ${}^1y_x$ and ${}^1y_{x-1}$ vanish by means of the equations

$$\begin{aligned} &\alpha B_x + {}^1\alpha {}^1B_x = 0, \\ &\alpha C_x + {}^1\alpha {}^1C_x + {}^1C_{x-1} B_{x-1} - C_{x-1}\, {}^1B_{x-1} = 0, \end{aligned}$$

and I have in this manner a differential equation between y_x and x alone; by an entirely similar process, one will find one of them between ${}^1y_x$ and x; and it would be the same thing if one has a greater number of equations and of variables.

It is easy to see that, if there was in each equation some terms such as T_x, $X_x, \dots$, T_x, X_x being some functions any whatsoever of x, they would be integrable in the same cases where they are it, these terms not being there.

When one has $n-1$ equations among n variables, these being able to have an infinity of different relations among them, the integration of these equations presents thus a great number of curious researches; but there is a case which merits a particular attention, in this that it is encountered sometimes and principally in the analyses of chances; it is the case in which these equations return to themselves.

XV.

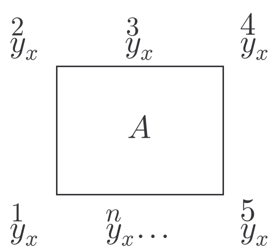
On the differential equations returning to themselves.

If one has the following equations, among the n variables $\overset{1}{y}_x,\ \overset{2}{y}_x,\ \overset{3}{y}_x, \dots$,

$$\begin{aligned}
&\overset{1}{y}_x = A\overset{2}{y}_{x-1},\\
&\overset{2}{y}_x = A\overset{3}{y}_{x-1},\\
&\overset{3}{y}_x = A\overset{4}{y}_{x-1},\\
&\dots\dots\dots\dots,\\
&\overset{n}{y}_x = A\overset{1}{y}_{x-1}.
\end{aligned}$$

These equations are those which I call *equations returning to themselves*.

In general, if one disposes on the perimeter of *fig.* A the n variables $\overset{1}{y}_x,\ \overset{2}{y}_x,\ \overset{3}{y}_x, \dots$, as the figure represents them,



and if then a function any whatsoever of one of these variables and of its finite differences are constantly equal to any function whatsoever of those which follow it and of their finite differences, the equation which results is that which I name an *equation returning to itself*. If, for example, each of these variables is equal to twice that which follows it, when one supposes x diminishing by unity, plus three times that which follows this last, when one supposes x diminishing by two units, one will have

$$\begin{aligned}
\overset{1}{y}_x &= 2\overset{2}{y}_{x-1} + 3\overset{3}{y}_{x-2},\\
\overset{2}{y}_x &= 2\overset{3}{y}_{x-1} + 3\overset{4}{y}_{x-2},\\
&\dots\dots\dots\dots,\\
\overset{n}{y}_x &= 2\overset{1}{y}_{x-1} + 3\overset{2}{y}_{x-2}.
\end{aligned}$$

One sees thence that, although in the order of the variable $\overset{1}{y}_x$ is the first, one would have been able however equally to begin with any other of these variables, and the equations would have been absolutely the same, this which is the particular character of this kind of equations. This put,

XVI.

Problem III. — I suppose that one has the returning equations

$$\begin{aligned}
\overset{1}{y}_x + A\overset{1}{y}_{x-1} + {}^1A\overset{1}{y}_{x-2} + \dots &= B\overset{2}{y}_x + {}^1B\overset{2}{y}_{x-1} + {}^2B\overset{2}{y}_{x-2} + \dots + X_x,\\
\overset{2}{y}_x + A\overset{2}{y}_{x-1} + {}^1A\overset{2}{y}_{x-2} + \dots &= B\overset{3}{y}_x + {}^1B\overset{3}{y}_{x-1} + {}^2B\overset{3}{y}_{x-2} + \dots + X_x,\\
\\
\overset{n}{y}_x + A\overset{n}{y}_{x-1} + {}^1A\overset{n}{y}_{x-2} + \dots &= B\overset{1}{y}_x + {}^1B\overset{1}{y}_{x-1} + {}^2B\overset{1}{y}_{x-2} + \dots + X_x;
\end{aligned}$$

it is necessary to determine $\overset{1}{y}_x$, $\overset{2}{y}_x$,...

The first equation gives

$$\begin{aligned}
&\overset{1}{y}_x + A\overset{1}{y}_{x-1} + {}^1A\overset{1}{y}_{x-2} + \dots + A\overset{1}{y}_{x-1} + A^2\overset{1}{y}_{x-2} + \dots + {}^1A\overset{1}{y}_{x-2}\\
&\quad = B(\overset{2}{y}_x + A\overset{2}{y}_{x-1} + {}^1A\overset{2}{y}_{x-2} + \dots) + {}^1B(\overset{2}{y}_{x-1} + A\overset{2}{y}_{x-2} + {}^1A\overset{2}{y}_{x-3} + \dots) + \dots\\
&\qquad + X_x + AX_{x-1} + {}^1AX_{x-2} + \dots
\end{aligned}$$

I substitute instead of $\overset{2}{y}_x + A\overset{2}{y}_{x-1} + \dots$, $\overset{2}{y}_{x-1} + A\overset{2}{y}_{x-2} + \dots$ their values which the second equation gives, this which gives me an equation among $\overset{1}{y}_x$, $\overset{1}{y}_{x-1}$,... and $\overset{3}{y}_x$, $\overset{3}{y}_{x-1}$, ...; by operating on this here as on the first, I will have an equation among $\overset{1}{y}_x$, $\overset{1}{y}_{x-1}$,... and $\overset{4}{y}_x$, $\overset{4}{y}_{x-1}$, ... and, by continuing to operate thus until the variable $\overset{q}{y}_x$, I will arrive to an equation of this form

$$\overset{1}{y}_x + b_q\overset{1}{y}_{x-1} + {}^1b_q\overset{1}{y}_{x\text{-}2} + \dots$$
$$= a_q(\overset{q}{y}_x + A\,\overset{q}{y}_{x-1} + {}^1A\,\overset{q}{y}_{x-2} + \dots) + {}^1a_q(\overset{q}{y}_{x-1} + A\,\overset{q}{y}_{x-2} + \dots) + \dots + \overset{q}{u}_x,$$

It is necessary to determine b_q, ${}^1b_q, \dots, a_q$, ${}^1a_q$, $\dots$, $\overset{q}{u}_x$.

For this I substitute into the preceding equation, instead of $\overset{q}{y}_x + A\,\overset{q}{y}_{x-1} + \dots$, $\overset{q}{y}_{x-1} + A\,\overset{q}{y}_{x-2} + \dots$, their values that the q^{th} of the returning equations gives, this which gives

$$\overset{1}{y}_x + b_q\overset{1}{y}_{x-1} + {}^1b_q\overset{1}{y}_{x\text{-}2} + \dots$$
$$= a_q(B\overset{q+1}{y_x} + {}^1B\,\overset{q+1}{y}_{x-1} + \dots + X_x) + {}^1a_q(B\,\overset{q+1}{y}_{x-1} + {}^1B\,\overset{q+1}{y}_{x-2} + \dots + X_{x-1}) + \dots + \overset{q}{u}_x,$$

whence I conclude

$$\left.\begin{array}{r}\overset{1}{y}_x + b_q\\ + A\end{array}\right|\left.\begin{array}{r}\overset{1}{y}_{x-1} + {}^1b_q\\ + Ab_q\\ + {}^1A\end{array}\right|\overset{1}{y}_{x-2} + \dots = \begin{array}{l} a_qB\Big(\overset{q+1}{y_x} + A\,\overset{q+1}{y}_{x-1} + {}^1A\,\overset{q+1}{y}_{x-2} + \dots \\ + ({}^1a_qB + a_q{}^1B)\Big(\overset{q+1}{y}_{x-1} + A\,\overset{q+1}{y}_{x-2} + \dots\Big) \\ + ({}^2a_qB + {}^1a_q{}^1B + a_q{}^2B)\Big(\overset{q+1}{y}_{x-2} + \dots\Big) \\ + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ + X_xa_q \\ + X_{x-1}({}^1a_q + Aa_q) \\ + X_{x-2}({}^2a_q + A^1a_q + {}^1Aa_q) \\ + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ + \overset{q}{u}_x + A\overset{q}{u}_{x-1} + {}^1A\overset{q}{u}_{x-2} + \dots; \end{array}$$

but one has

$$\overset{1}{y}_x + b_{q+1}\overset{1}{y}_{x-1} + {}^1b_{q+1}\overset{1}{y}_{x-2} + \dots = \begin{array}{l} a_{q+1}\Big(\overset{q+1}{y_x} + A\,\overset{q+1}{y}_{x-1} + \dots\Big) \\ + {}^1a_{q+1}\Big(\overset{q+1}{y}_{x-1} + \dots\Big) \\ + \dots\dots\dots\dots\dots\dots\dots \\ + \overset{q+1}{u_x}; \end{array}$$

whence one has, by comparing,

$$
\begin{aligned}
&b_{q+1} = b_q + A,\\
&{}^1b_{q+1} = {}^1b_q + Ab_q + {}^1A,\\
&\dots\dots\dots\dots\dots\dots;\\
&a_{q+1} = a_q B,\\
&{}^1a_{q+1} = {}^1a_q B + a_q {}^1B,\\
&\dots\dots\dots\dots\dots\dots;
\end{aligned}
$$

$$
(\Lambda) \quad \begin{cases} \overset{q+1}{u}_x = \overset{q}{u}_x + A\overset{q}{u}_{x-1} + {}^1A\overset{q}{u}_{x-2} + \dots \\ \qquad + X_x a_q + X_{x-1}({}^1a_q + Aa_q) + X_{x-2}({}^2a_q + A^1a_q + {}^1Aa_q) + \dots \end{cases}
$$

By means of these equations, one will determine easily a_q, ${}^1a_q$,…, b_q, ${}^1b_q$, …; in order to determine $\overset{q}{u}_x$, I observe that one has

$$
\overset{q}{u}_x = f_q X_x + {}^1f_q X_{x-1} + {}^2f_q X_{x-2} + \dots;
$$

I substitute this value into equation (A), this which gives

$$
\begin{aligned}
\overset{q+1}{u}_x = {}& X_x(f_q + a_q) + X_{x-1}({}^1f_q + {}^1a_q + Aa_q + Af_q)\\
&+ X_{x-2}({}^2f_q + {}^2a_q + A^1a_q + {}^1Aa_q + {}^1Af_q + A^1f_q)\\
&+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots;
\end{aligned}
$$

but one has

$$
\overset{q+1}{u}_x = f_{q+1}X_x + {}^1f_{q+1}X_{x-1} + {}^2f_{q+1}X_{x-2}\dots;
$$

therefore

$$
\begin{aligned}
&f_{q+1} = f_q + a_q,\\
&{}^1f_{q+1} = {}^1f_q + {}^1a_q + Af_q,\\
&\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

By means of these equations one will determine f_q, ${}^1f_q$, …, and hence $\overset{q}{u}_x$. I suppose now $q = n$, and one will have

$$\overset{1}{y}_x + b_q\overset{1}{y}_{x-1} + \dots = a_n(\overset{n}{y}_x + A\overset{n}{y}_{x-1} + \dots)$$
$$+ {}^1a_n(\overset{n}{y}_{x-1} + {}^1A\overset{n}{y}_{x-2} + \dots)$$
$$+ \dots\dots\dots\dots\dots\dots\dots\dots$$
$$+ \overset{n}{u}_x;$$

but one has

$$\overset{n}{y}_x + A\overset{n}{y}_{x-1} + \dots = B\overset{1}{y}_x + {}^1B\overset{1}{y}_{x-1} + \dots + X_x;$$

therefore

$$\begin{aligned}\overset{1}{y}_x + b_n\overset{1}{y}_{x-1} + \dots = {} & a_n(B\overset{1}{y}_x + {}^1B\overset{1}{y}_{x-1} + \dots) && + a_nX_x \\ & + {}^1a_n(B\overset{1}{y}_{x-1} + \dots) && + {}^1a_nX_{x-1} \\ & + \dots\dots\dots\dots\dots && + \dots\dots \\ & && + \overset{n}{u}_x,\end{aligned}$$

and, by ordering the different terms of this equation

$$\overset{1}{y}_x(1 - a_nB) + \overset{1}{y}_{x-1}(b_n - a_n{}^1B - {}^1a_nB) + \overset{1}{y}_{x-2}({}^1b_n - a_n{}^2B + {}^1a_n{}^1B - {}^2a_nB) + \dots$$
$$- \overset{n}{u}_x - a_nX_x - {}^1a_nX_{x-1} - \dots = 0$$

one will have an equation entirely similar for $\overset{2}{y}_x$, $\overset{3}{y}_x$, $\dots$

XVII.

PROBLEM IV. — *I suppose now that the returning equations contain three variables, and that one has*

$$\overset{1}{y}_x + A\overset{1}{y}_{x-1} + {}^1A\overset{1}{y}_{x-2} + \dots = B\overset{2}{y}_x + {}^1B\overset{2}{y}_{x-1} + {}^2B\overset{2}{y}_{x-2} + \dots$$
$$+ C\overset{3}{y}_x + {}^1C\overset{3}{y}_{x-1} + {}^2C\overset{3}{y}_{x-2} + \dots$$
$$+ X_x,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$
$$\overset{n}{y}_x + A\overset{n}{y}_{x-1} + {}^1A\overset{n}{y}_{x-2} + \dots = B\overset{1}{y}_x + {}^1B\overset{1}{y}_{x-1} + {}^2B\overset{1}{y}_{x-2} + \dots$$
$$+ C\overset{2}{y}_x + {}^1C\overset{2}{y}_{x-1} + \dots$$
$$+ X_x,$$

it is necessary to determine $\overset{1}{y}_x$, $\overset{2}{y}_x, \dots$

By following the process of the preceding problem, one will arrive to an equation of this form

$$\begin{aligned}\overset{1}{y}_x + b_q\overset{1}{y}_{x-1} + {}^1b_q\overset{1}{y}_{x-2} + \dots = {} & a_q(\overset{q}{y}_x + A\overset{q}{y}_{x-1} + {}^1A\overset{q}{y}_{x-2} + \dots) \\ & + {}^1a_q(\overset{q}{y}_{x-1} + A\overset{q}{y}_{x-2} + {}^1A\overset{q}{y}_{x-3} + \dots) \\ & + \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & + c_q(\overset{q+1}{y_x} + A\overset{q+1}{y}_{x-1} + {}^1A\overset{q+1}{y}_{x-2} + \dots) \\ & + {}^1c_q(\overset{q+1}{y}_{x-1} + A\overset{q+1}{y}_{x-2} + \dots) \\ & + \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & + \overset{q}{u}_x.\end{aligned}$$

I substitute now into this equation, instead of

$$\overset{q}{y}_x + A\overset{q}{y}_{x-1} + \dots, \quad \overset{q}{y}_{x-1} + A\overset{q}{y}_{x-2} + \dots,$$

their values that the q^{th} equation gives, this which produces the following

$$\begin{aligned}\overset{1}{y}_x + b_q\overset{1}{y}_{x-1} + {}^1b_q\overset{1}{y}_{x-2} + \dots = {} & a_q(B\overset{q+1}{y_x} + {}^1B\overset{q+1}{y}_{x-1} + {}^2B\overset{q+1}{y}_{x-2} + \dots) \\ & + {}^1a_q(B\overset{q+1}{y}_{x-1} + {}^1B\overset{q+1}{y}_{x-2} + {}^2B\overset{q+1}{y}_{x-3} + \dots) \\ & + \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & + a_q(C\overset{q+2}{y_x} + {}^1C\overset{q+2}{y}_{x-1} + \dots) \\ & + {}^1a_q(C\overset{q+2}{y}_{x-1} + \dots) \\ & + \dots\dots\dots\dots\dots\dots \\ & + c_q(\overset{q+1}{y_x} + A\overset{q+1}{y}_{x-1} + \dots) \\ & + {}^1c_q(\overset{q+1}{y}_{x-1} + \dots) \\ & + \dots\dots\dots\dots\dots\dots \\ & + a_qX_x + {}^1a_qX_{x-1} + \dots \\ & + \overset{q}{u}_x;\end{aligned}$$

whence one will conclude easily

$$\left.\begin{array}{r}\overset{1}{y}_x + b_q \\ + A\end{array}\right| \left.\begin{array}{r}\overset{1}{y}_{x-1} + {}^1b_q \\ + Ab_q \\ + {}^1A\end{array}\right| \overset{1}{y}_{x-2} + \dots$$

$$\begin{aligned} &= (a_qB + c_q)(\overset{q+1}{y_x} + A\overset{q+1}{y}_{x-1} + \dots) \\ &\quad + ({}^1a_qB + a_q{}^1B + {}^1c_q + Ac_q)(\overset{q+1}{y}_{x-1} + A\overset{q+1}{y}_{x-2} + \dots) \\ &\quad + ({}^2a_qB + {}^1a_q{}^1B + a_q{}^2B + {}^2c_q + A^1c_q + {}^1Ac_q)(\overset{q+1}{y}_{x-2} + A\overset{q+1}{y}_{x-3} + \dots) \\ &\quad + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\quad + a_qC(\overset{q+2}{y_x} + A\overset{q+2}{y}_{x-1} + \dots) \\ &\quad + ({}^1a_qC + a_q{}^1C)(\overset{q+2}{y}_{x-1} + A\overset{q+2}{y}_{x-2} + \dots) \\ &\quad + ({}^2a_qC + {}^1a_q{}^1C + a_q{}^2C)(\overset{q+2}{y}_{x-2} + A\overset{q+2}{y}_{x-3} + \dots) \\ &\quad + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\quad + \overset{q}{u}_x + A\overset{q}{u}_{x-1} + {}^1A\overset{q}{u}_{x-2} + \dots \\ &\quad + X_xa_q + X_{x-1}({}^1a_q + Aa_q) + \dots; \end{aligned}$$

now one has

$$\begin{aligned} \overset{1}{y}_x + b_{q+1}\overset{1}{y}_{x-1} + \dots = a_{q+1}&(\overset{q+1}{y_x} + Ay^{q+1}_{x-1} + \dots) \\ &+ \dots\dots\dots\dots\dots\dots \\ &+ c_{q+1}(\overset{q+2}{y_x} + A\overset{q+2}{y}_{x-1} + \dots) \\ &+ \dots\dots\dots\dots\dots\dots \\ &+ \overset{q+1}{u_x}; \end{aligned}$$

whence one will have, by comparing,

$$\begin{aligned} b_{q+1} &= b_q + A, \\ {}^1b_{q+1} &= {}^1b_q + Ab_q + {}^1A, \\ \dots&\dots\dots\dots\dots\dots; \end{aligned}$$

thus one will determine b_q, ${}^1b_q$, $\dots$; next

$$a_{q+1} = a_qB + c_q \quad \text{and} \quad c_{q+1} = a_qC, \quad \text{hence} \quad a_{q+1} = a_qB + a_{q-1}C;$$

whence one will have a_q and c_q. Moreover, one will have

$$
\begin{aligned}
{}^1a_{q+1} &= {}^1a_qB + a_q{}^1B + {}^1c_q + Ac_q,\\
{}^1c_{q+1} &= {}^1a_qC + a_q{}^1C.
\end{aligned}
$$

Therefore

$$
{}^1a_{q+1} = {}^1a_qB + a_q{}^1B + {}^1c_{q-1}{}^1C + a_{q-1}{}^1C + Ac_q;
$$

whence one will have ${}^1a_q$ and ${}^1c_q$, and thus of the rest; finally one will determine $\overset{q}{u}_x$, as in the preceding problem.

If one supposes presently $q = n$, one will have

$$
\begin{aligned}
\overset{1}{y}_x(1 - c_n) + \overset{1}{y}_{x-1}(1 - Ac_n - {}^1c_n) + \dots = {} & a_n(\overset{n}{y}_x + A\overset{n}{y}_{x-1} + \dots)\\
& + {}^1a_n(\overset{n}{y}_{x-1} + A\overset{n}{y}_{x-2} + \dots)\\
& + \dots\dots\dots\dots\dots\dots\dots\dots\\
& + \overset{n}{u}_x.
\end{aligned}
$$

One will form some entirely similar equations among $\overset{n-1}{y_x}$ and $\overset{n}{y}_x, \overset{n-2}{y_x}$ and $\overset{n-1}{y_x}, \dots,$ and one will have a number n of returning equations in two variables, such as I have considered in the preceding problem.

The same method would succeed equally if the returning equations contained four or a greater number of variables.

XVIII.
On the integral calculus in the finite and partial differences.

I suppose that ${}_ny_x$ represents any function whatsoever of two variables x and n; I can in this function make n vary by regarding x as constant; I can make x vary by regarding n as constant; finally, I can vary n and x all together, their variations being in any ratio whatsoever; now, if there exists among ${}_ny_x$ and these different variations any equation whatsoever, it will be that which I name an *equation in the finite and partial differences*.

${}_ny_x$ represents always a function of two variables x and n:

${}_{n-1}y_x$, ${}_{n-2}y_x$, $\dots$ signify that n has diminished by one, by two, $\dots$ units in this function;

${}_ny_{x-1}$, ${}_ny_{x-2}$, $\dots$ signify that x has diminished by one, by two, $\dots$ units in this function;

${}_{n-1}y_{x-2}, \dots$ signifies that n has diminished by one unit, and x by two units, and thus in sequence.

An equation in the partial differences is therefore an equation among these different quantities; such as this here:

$$ {}_{n}y_{x} = a.{}_{n}y_{x-1} + b.{}_{n-1}y_{x-1}. $$

The equations in the finite differences have been found by the consideration of the sequences (Art. II). This is similarly the consideration of certain sequences that I have named *récurro-récurrentes* (*see* volume VI of *Savants étranges*), which has led me to the finite and partial differences; here is how: I suppose that one has the sequences

$$ (i) \qquad \begin{cases} {}_1y_1, & {}_1y_2, & {}_1y_3, & {}_1y_4, & {}_1y_5, & \dots, & {}_1y_x, & \dots, \\ {}_2y_1, & {}_2y_2, & {}_2y_3, & {}_2y_4, & {}_2y_5, & \dots, & {}_2y_x, & \dots, \\ {}_3y_1, & {}_3y_2, & {}_3y_3, & {}_3y_4, & {}_3y_5, & \dots, & {}_3y_x, & \dots, \\ \dots, & \dots, & \dots, & \dots, & \dots, & \dots, & & \dots, \\ {}_ny_1, & {}_ny_2, & {}_ny_3, & {}_ny_4, & {}_ny_5, & \dots, & {}_ny_x, & \dots, \end{cases} $$

If any term whatsoever ${}_{n}y_{x}$ of these sequences is constantly equal to any number whatsoever of the preceding terms taken in many of these sequences, and each multiplied by a function of x and of n, these sequences are those that I have called *récurro-récurrentes*, and the equation which expresses the law according to which they are formed is an equation in the finite and partial differences.

I will observe here that the sequences (*i*) can be considered not only in the horizontal sense, but further in the vertical sense, and, rather than in the first sense x is their index, n will be it in the second.

I will suppose in the following, as I have done it above in the equations in the ordinary differences, that the differences of x and of n are constants and equal to unity; if they are constants without being equal to unity, it will always be possible to render them such, by the introduction of new variables; I will suppose moreover (this which is yet permitted) that the smallest values that x and n can receive are unity; and each time that I myself will depart from this assumption, the state of the question will make it known. This put:

If one has an equation in the partial differences such that

$$ {}_{n}y_{x} = 2.{}_{n}y_{x-1} + 2.{}_{n-1}y_{x-1}, $$

it begins to hold only when x and n are greater than unity, as in the ordinary differences the equation ${}_1y_x = a.{}_1y_{x-1}$ holds only when x is greater than 1; so that ${}_1y_1$ remains arbitrary, and one determines by means of this equation only the values of ${}_1y_2$, ${}_1y_3, \ldots$; likewise, in the equation

$$ {}_ny_x = 2.{}_ny_{x-1} + 2.{}_{n-1}y_{x-1}, $$

${}_1y_x$ and ${}_ny_1$ are arbitrary; thus the general expression of ${}_ny_x$ contains an arbitrary function.

In general, the number of arbitrary functions that the integral of an equation in the partial differences contains will be determined by the degree of the difference of that of the two quantities x and n which varies the least; thus, in the equation

$$ {}_ny_x = {}_ny_{x-1} + 3.{}_{n-1}y_{x-1}, $$

the number of arbitrary functions which the integral contains is 1, because, n being here that of the two variables of which the difference is the least, it varies only by one unit; indeed, it is clear that, if one knows ${}_1y_x$, one can determine ${}_2y_x$, ${}_3y_x$, ${}_4y_x, \ldots$ by means of the equation

$$ {}_ny_x = {}_ny_{x-1} + 3.{}_{n-1}y_{x-1}; $$

there is therefore then only ${}_1y_x$ arbitrary.

XIX.

PROBLEM V. — *The equation in the finite and partial differences*

$$ {}_ny_x = {}_nH_x.{}_{n-1}y_{x-1} + {}^1_nH_x.{}_{n-2}y_{x-2} + {}^2_nH_x.{}_{n-1}y_{x-1} + \ldots + {}_nP_x $$

being given, one proposes to integrate it.

Since, in each term of this equation, the variable n decreases according to the same law as the variable x, I can suppose $x = n + K$, K being any constant whatsoever; ${}_ny_x$, ${}_nH_x$, ${}^1_nH_x, \ldots$ become then functions of x and of K; I represent in this case ${}_ny_x$ by u_x; ${}_nH_x$, ${}^1_nH_x, \ldots$ by L_x, ${}^1L_x, \ldots,$ finally ${}_nP_x$ by X_x; the proposed equation becomes therefore

$$ u_x = L_xu_{x-1} + {}^1L_xu_{x-2} + {}^2L_xu_{x-1} + \ldots + X_x, $$

an equation in the ordinary differences, and of which the integral has this form by the preceding Articles, by restoring instead of K its value $x - n$,

$$u_x = C.{}_n z_x + {}^1C.{}_n^1 z_x + {}^2C.{}_n^2 z_x + \ldots + {}_n R_x;$$

C, ${}^1C$, ${}^2C, \ldots$ are some arbitrary constants, which can be functions of K or of $x - n$; one will have therefore

$${}_n y_x = {}_n z_x.\phi(x-n) + {}_n^1 z_x.{}^1\phi(x-n) + {}_n^2 z_x.{}^2\phi(x-n) + \ldots + {}_n R_x;$$

one will determine the arbitrary functions $\phi(x-n)$, ${}^1\phi(x-n)$, $\ldots$ by means of the values of ${}_n y_x$, in as many particular assumptions for x as there are of these arbitrary functions.

The proposed equation in the partial differences is therefore generally integrable, this which comes from this that in each term n and x vary in the same manner; but, if one excepts this case and some others quite rare, it is impossible to have an integral entirely rid of any sign of integration. In order to show it by a quite simple example, I suppose that one has to integrate the equation

$${}_n y_x = {}_n y_{x-1} + {}_{n-1} y_{x-1};$$

by supposing ${}_1 y_x = \phi(x)$, one will have

$${}_2 y_x - {}_2 y_{x-1} = \phi(x-1) \quad \text{or} \quad \Delta.{}_2 y_x = \phi(x),$$

hence ${}_2 y_x = \Sigma\phi(x)$; one will find similarly

$${}_3 y_x = \Sigma^2\phi(x), \qquad {}_4 y_x = \Sigma^3\phi(x),$$

and generally

$${}_n y_x = \Sigma^{n-1}\phi(x);$$

such is therefore the complete value of ${}_n y_x$ by taking care to add to each integration an arbitrary constant.

One can simplify this value and reduce it to some quantities affected with the simple sign of integration, in the following manner.

It is necessary to reduce the double integral $\Sigma^2\phi(x)$ to simple integrals; I make for this

$$\Sigma^2\phi(x) = z_x\,\Sigma\phi(x) - \Sigma t_x\,\phi(x);$$

by differentiating, there comes

$$\Sigma\phi(x) = (z_x + \Delta z_x)[\phi(x) + \Sigma\phi(x)] - z_x \Sigma\phi(x) - t_x\,\phi(x)$$

or

$$\Sigma\phi(x) = (z_x + \Delta z_x - t_x)\phi(x) + \Delta z_x\,\Sigma\phi(x).$$

Therefore $\Delta z_x = 1$ and $t_x = z_x + \Delta z_x;$ I can therefore suppose $z_x = x$ and $t_x = x + 1$, this which gives

$$\Sigma^2\phi(x) = x\,\Sigma\phi(x) - \Sigma(x+1)\phi(x);$$

one will reduce, by a similar process, $\Sigma^3\phi(x)$ to some quantities affected by a single sign of integration; but it will be impossible to rid it of it entirely.

Here is now a method to integrate equations in the partial differences, in which the inconvenience of the quantities affected by many signs of integration is not at all to fear.

XX.

PROBLEM VI. — *The equation in the finite and partial differences*

$$(h)\qquad \begin{cases} {}_ny_x = +A_n \cdot {}_ny_{x-1} + {}^1A_n \cdot {}_ny_{x-2} + {}^2A_n \cdot {}_ny_{x-3} + \dots + N_n \\ \qquad + B_n \cdot {}_{n-1}y_x + {}^1B_n \cdot {}_{n-1}y_{x-1} + {}^2B_n \cdot {}_{n-1}y_{x-2} + \dots \end{cases}$$

being given, one proposes to integrate it.

For this I seek to restore the integration to that of an equation in the ordinary differences. I suppose therefore that one has ${}_1y_x = \phi(x)$; equation (*h*) will give the following

$$(1)\qquad {}_2y_x = A_2 \cdot {}_2y_{x-1} + {}^1A_2 \cdot {}_2y_{x-2} + \dots + N_2 + B_2\phi(x) + {}^1B_2\phi(x-1) + \dots,$$

next

$${}_3y_x = A_3 \cdot {}_3y_{x-1} + {}^1A_3 \cdot {}_3y_{x-2} + \dots + N_3 + B_3 \cdot {}_2y_x + {}^1B_3 \cdot {}_2y_{x-1} + \dots;$$

whence it is easy to conclude

$$
\begin{aligned}
{}_3y_x - A_3\cdot{}_3y_{x-1} - {}^1A_3\cdot{}_3y_{x-2} - \ldots - A_2({}_3y_{x-1} - A_3\cdot{}_3y_{x-2} - \ldots) - {}^1A_1({}_2y_{x-2} - \ldots)\\
= B_3({}_2y_x - A_2\cdot{}_2y_{x-1} - \ldots) + {}^1B_3({}_2y_{x-1} - A_2\cdot{}_2y_{x-2} - \ldots) + \ldots\\
+ N_2(1 - A_2 - {}^1A_2 - \ldots).
\end{aligned}
$$

If one substitutes, instead of

$$
\begin{aligned}
&{}_2y_x - A_2\cdot{}_2y_{x-1} - \ldots,\\
&{}_2y_{x-1} - A_2\cdot{}_2y_{x-2} - \ldots,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}
$$

their values drawn from equation (1), one will have an equation of this form:

$$
{}_3y_x - a_3\cdot{}_3y_{x-1} - {}^1a_3\cdot{}_3y_{x-2} - \ldots = {}_3u_x.
$$

This equation is in the ordinary differences; in order to integrate it by the preceding Articles, it is necessary to know ${}_3u_x$ and the roots of the equation

$$
1 = \frac{a_3}{f} + \frac{{}^1a_3}{f^2} + \frac{{}^2a_1}{f^3} + \ldots;
$$

now this equation is the same as this here

$$
0 = 1 - \frac{A_3}{f} + \frac{{}^1A_3}{f^2} - \ldots - \frac{A_2}{f}\left(1 - \frac{A_3}{f} - \ldots\right) - \frac{{}^1A_2}{f^1}\left(1 - \frac{A_3}{f} - \ldots\right)
$$

and, hence, it is equal to the following

$$
0 = \left(1 - \frac{A_2}{f} - \frac{{}^1A_2}{f^2} - \frac{{}^2A_2}{f^3} - \ldots\right)\left(1 - \frac{A_3}{f} - \frac{{}^1A_3}{f^2} - \ldots\right).
$$

By following the same process for ${}_4y_x$, ${}_5y_x$ and generally for ${}_ny_x$, one will transform equation (*h*) of the Problem in the following

$$
{}_ny_x = a_n\cdot{}_ny_{x-1} + {}^1a_n\cdot{}_ny_{x-2} + \ldots + {}_nu_x, \tag{2}
$$

that it will be easy to integrate it when one will know ${}_nu_x$ and the roots of the equation

$$1 - \frac{a_n}{f} + \frac{{}^1a_n}{f^2} + \frac{{}^2a_n}{f^3} - \dots;$$

one will see easily that this equation is the same as this here

$$0 = \left(1 - \frac{A_2}{f} - \frac{{}^1A_2}{f^2} - \frac{{}^2A_2}{f^3} - \dots\right)\left(1 - \frac{A_3}{f} - \frac{{}^1A_3}{f^2} - \dots\right)\dots\left(1 - \frac{A_n}{f} - \frac{{}^1A_n}{f^2} - \dots\right),$$

whence it is easy to conclude a_n, ${}^1a_n, \dots$.

In order to determine presently the value of ${}_nu_x$, I observe that, from the equation

(2) $${}_ny_x = a_n \cdot {}_ny_{x-1} + {}^1a_n \cdot {}_ny_{x-2} + \dots + {}_nu_x,$$

one draws

$$\begin{aligned} B_n \cdot {}_{n-1}y_x &= B_n . a_{n-1} \cdot {}_{n-1}y_{x-1} + B_n . {}^1a_{n-1} \cdot {}_{n-1}y_{x-2} + \dots + B_n \cdot {}_{n-1}u_x, \\ {}^1B_n \cdot {}_{n-1}y_{x-1} &= {}^1B_n . a_{n-1} \cdot {}_{n-1}y_{x-2} + {}^1B_n . {}^1a_{n-1} \cdot {}_{n-1}y_{x-3} + \dots + {}^1B_n \cdot {}_{n-1}u_{x-1}, \\ \dots&\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

If one adds all these equations member by member, one will have

$$\begin{aligned} &B_n \cdot {}_{n-1}y_x + {}^1B_n \cdot {}_{n-1}y_{x-1} + \dots \\ &\quad = a_{n-1}(B_n \cdot {}_{n-1}y_{x-1} + {}^1B_n \cdot {}_{n-1}y_{x-2} + \dots) \\ &\quad\quad + {}^1a_{n-1}(B_n \cdot {}_{n-1}y_{x-2} + \dots) \\ &\quad\quad + \dots\dots\dots\dots\dots\dots \\ &\quad\quad + B_n \cdot {}_{n-1}u_x + {}^1B_n \cdot {}_{n-1}u_{x-1} + \dots \end{aligned}$$

Now, if one substitutes, instead of

$$\begin{aligned} &B_n \cdot {}_{n-1}y_x + {}^1B_n \cdot {}_{n-1}y_{x-1} + \dots, \\ &B_n \cdot {}_{n-1}y_{x-1} + {}^1B_n \cdot {}_{n-1}y_{x-1} + \dots, \end{aligned}$$

their values given by the equation of the problem, one will have

$$
\begin{aligned}
{}_n y_x - A_n \cdot {}_n y_{x-1} - \ldots - N_n = {} & a_{n-1}({}_n y_{x-1} - A_n \cdot {}_n y_{x-2} - \ldots - N_n) \\
& + {}^1a_{n-1}({}_n y_{x-2} - \ldots - N_n) \\
& + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& + B_n \cdot {}_{n-1}u_x + {}^1B_n \cdot {}_{n-1}u_{x-1} + \ldots
\end{aligned}
$$

By ordering the different terms of this equation, one will have

$$
\begin{aligned}
{}_n y_x = {} & {}_n y_{x-1}(a_{n-1} + A_n) \\
& + {}_n y_{x-2}({}^1a_{n-1} - a_{n-1}.A_n + {}^1A_n) \\
& + {}_n y_{x-3}({}^2a_{n-1} - {}^1a_{n-1}.A_n + a_{n-1}.{}^1A_n + {}^2A_n) \\
& + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& + B_n \cdot {}_{n-1}u_x + {}^1B_n \cdot {}_{n-1}u_{x-1} + \ldots \\
& + N_n(1 - a_{n-1} - {}^1a_{n-1} - \ldots).
\end{aligned}
$$

If one compares now term by term this last equation with equation (2), one will have the following:

$$
\begin{aligned}
a_n &= a_{n-1} + A_n, \\
{}^1a_n &= {}^1a_{n-1} - a_{n-1}.A_n + {}^1A_n, \\
{}^2a_n &= {}^2a_{n-1} - {}^1a_{n-1}.A_n - a_{n-1}.{}^1A_n + {}^2A_n, \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

One could, by integrating these equations, determine a_n, ${}^1a_n$, $\ldots$, if it was not much more simple to conclude them by the preceding method.

Finally one will have

(3) $$ {}_n u_x = N_n(1 - a_{n-1} - {}^1a_{n-1} - {}^2a_{n-1} - \ldots) + B_n \cdot {}_{n-1}u_x + {}^1B_n \cdot {}_{n-1}u_{x-1} + \ldots $$

In order to integrate this last equation, I observe that, since ${}_1y_x = \phi(x)$, one will have ${}_1u_x = \phi(x)$; whence I conclude

$$ {}_2u_x = N_2(1 - a_1 - \ldots) + B_2\phi(x) + {}^1B_2\phi(x-1) + \ldots; $$

one would have in the same manner ${}_3u_x$, ${}_4u_x$, $\ldots$, and one sees that by preceding thus one will have generally

(4) $$ {}_n u_x = b_n\phi(x) + {}^1b_n\phi(x-1) + {}^2b_n\phi(x-2) + \ldots + C_n; $$

therefore

$$
\begin{aligned}
{}_{n-1}u_x &= b_{n-1}\phi(x) + {}^1b_{n-1}\phi(x-1) + \dots + C_{n-1},\\
{}_{n-1}u_{x-1} &= b_{n-1}\phi(x-1) + {}^1b_{n-1}\phi(x-2) + \dots + C_{n-1},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

If one substitutes these values into equation (3), one will have

$$
\begin{aligned}
{}_nu_x = N_n(1 - a_{n-1} - {}^1a_{n-1} - \dots) + C_{n-1}(B_n + {}^1B_n + \dots)\\
+ b_{n-1}B_n\phi(x) + \phi(x-1)({}^1b_{n-1}B_n + b_{n-1}{}^1B_n + \dots;
\end{aligned}
$$

whence, by comparing with equation (4), one will have

$$
\begin{aligned}
b_n &= B_n.b_{n-1},\\
{}^1b_n &= B_n.{}^1b_{n-1} + {}^1B_n.b_{n-1},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
C_n &= C_{n-1}(B_n + {}^1B_n + \dots) + N_n(1 - a_{n-1} - {}^1a_{n-1} - \dots).
\end{aligned}
$$

By integrating these different equations and adding the appropriate constants, one will have the values of b_n, ${}^1b_n$, $\dots$, C_n, and hence that of ${}_nu_x$. The constants must be such, that by supposing $n = 1$ one has ${}_nu_x = \phi(x)$; so that one must have $C_1 = 0$, $b_1 = 1$, ${}^1b_1 = 0$, ${}^2b_1 = 0$, $\dots$.

By integrating equation (2) to which the equation of the problem is reduced, this operation introduces in the expression of ${}_ny_x$ some arbitrary constants, which can be functions of n; but these functions are not arbitrary, since the integral of equation (*h*) can contain no other arbitrary function than $\phi(x)$; one will determine them in this manner.

If one names p_n, ${}^1p_n$, ${}^2p_n, \dots$ the roots of the equation

$$1 = \frac{a_n}{f} + \frac{{}^1a_n}{f^2} + \frac{{}^2a_n}{f^3} + \dots;$$

one will have, by Article X,

$${}_ny_x = C_n.p_n^x + {}^1C_n.{}^1p_n^x + {}^2C_n.{}^2p_n^x + \dots + {}_nL_x.$$

If one substitutes this expression of ${}_ny_x$ into equation (*h*), one will draw from it, by comparing the terms homologous with respect to x, as many differential equations as there are functions C_n, ${}^1C_n$, $\ldots$, and, by integrating these equations, one will determine these functions.

Instead of making ${}_1y_x = \phi(x)$, one can imagine a differential equation any whatsoever between ${}_1y_x$ and x; I suppose that this equation is that of a recurrent sequence, so that one has

$$ {}_1y_x = F.{}_1y_{x-1} + {}^1F.{}_1y_{x-2} + \ldots + L, $$

F, ${}^1F$, $\ldots$ and L being constants; by following the method of the problem, one will arrive to the following equation

$$ (5) \qquad {}_ny_x = a_n.{}_ny_{x-1} + {}^1a_n.{}_ny_{x-2} + {}^2a_n.{}_ny_{x-3} + \ldots + u_n, $$

and one will find that the equation

$$ 1 = \frac{a_n}{f} + \frac{{}^1a_n}{f^2} + \frac{{}^2a_n}{f^3} + \ldots $$

is the same as this here:

$$ 0 = \left(1 - \frac{F}{f} - \frac{{}^1F}{f^2} - \ldots\right)\left(1 - \frac{A_2}{f} - \frac{{}^1A_2}{f^2} - \ldots\right)\ldots\left(1 - \frac{A_n}{f} - \frac{{}^1A_n}{f^2} - \ldots\right) $$

One will have next

$$ u_n = u_{n-1}(B_n + {}^1B_n + \ldots) + N_n(1 - a_{n-1} - {}^1a_{n-1} - \ldots), $$

whence it will be easy to conclude the value of ${}_ny_x$.

The case in which the equation between ${}_1y_x$ and x is that of a recurrent sequence is the one which is encountered most frequently in the application of this theory.

One can observe here that the quantities B_n, ${}^1B_n$, $\ldots$ enter not at all into the formation of a_n, ${}^1a_n$, $\ldots$, but simply in that of u_n; whence it follows that, when this quantity is null (this which must happen very often), equation (5) will remain the same thing as the quantities B_n, ${}^1B_n$, $\ldots$ are; thence there results that, in this case, these quantities influence in the solution of the problem only on the determination of the arbitrary constants which come from the integration of equation (5).

XXI.

In order to clarify the preceding theory with some examples, I suppose that one has the two equations

$$
\begin{aligned}
{}_1y_x &= 2.{}_1y_{x-1}, \\
{}_ny_x &= 2.{}_ny_{x-1} + 2.{}_{n-1}y_{x-1}.
\end{aligned}
$$

If in the first equation one makes ${}_1y_1 = 1$, one will form in its way the following sequence $1, 2, 4, 8, 16, \dots$ The second equation gives

$$
{}_2y_x = 2.{}_2y_{x-1} + 2.{}_1y_{x-1},
$$

and, if one supposes ${}_2y_1 = 0$, one will have ${}_2y_2 = 2$, ${}_2y_3 = 8, \dots;$ one will form in this manner the sequence $0, 2, 8, 24, \dots$ By continuing thus and supposing always ${}_3y_1 = 0$, ${}_4y_1 = 0$, ${}_5y_1 = 0$, $\dots$ one will form the *récurro-récurrentes* sequences:

	1	2	3	4	5	6	7	8	$\dots$	x
1	1	2	4	8	16	32	64	128	$\dots$	.
2	0	2	8	24	64	160	384	896	$\dots$	.
3	0	0	4	24	96	320	960	2688	$\dots$	.
4	0	0	0	8	64	320	1280	4480	$\dots$	.
5	0	0	0	0	16	160	960	4880	$\dots$	.
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	.	.	.	.	.	.	.	.	.	.

It is necessary presently to determine the general term of these sequences or, this which reverts to the same, the expression of ${}_ny_x$.

For this, I observe that one has, by the preceding Article,

$$
{}_ny_x = a_n.{}_ny_{x-1} + {}^1a_n.{}_ny_{x-2} + \dots + u_n,
$$

next the equation

$$
1 = \frac{a_n}{f} + \frac{{}^1a_n}{f^2} + \dots
$$

is in this case this here

$$
0 = \left(1 - \frac{2}{f}\right)^n,
$$

of which all the roots are equal to 2; one has, moreover, $u_n = 2u_{n-1}$; therefore $u_n = H.2^n$. Now, putting $n = 1$, one has $u_n = 0$, therefore $H = 0$; one will have thus, by Article IX,

$$ {}_ny_x = 2^{x-1}\Big[C_n\frac{(x-1)(x-2)\dots(x-n+1)}{1.2.3\dots(n-1)} + D_n\frac{(x-1)(x-2)\dots(x-n+2)}{1.2.3\dots(n-2)} + E_n\frac{(x-1)\dots(x-n+3)}{1.2.3\dots(n-3)} + \cdots\Big]. $$

In order to determine the arbitrary constants C_n, D_n, …, one will substitute this value of ${}_ny_x$ into the equation

$$ {}_ny_x = 2.{}_ny_{x-1} + 2.{}_{n-1}y_{x-1}, $$

by observing that

$$ \frac{(x-1)(x-2)\dots(x-n+1)}{1.2.3\dots(n-1)} = \frac{(x-2)(x-3)\dots(x-n)}{1.2.3\dots(n-1)} + \frac{(x-2)\dots(x-n+1)}{1.2.3\dots(n-2)}, $$
$$ \frac{(x-1)(x-2)\dots(x-n+2)}{1.2.3\dots(n-2)} = \frac{(x-2)\dots(x-n+1)}{1.2.3\dots(n-2)} + \frac{(x-2)\dots(x-n+2)}{1.2.3\dots(n-3)}, $$
$$ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, $$

and one will have

$$ C_n\frac{(x-2)(x-3)\dots(x-n)}{1.2.3\dots(n-1)} + (C_n+D_n)\frac{(x-2)\dots(x-n+1)}{1.2.3\dots(n-2)} + (D_n+E_n)\frac{(x-2)\dots(x-n+2)}{1.2.3\dots(n-3)} + \dots $$
$$ = C_n\frac{(x-2)\dots(x-n)}{1.2.3\dots(n-1)} + (D_n+C_{n-1})\frac{(x-2)\dots(x-n+1)}{1.2.3\dots(n-2)} + (E_n+D_{n-1})\frac{(x-3)\dots(x-n+3)}{1.2.3\dots(n-3)} + \dots $$

By comparing term by term, one will have:

1° $C_n = C_{n-1}$; therefore $C_n = A$. Now, putting $n = 1$, the quantity

$$\frac{1(x-1)(x-2)\dots(x-n+1)}{1.2.3\dots(n-1)}$$

is reduced to its first factor 1, and the quantities following

$$\frac{1(x-1)(x-2)\dots(x-n+2)}{1.2.3\dots(n-2)}, \quad \dots$$

become nulls; therefore ${}_1y_x = A.2^{x-1}$. Now one has ${}_1y_1 = 1$; therefore $A = 1 = C_n$.

2° $D_n = D_{n-1}$, hence $D_n = A$ and ${}_2y_x = 2^{x-1}\left(\frac{x-1}{1} + A\right)$. Now putting $x = 1$, one has ${}_2y_1 = 0$ by the formation of the previous sequences; therefore $A = 0$ and $D_n = 0$.

One will find similarly $E_n = 0$, $F_n = 0$, $\dots$; therefore

$${}_ny_x = 2^{x-1}\frac{(x-1)(x-2)\dots(x-n+1)}{1.2.3\dots(n-1)}.$$

Let, for example, $x = 8$ and $n = 5$; one will have

$${}_5y_8 = 2^7\frac{7.6.5.4}{1.2.3.4} = 4480.$$

I take further for example the two equations

$$\begin{aligned} &{}_1y_x = 2.{}_1y_{x-1}, \\ &{}_ny_x = (n+1).{}_ny_{x-1} + {}_{n-1}y_{x-1}. \end{aligned}$$

If one supposes

$${}_1y_1 = 1, \quad {}_2y_1 = 0, \quad {}_3y_1 = 0, \quad {}_4y_1 = 0, \quad \dots,$$

one will form the following sequences:

	1	2	3	4	5	6	7	8	…	x
1	1	2	4	8	16	32	64	.	…	.
2	0	1	5	19	65	211	665	.	…	.
3	0	0	1	9	55	285	1351	.	…	.
4	0	0	0	1	14	125	910	.	…	.
5	0	0	0	0	1	20	245	.	…	.
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
n	.	.	.	.	.	.	.	.	.	.

In order to find now the general term of these sequences, or the expression of ${}_ny_x$, I observe that one has, by the previous Article,

$${}_ny_x = a_n . {}_ny_{x-1} + {}^1a_n . {}_ny_{x-2} + {}^2a_n . {}_ny_{x-3} + \ldots + u_n,$$

and that the equation

$$1 = \frac{a_n}{f} + \frac{{}^1a_n}{f^2} + \frac{{}^2a_n}{f^3} + \ldots$$

is the same as this here

$$0 = \left(1 - \frac{2}{f}\right)\left(1 - \frac{3}{f}\right)\left(1 - \frac{4}{f}\right)\cdots\left(1 - \frac{n+1}{f}\right);$$

finally, that one has

$$u_n = 2u_{n-1};$$

whence, by integrating,

$$u_n = H2^n.$$

Now, putting $n = 1$, one has $u_1 = 0$; therefore

$$H = 0 \quad \text{and} \quad u_n = 0.$$

By integrating, one will have therefore

$${}_ny_x = C_n 2^{x-1} + {}^1C_n 3^{x-1} + {}^2C_n 4^{x-1} + \ldots + {}^{n-1}C_n (n+1)^{x-1},$$

an equation in which it is necessary presently to determine the arbitrary constants

C_n, ${}^1C_n$, ... For this, I substitute this value of ${}_ny_x$ into the equation

$$ {}_ny_x = (n+1).{}_ny_{x-1} + {}_{n-1}y_{x-1}, $$

this which gives

$$ \begin{aligned} &C_n 2^{x-1} + {}^1C_n 3^{x-1} + \dots \\ &\quad = (n+1)C_n 2^{x-2} + (n+1).{}^1C_n 3^{x-2} + \dots + C_{n-1}2^{x-2} + {}^1C_{n-1}3^{x-2} + \dots; \end{aligned} $$

whence, by comparing term by term, I will have

$$ \begin{aligned} 2.C_n &= (n+1).C_n + C_{n-1}, \\ 3.{}^1C_n &= (n+1).{}^1C_n + {}^1C_{n-1}, \\ &\dots\dots\dots\dots\dots\dots \end{aligned} $$

It is clear that the first equation begins to hold only when $n = 2$; the second, when $n = 3$; the third, when $n = 4$, ... By integrating the first, one will have

$$ C_n = \frac{C_1}{(1-2)(1-3)(1-4)\dots(1-n)}. $$

Now, since one has ${}_1y_x = 2^{x-1}$, one will have $C_1 = 1$; therefore

$$ C_n = \pm \frac{1}{1.2.3\dots(n-1)}, $$

the $+$ sign holding if n is odd, and the $-$ sign if it is even.

One will have similarly

$$ {}^1C_n = \frac{{}^1C_2}{(2-3)(2-4)\dots(2-n)}. $$

Now, putting $n = 2$, one has

$$ {}_2y_x = C_2 2^{x-1} + {}^1C_2 3^{x-1} = {}^1C_2 3^{x-1} - 2^{x-1}. $$

Therefore, since ${}_2y_1 = 0$, one will have ${}^1C_2 = 1$; hence

$$ {}^1C_n = \mp \frac{1}{1.2.3\dots(n-2)}, $$

the $+$ sign having place if n is even, and the $-$ sign if it is odd. One will find, by a similar calculation,

$$
\begin{aligned}
{}^2C_n &= \pm \frac{1}{1.2} \frac{1}{1.2.3\ldots(n-3)},\\
{}^3C_n &= \mp \frac{1}{1.2.3} \frac{1}{1.2.3\ldots(n-4)},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

Therefore

$$
\begin{aligned}
{}_ny_x = \pm \frac{1}{1.2.3\ldots(n-1)}\Bigg[2^{x-1} - \frac{n-1}{1}3^{x-1} + \frac{(n-1)(n-2)}{1.2}4^{x-1} \\
- \frac{(n-1)(n-2)(n-3)}{1.2.3}5^{x-1} + \ldots \pm (n+1)^{x-1}\Bigg],
\end{aligned}
$$

the $+$ sign having place if n is odd, and the $-$ sign if it is even. Let $n = 4$ and $x = 7$; one will have

$$
{}_4y_7 = -\frac{1}{1.2.3}(2^6 - 3.3^6 + 4.4^6 - 5^6) = 910.
$$

XXII.

PROBLEM VII. — *The differential equation*

$$
\begin{aligned}
{}_ny_x + A_n.{}_ny_{x-1} + {}^1A_n.{}_ny_{x-2} + \ldots + N_n = B_n.{}_{n-1}y_x + {}^1B_n.{}_{n-1}y_{x-1} + \ldots \\
+ C_n.{}_{n-2}y_x + {}^1C_n.{}_{n-2}y_{x-1} + \ldots
\end{aligned}
$$

being given, one proposes to integrate it.

In following the analysis of the preceding Problem, I make ${}_1y_x = \phi(x)$ and ${}_2y_x = {}^1\phi(x)$; the proposed equation will give therefore

$$
\begin{aligned}
{}_3y_x + A_3.{}_3y_{x-1} + {}^1A_3.{}_3y_{x-2} + \ldots + N_3 = B_3.{}^1\phi(x) + {}^1B_3.{}^1\phi(x-1) + \ldots \\
+ C_3.\phi(x) + {}^1C_3.\phi(x-1) + \ldots
\end{aligned}
$$

and

$$
\begin{aligned}
{}_4y_x + A_4 \cdot {}_4y_{x-1} + {}^1A_4 \cdot {}_4y_{x-2} + \ldots + N_4 \\
= B_4 \cdot {}_3y_x + {}^1B_4 \cdot {}_3y_{x-1} + \ldots \\
+ C_4 . {}^1\phi(x) + {}^1C_4 . {}^1\phi(x-1) + \ldots
\end{aligned}
$$

whence one will draw

$$
\begin{aligned}
&{}_4y_x + A_4 \cdot {}_4y_{x-1} + {}^1A_4 \cdot {}_4y_{x-2} + \ldots + N_4 \\
&\quad + A_4({}_4y_{x-1} + A_4 \cdot {}_4y_{x-2} + \ldots) \\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= B_4({}_3y_x + A_3 \cdot {}_3y_{x-1} + \ldots) \\
&\quad + {}^1B_4({}_3y_{x-1} + A_3 \cdot {}_3y_{x-2} + \ldots) \\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\quad + C_4 . {}^1\phi(x) + {}^1C_3 . {}^1\phi(x-1) + \ldots \\
&\quad + A_3 . C_4 . {}^1\phi(x-1) + \ldots
\end{aligned}
$$

Now, if one substitutes into this equation, instead of

$$
\begin{aligned}
&{}_3y_x + A_3 \cdot {}_3y_{x-1} + \ldots, \\
&{}_3y_{x-1} + A_3 \cdot {}_3y_{x-2} + \ldots,
\end{aligned}
$$

their values, one will have an equation of this form

$$
{}_4y_x = a_4 \cdot {}_4y_{x-1} + {}^1a_4 \cdot {}_4y_{x-2} + {}^2a_4 \cdot {}_4y_{x-3} + \ldots + {}_4u_x
$$

This equation will be integrated by that which precedes, as soon as one will know ${}_4u_x$ and the roots of the equation

$$
1 = \frac{a_4}{f} + \frac{{}^1a_4}{f^2} + \frac{{}^2a_4}{f^3} + \ldots
$$

Now it is easy to see that this equation is the same as this one here

$$
0 = \left(1 + \frac{A_3}{f} + \frac{{}^1A_3}{f^2} + \ldots\right)\left(1 + \frac{A_4}{f} + \frac{{}^1A_4}{f^2} + \ldots\right).
$$

By following the same process for ${}_5y_x$, ${}_6y_x$, $\ldots$, and generally for ${}_ny_x$, one will arrive to an equation of this form

(A) $$ {}_n y_x = a_n \cdot {}_n y_{x-1} + {}^1a_n \cdot {}_n y_{x-2} + {}^2a_n \cdot {}_n y_{x-3} + \dots + {}_n u_x, $$

an equation which will be easily integrable when one will know ${}_n u_x$ and the roots of the equation

$$ 1 = \frac{a_n}{f} + \frac{{}^1a_n}{f^2} + \frac{{}^2a_n}{f^3} + \dots $$

Now one will find easily that this equation is the same as this one

$$ 0 = \left(1 + \frac{A_3}{f} + \frac{{}^1A_3}{f^2} + \dots\right)\left(1 + \frac{A_4}{f} + \frac{{}^1A_4}{f^2} + \dots\right)\dots\left(1 + \frac{A_n}{f} + \frac{{}^1A_n}{f^2} + \dots\right), $$

whence it is easy to conclude a_n, ${}^1a_n, \dots$

In order to determine presently ${}_n u_x$, I observe that the equation of the Problem gives the following:

$$ \begin{aligned} &{}_n y_x - a_n \cdot {}_n y_{x-1} - {}^1a_n \cdot {}_n y_{x-2} - \dots + N_n(1 - a_n - {}^1a_n - \dots) \\ &\qquad + A_n({}_n y_{x-1} - {}^1a_n \cdot {}_n y_{x-2} - \dots) \\ &\qquad + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &= B_n({}_{n-1} y_x - a_n \cdot {}_{n-1} y_{x-1} - \dots) \\ &\qquad + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\qquad + C_n({}_{n-2} y_x - a_n \cdot {}_{n-2} y_{x-1} - \dots) \\ &\qquad + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned} $$

Now one has, by equation (A),

$$ \begin{aligned} {}_n y_x - a_n \cdot {}_n y_{x-1} - \dots &= {}_n u_x, \\ {}_n y_{x-1} - a_n \cdot {}_n y_{x-2} - \dots &= {}_n u_{x-1}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots&; \end{aligned} $$

moreover,

$$ \begin{aligned} &{}_{n-1} y_x - a_n \cdot {}_{n-1} y_{x-1} - {}^1a_n \cdot {}_{n-1} y_{x-2} - \dots \\ &\quad = {}_{n-1} y_x - a_{n-1} \cdot {}_{n-1} y_{x-1} - \dots + A_n({}_{n-1} y_{x-1} - a_{n-1} \cdot {}_{n-1} y_{x-2} - \dots) + \dots \\ &\quad = {}_{n-1} u_x + A_n \cdot {}_{n-1} u_{x-1} + \dots; \end{aligned} $$

similarly,

$$
\begin{aligned}
&{}_{n-2}y_x - a_{n-1}\cdot{}_{n-2}y_{x-1} - \dots \\
&\quad = {}_{n-2}y_x - a_{n-2}\cdot{}_{n-2}y_{x-1} - \dots + A_{n-1}({}_{n-2}y_{x-1} - \dots) + \dots \\
&\quad = {}_{n-2}u_x + A_{n-1}\cdot{}_{n-2}u_{x-1} + \dots
\end{aligned}
$$

and

$$
\begin{aligned}
&{}_{n-2}y_x - a_n\cdot{}_{n-2}y_{x-1} - \dots \\
&\quad = {}_{n-2}y_x - a_{n-1}\cdot{}_{n-2}y_{x-1} - \dots + A_n({}_{n-2}y_{x-1} - \dots) + \dots \\
&\quad = {}_{n-2}u_x + A_{n-1}\cdot{}_{n-2}u_{x-1} + \dots + A_n({}_{n-2}u_{x-1} + A_{n-1}\cdot{}_{n-2}u_{x-2} + \dots) + \dots;
\end{aligned}
$$

therefore

$$
\text{(V)}\quad \left\{
\begin{aligned}
&{}_nu_x + A_n\cdot{}_nu_{x-1} + {}^1A_n\cdot{}_nu_{x-2} + \dots + N_n(1 - a_n - {}^1a_n - \dots) \\
&\quad = B_n({}_{n-1}u_x + A_n\cdot{}_{n-1}u_{x-1} + \dots) + {}^1B_n({}_{n-1}u_{x-1} + \dots) + \dots \\
&\qquad + C_n[{}_{n-2}u_x + A_{n-1}\cdot{}_{n-2}u_{x-1} + \dots + A_n({}_{n-2}u_{x-1} + A_{n-1}\cdot{}_{n-2}u_{x-2}\dots) \\
&\qquad\qquad + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots] \\
&\qquad + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
\right.
$$

In order to integrate this equation, one will observe that the value of ${}_nu_x$ must have this form

$$
\begin{aligned}
{}_nu_x &= b_n\phi(x) + {}^1b_n\phi(x-1) + {}^2b_n\phi(x-2) + \dots \\
&\quad + c_n{}^1\phi(x) + {}^1c_n{}^1\phi(x-1) + {}^2c_n{}^1\phi(x-2) + \dots + g_n.
\end{aligned}
$$

There is no longer now a question but to determine b_n, ${}^1b_n$, $\dots$, c_n, ${}^1c_n$, $\dots$, g_n. For this, one will substitute this value of ${}_nu_x$ into equation (V), this which gives

$$
\begin{aligned}
b_n\phi(x) &+ \phi(x-1)({}^1b_n + A_n.b_n) + \dots \\
&\quad + c_n{}^1\phi(x) + {}^1\phi(x-1)({}^1c_n + A_n.c_n) + \dots \\
= \phi(x)&(B_nb_{n-1} + C_nb_{n-2}) \\
&\quad + \phi(x-1)[B_n{}^1b_{n-1} + B_nA_nb_{n-1} + {}^1B_nb_{n-1} \\
&\qquad + C_n{}^1b_{n-2} + C_nA_{n-1}b_{n-2} + C_nA_nb_{n-2} + {}^1C_nb_{n-2}] \\
&\quad + \dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\quad + {}^1\phi(x)(B_nc_{n-1} + C_nc_{n-2}) \\
&\quad + {}^1\phi(x-1)[B_n{}^1c_{n-1} + B_nA_nc_{n-1} + {}^1B_nc_{n-1} \\
&\qquad + C_n{}^1c_{n-2} + C_nA_{n-1}c_{n-2} + C_nA_nc_{n-2} + {}^1C_nc_{n-2}] \\
&\quad + \dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

whence one will have

$$
\begin{aligned}
b_n &= B_nb_{n-1} + C_nb_{n-2} \\
{}^1b_n &= B_n{}^1b_{n-1} + C_n^1b_{n-2} + b_{n-1}(B_nA_n + {}^1B_n + C_nA_{n-1}) + b_{n-2}(C_nA_n + {}^1C_n), \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
c_n &= B_nc_{n-1} + C_nc_{n-2}, \\
&\dots\dots\dots\dots\dots ;
\end{aligned}
$$

by integrating, one will have the values of b_n, ${}^1b_n$, $\dots$, c_n, ${}^1c_n$, $\dots$

These equations ascend to the second differences, their integral must contain two arbitrary constants. Now, by supposing $n = 1$,

$$ {}_ny_x = \phi(x). $$

One must therefore have then

$$
\begin{aligned}
b_n &= 1, \quad {}^1b_n = 0, \quad {}^2b_n = 0, \quad \dots, \\
c_n &= 0, \quad {}^1c_n = 0, \quad {}^2c_n = 0, \quad \dots,
\end{aligned}
$$

Moreover, by supposing $n = 2$,

$$ {}_ny_x = {}^1\phi(x). $$

Therefore then

$$b_n = 0, \quad {}^1b_n = 0, \quad {}^2b_n = 0, \quad \dots,$$
$$c_n = 1, \quad {}^1c_n = 0, \quad {}^2c_n = 0, \quad \dots,$$

By means of these conditions, it will be easy to determine the arbitrary constants. Knowing thus the expression of ${}_nu_x$, there is no longer a question but to integrate equation (A), and the arbitrary constants that the integration introduces, which can be functions of n, will be determined by the method that I have given (Art. XX).

If, instead of the two equations

$$\begin{aligned} {}_1y_x &= \phi(x) \\ {}_2y_x &= {}^1\phi(x). \end{aligned}$$

one had the two following

$$\begin{aligned} &{}_1y_x + E.{}_1y_{x-1} + {}^1E.{}_1y_{x-2} + \dots + K = 0, \\ &{}_2y_x + H.{}_2y_{x-1} + {}^1H.{}_2y_{x-2} + \dots + L = F.{}_1y_x + {}^1F.{}_1y_{x-1} + \dots, \end{aligned}$$

one will arrive, by the preceding method, to an equation of this form

$${}_ny_x = a_n.{}_ny_{x-1} + {}^1a_n.{}_ny_{x-2} + \dots + {}_nu_x,$$

and one will find that the equation

$$1 = \frac{a_n}{f} + \frac{{}^1a_n}{f^2} + \dots$$

is the same as this one here:

$$\begin{aligned} 0 = &\left(1 - \frac{E}{f} + \frac{{}^1E}{f^2} + \dots\right)\left(1 - \frac{H}{f} + \frac{{}^1H}{f^2} + \dots\right) \\ &\times \left(1 - \frac{A_3}{f} + \dots\right)\dots\left(1 - \frac{A_n}{f} + \dots\right). \end{aligned}$$

In order to determine u_n, one must observe that in this case equation (V) becomes

$$\begin{aligned}u_n(1 + A_n + {}^1A_n + \dots) &+ N_n(1 - a_n - {}^1a_n - \dots)\\ &= u_{n-1}(1 + A_n + \dots)(B_n + {}^1B_n + \dots)\\ &\quad + u_{n-1}(1 + A_{n-1} + \dots)(1 + A_n + \dots)(C_n + \dots);\end{aligned}$$

now

$$1 - a_n - {}^1a_n - \dots = (1 - a_{n-1} - {}^1a_{n-1} - \dots)(1 + A_n + {}^1A_n + \dots);$$

therefore

$$\begin{aligned}u_n = N_n&(a_{n-1} + {}^1a_{n-1} + \dots - 1)\\ &+ u_{n-1}(B_n + {}^1B_n + \dots) + u_{n-2}(1 + A_{n-1} + \dots)(C_n + {}^1C_n + \dots).\end{aligned}$$

This equation being differential of the second order contains two arbitrary constants; they will be determined by means of the values of u_1 and u_2. Now one has

$$\begin{aligned}&u_1 = -L,\\ &u_2 = -L(1 + E + {}^1E + \dots) - K(F + {}^1F + \dots).\end{aligned}$$

XXIII.

Although, in the last two problems, the equations in the partial differences considered with respect to the variable n do not pass the second order, one sees however that the method will succeed generally, whatever be the degree of the difference of the variables. Thus method supposes in truth that ${}_1y_x$ or ${}_1y_x$ and ${}_2y_x, \dots$ according to the degree of the difference of n, are given as functions of x, or by some linear equations among x and these quantities; now it can happen that this is not. I suppose, for example, that one has the following equations:

$$\begin{aligned}&{}_1y_x = {}_2y_{x-1},\\ &{}_2y_x = {}_1y_{x-1} + {}_3y_{x-1},\\ &\dots\dots\dots\dots\dots\dots\dots,\\ &{}_ny_x = {}_{n-1}y_{x-1} + {}_{n+1}y_{x-1},\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots,\\ &{}_my_x = {}_{m-1}y_{x-1}.\end{aligned}$$

The equation

$$ {}_{n}y_x = {}_{n-1}y_{x-1} + {}_{n+1}y_{x-1} $$

is in the partial differences; but it differs from the preceding equations:

1° In this that ${}_1y_x$ and ${}_2y_x$ are not at all given as functions of x, or by two differential equations;

2° In this that it ceases to hold when $n = m$.

As this kind of equations are encountered sometimes, and principally in the analysis of hazards, I am going to give here the manner to integrate them.

I observe for this that, if one was able to reduce the equation

$$ {}_{n}y_x = {}_{n-1}y_{x-1} + {}_{n+1}y_{x-1}, $$

which is of the third order with respect to n, to another of the second order, the problem would be resolved; I suppose indeed that the equation of the second order is

$$ {}_{n}y_x = a_n \cdot {}_{n}y_{x-1} + {}^{1}a_n \cdot {}_{n}y_{x-2} + \ldots + u_n + b_n \cdot {}_{n+1}y_x + {}^{1}b_n \cdot {}_{n+1}y_{x-1} + \ldots $$

In the case $n = m - 1$, one will have

$$ {}_{m-1}y_x = a_{m-1} \cdot {}_{m-1}y_{x-1} + {}^{1}a_{m-1} \cdot {}_{m-1}y_{x-2} + \ldots + u_{m-1} + b_{m-1} \cdot {}_{m}y_x + \ldots, $$

whence, eliminating ${}_{m-1}y_x$ by means of the equation ${}_{m}y_x = {}_{m-1}y_{x-1}$, one will have an equation in the ordinary differences between x and ${}_{m}y_x$.

All difficulty consists therefore to lower the equation from the third order, with respect to n,

$$ {}_{n}y_x = {}_{n-1}y_{x-1} + {}_{n+1}y_{x-1} $$

to one of the second order; this is the object of the following problem.

PROBLEM VIII. — *The equation in the partial differences of the second order, with respect to* n*,*

$$ (\gamma) \qquad \begin{cases} {}_{n}y_x = A_n \cdot {}_{n}y_{x-1} + {}^{1}A_n \cdot {}_{n}y_{x-2} + \ldots + N_n \\ \qquad + B_n \cdot {}_{n+1}y_x + {}^{1}B_n \cdot {}_{n+1}y_{x-1} + {}^{2}B_n \cdot {}_{n+1}y_{x-2} + \ldots \\ \qquad + C_n \cdot {}_{n+1}y_x + {}^{1}C_n \cdot {}_{n+1}y_{x-1} + {}^{2}C_n \cdot {}_{n+1}y_{x-2} + \ldots \end{cases} $$

being given, it is necessary to lower it to another of the first order with respect to n.

It is necessary for this that, under a particular assumption for n, this equation is reduced to one of the first order. I suppose therefore that, by making $n = 1$, one has this here

$$(\eta) \qquad {}_1y_x = F\cdot{}_1y_{x-1} + {}^1F\cdot{}_1y_{x-2} + \ldots + L + H\cdot{}_2y_x + {}^1H\cdot{}_2y_{x-1} + \ldots$$

It is easy to see, this put, that equation (γ) can always be transformed into the following (θ), of the second order with respect to n,

$$(\theta) \qquad \begin{cases} {}_ny_x = a_n\cdot{}_ny_{x-1} + {}^1a_n\cdot{}_ny_{x-2} + {}^2a_n\cdot{}_ny_{x-1} + \ldots + u_n \\ \qquad + b_n\cdot{}_{n+1}y_x + {}^1b_n\cdot{}_{n+1}y_{x-1} + {}^2b_n\cdot{}_{n+1}y_{x-2} + \ldots, \end{cases}$$

from which one will determine the coefficients a_n, ${}^1a_n$, $\ldots$, b_n, ${}^1b_n$, $\ldots$ in this manner: the equation (θ) gives this here

$$\begin{aligned} C_n\cdot{}_{n+1}y_x &= C_n(a_{n-1}\cdot{}_{n-1}y_{x-1} + {}^1a_{n-1}\cdot{}_{n-1}y_{x-2} + {}^2a_n\cdot{}_{n-1}y_{x-3} + \ldots + u_{n-1} \\ &\quad + b_{n-1}\cdot{}_ny_x + {}^1b_{n-1}\cdot{}_ny_{x-1} + {}^2b_{n-1}\cdot{}_ny_{x-2} + \ldots), \\ {}^1C_n\cdot{}_{n-1}y_{x-1} &= {}^1C_n(a_{n-1}\cdot{}_{n-1}y_{x-2} + {}^1a_{n-1}\cdot{}_{n-1}y_{x-3} + \ldots + u_{n-1} \\ &\quad + b_{n-1}\cdot{}_ny_{x-1} + {}^1b_{n-1}\cdot{}_ny_{x-2} + \ldots) \end{aligned}$$

. . . .

If one adds these different equations member by member, and if one substitutes in their sum, instead of

$$\begin{gathered} C_n\cdot{}_{n-1}y_x + {}^1C_n\cdot{}_{n-1}y_{x-1} + \ldots, \\ C_n\cdot{}_{n-1}y_{x-1} + {}^1C_n\cdot{}_{n-1}y_{x-2} + \ldots, \end{gathered}$$

their values which furnish equation (γ), one will have, after having ordered,

$$
\begin{aligned}
{}_{n}y_x = \frac{1}{1-b_{n-1}C_n}[&{}_{n}y_{x-1}(a_{n-1}+A_n+b_{n-1}{}^{1}C_n+{}^{1}b_{n-1}C_n)\\
&+{}_{n}y_{x-2}({}^{1}a_{n-1}-a_{n-1}A_n+{}^{1}A_n\\
&\qquad+b_{n-1}{}^{2}C_n+{}^{1}b_{n-1}{}^{1}C_n+{}^{2}b_{n-1}C_n)\\
&+{}_{n}y_{x-3}({}^{2}a_{n-1}-{}^{1}a_{n-1}A_n-a_{n-1}{}^{1}A_n+{}^{2}A_n\\
&\qquad+b_{n-1}{}^{3}C_n+{}^{1}b_{n-1}{}^{2}C_n+{}^{2}b_{n-1}{}^{1}C_n+{}^{3}b_{n-1}C_n)\\
&+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&+{}_{n+1}y_xB_n\\
&+{}_{n+1}y_{x-1}({}^{1}B_n-a_{n-1}B_n)\\
&+{}_{n+1}y_{x-2}({}^{2}B_n-a_{n-1}{}^{1}B_n-{}^{1}a_{n-1}B_n)\\
&+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&+u_{n-1}(C_n+{}^{1}C_n+{}^{2}C_n+\dots)\\
&\qquad+N_n(1-a_{n-1}-{}^{1}a_{n-1}-{}^{2}a_{n-1}-\dots)].
\end{aligned}
$$

By comparing this equation with equation (θ), one will have

1° $$b_n = \frac{B_n}{1-C_nb_{n-1}}.$$

In order to integrate this equation, I make $b_n = \frac{z_{n-1}}{z_n}$; this which gives

$$0 = z_{n-1} + C_nz_{n-2} + B_nz_n,$$

a linear equation in the ordinary differences.

2° $${}^{1}b_n = \frac{{}^{1}B_n - a_{n-1}B_n}{1-C_nb_{n-1}},$$

3° $$a_n = \frac{A_n + a_{n-1} + {}^{1}b_{n-1}C_n + b_{n-1}{}^{1}C_n}{1-C_nb_{n-1}}.$$

From the first of these equations, one will have

$$ {}^1b_{n-1} = \frac{{}^1B_{n-1} - a_{n-2}B_{n-1}}{1 - C_{n-1}b_{n-2}}; $$

substituting this value of b_{n-1} in the second, one will have

$$ a_n = \frac{A_n + a_{n-1} + C_n \frac{{}^1B_{n-1} - a_{n-2}B_{n-1}}{1 - C_{n-1}b_{n-2}} + b_{n-1}\,{}^1C_n}{1 - C_n b_{n-1}}, $$

whence one will have a_n, hence ${}^1b_n$, and thus the rest.

Finally, one will determine u_n by this equation

$$ u_n = u_{n-1}\frac{C_n + {}^1C_n + \ldots + N_n(1 - a_{n-1} - {}^1a_{n-1} - \ldots)}{1 - C_n b_{n-1}}. $$

Equation (γ) of the second order with respect to n will be lowered to another (θ) of the first order; and one sees that the preceding method will succeed generally, whatever be the order of the proposed.

XXIV.

On the equations in finite and partial differences in four variables.

Until now I have considered the equations in the partial differences among three variables ${}_ny_x$, n and x; I am going presently to say a word on those which contain a greater number of them.

I suppose that ${}_{m,n}y_x$ represents a function of three variables x, m and n, of which I regard the differences as constants and equal to unity; I am able, in this function, to make m, n and x vary separately, or two of these quantities at once, or all three together in any relation whatsoever; now, if there exists an equation among these different variations, it will be that which I name an *equation in the partial differences in four variables*. This put,

PROBLEM IX. — *I suppose that one has the equation in the partial differences in four variables*

$$(\Omega)\quad\begin{cases} {}_{m,n}y_x + {}_mA_n\cdot{}_{m,n}y_{x-1} + {}_m^1A_n\cdot{}_{m,n}y_{x-2} + \ldots + {}_mN_n \\ \qquad + {}_mB_n\cdot{}_{m,n-1}y_x + {}_m^1B_n\cdot{}_{m,n-1}y_{x-1} + {}_m^2B_n\cdot{}_{m,n-1}y_{x-2} + \ldots \\ \qquad = {}_mC_n\cdot{}_{m-1,n}y_x + {}_m^1C_n\cdot{}_{m-1,n}y_{x-1} + {}_m^2C_n\cdot{}_{m-1,n}y_{x-2} + \ldots; \end{cases}$$

one proposes to determine ${}_{m,n}y_x$.

I suppose that, in the case of $n = 1$, one has, or one can have the following equation

$${}_{m,1}y_x + D_m\cdot{}_{m,1}y_{x-1} + {}^1D_m\cdot{}_{m,1}y_{x-2} + \ldots + L_n = 0,$$

and that, in the case of $m = 1$, one has, or one can have this here

$${}_{1,n}y_x + E_n\cdot{}_{1,n}y_{x-1} + {}^1E_n\cdot{}_{1,n}y_{x-2} + \ldots + {}^1H_n = 0;$$

one will be able, in this case, to transform equation (Ω) into the following

$$(\varpi)\quad {}_{m,n}y_x = {}_ma_n\cdot{}_{m,n}y_{x-1} + {}_m^1a_n\cdot{}_{m,n}y_{x-2} + {}_m^2a_n\cdot{}_{m,n}y_{x-3} + \ldots + {}_mu_n,$$

from which one will determine the coefficients in this manner.

This equation gives

$${}_mC_n\cdot{}_{m-1,n}y_x = {}_mC_n({}_{m-1}a_n\cdot{}_{m-1,n}y_{x-1} + {}_{m-1}^1a_n\cdot{}_{m-1,n}y_{x-2} + \ldots + {}_{m-1}u_n),$$
$${}_m^1C_n\cdot{}_{m-1,n}y_{x-1} = {}_m^1C_n({}_{m-1}a_n\cdot{}_{m-1,n}y_{x-2} + {}_{m-1}^1a_n\cdot{}_{m-1,n}y_{x-3} + \ldots + {}_{m-1}u_n),$$
....

If one adds all these equations member by member, and if one eliminates the quantities

$${}_mC_n\cdot{}_{m-1,n}y_x + {}_m^1C_n\cdot{}_{m-1,n}y_{x-1} + \ldots$$
$${}_mC_n\cdot{}_{m-1,n}y_{x-1} + {}_m^1C_n\cdot{}_{m-1,n}y_{x-2} + \ldots$$

by means of equation (Ω), one will have

$$(\sigma)\quad \left\{\begin{aligned}
{}_{m,n}y_x = {}_{m,n}y_{x-1}({}_{m-1}a_n - {}_mA_n)&\\
&+ {}_{m,n}y_{x-2}({}_{m-1}^{\;\;1}a_n + {}_{m-1}a_n\cdot{}_mA_n - {}_m^1A_n)\\
&+ {}_{m,n}y_{x-3}({}_{m-1}^{\;\;2}a_n + {}_{m-1}^{\;\;1}a_n\cdot{}_mA_n + {}_{m-1}a_n\cdot{}_mA_n - {}_m^2A_n)\\
&+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&- {}_{m,n-1}y_x\cdot{}_mB_n\\
&+ {}_{m,n-1}y_{x-1}(-{}_m^1B_n + {}_{m-1}a_n\cdot{}_mB_n)\\
&+ {}_{m,n-1}y_{x-2}(-{}_m^2B_n + {}_{m-1}^{\;\;1}a_n\cdot{}_mB_n + {}_{m-1}a_n\cdot{}_m^1B_n)\\
&+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&+ {}_{m-1}u_n({}_mC_n + {}_m^1C_n + \dots)\\
&- {}_mN_n(1 - {}_{m-1}a_n - {}_{m-1}^{\;\;1}a_n - \dots).
\end{aligned}\right.$$

This equation is in the partial differences among three variables by considering m as a constant, and it is contained in that of Problem IV of Article XX. Now, since equation (σ) can be transformed into equation (ϖ), one will have, by Article XX, the following equations:

$$\begin{aligned}
&{}_ma_n = {}_ma_{n-1} + {}_{m-1}a_n - {}_mA_n,\\
&{}_m^1a_n = {}_m^1a_{n-1} + {}_{m-1}^{\;\;1}a_n + {}_{m-1}a_n\cdot{}_mA_n - {}_m^1A_n - {}_ma_{n-1}({}_{m-1}a_n - {}_mA_n),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&{}_mu_n = {}_mu_{n-1}(-{}_mB_n - {}_m^1B_n + {}_{m-1}a_n\cdot{}_mB_n + \dots)\\
&\qquad + {}_{m-1}u_n({}_mC_n + {}_m^1C_n + \dots)(1 - {}_ma_{n-1} - {}_m^1a_{n-1} - \dots)\\
&\qquad - {}_mN_n(1 - {}_{m-1}a_n - {}_{m-1}^{\;\;1}a_n - \dots)(1 - {}_ma_{n-1} - {}_m^1a_{n-1} - \dots).
\end{aligned}$$

These equations are in the partial differences in three variables; in order to integrate them, I observe that they are all contained in this here:

$$(b)\qquad {}_ny_x = {}_nR_x\cdot{}_ny_{x-1} + {}_nT_x\cdot{}_{n-1}y_x + {}_nM_x.$$

I suppose therefore that, in the case of $n = 1$, one has ${}_1y_x = \phi(x)$. This put, one will be able always to transform equation (*b*) into the following

$$(l)\qquad {}_ny_x = {}_nb_x\cdot{}_ny_{x-1} + {}_n^1b_x\cdot{}_ny_{x-2} + {}_n^2b_x\cdot{}_ny_{x-3} + {}_nz_x,$$

whence one will have this here:

$$ {}_{n-1}y_x \cdot {}_nT_x = {}_nT_x({}_{n-1}b_x \cdot {}_ny_{x-1} + {}_{n-1}^{\;\;1}b_x \cdot {}_ny_{x-2} + \dots) + {}_nT_x \cdot {}_{n-1}z_x] $$

If one substitutes, instead of ${}_nT_x \cdot {}_{n-1}y_x$, ${}_nT_x \cdot {}_{n-1}y_{x\text{-}1}$, $\dots$, their values drawn from equation (*b*), one will have

$$
\begin{aligned}
{}_ny_x = {}&{}_nR_x \cdot {}_ny_{x-1} + {}_nM_x \\
&+ {}_{n-1}b_x({}_ny_{x-1} - {}_nR_{x-1} \cdot {}_ny_{x-2} - {}_nM_{x-1})\frac{{}_nT_x}{{}_nT_{x-1}} \\
&+ {}_{n-1}^{\;\;1}b_x({}_ny_{x-2} - {}_nR_{x-2} \cdot {}_ny_{x-3} - {}_nM_{x-2})\frac{{}_nT_x}{{}_nT_{x-2}} \\
&+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&+ {}_nT_x \cdot {}_{n-1}z_x
\end{aligned}
$$

whence one will draw, by comparing this equation with equation (*l*),

$$
\begin{aligned}
&{}_nb_x = {}_{n-1}b_x\frac{{}_nT_x}{{}_nT_{x-1}} + {}_nR_x, \\
&\dots\dots\dots\dots\dots\dots\dots\dots, \\
&{}_nz_x = {}_{n-1}z_x \cdot {}_nT_x + {}_nM_x - {}_{n-1}b_x \cdot {}_nM_{x-1}\frac{{}_nT_x}{{}_nT_{x-1}} - \dots,
\end{aligned}
$$

equations which are integrated easily by Problem I by regarding n as the only variable.

One could make some analogous researches on the partial differences in five, six, etc. variables, and one sees that the preceding method will succeed generally, whatever be the number of these variables.