QUATRIÈME SUPPLÉMENT.

Pierre Simon Laplace

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1. U being any function whatever of a variable t, if one develops it according to the powers of t, the coefficient of t^x, in this development, will be a function of x that I will designate by y_x; U is that which I have named *generating function* of y_x. If one multiplies U be a function T of t, similarly developed according to the ascending powers of t, the product UT will be a new generating function of a function of x, derived from the function y_x according to a law which will depend on the function T. If T is equal to $\frac{1}{t} - 1$, it is easy to see that the derivative will be $y_{x+1} - y_x$, or the finite difference of y_x. We designated generally, whatever be T, this derivative by δy_x. If one multiplies the product UT by T, the derivative of the product UT^2 will be a derivative of δy_x similar to the derivative of δy_x in y_x; one will be able therefore to designate by $\delta^2 y_x$ this second derivative; whence it is clear generally that UT^n will be the generating function of $\delta^n y_x$.

If one multiplies U by another function Z of t, similarly developed according to the ascending powers of t, and if one designates by the characteristic Δ that which we have named δ relative to the function T, UZ^n will be the generating function of $\Delta^n y_x$.

One is able to imagine T as a function of Z. By developing this function into series with respect to the ascending powers of Z, one will have an expression of T of this form

$$T = A^{(0)} + A^{(1)} Z + A^{(2)} Z^2 + \cdots$$

By multiplying this equation by U and passing again from the generating functions to the coefficients, one will have

$$\delta y_x = A^{(0)} y_x + A^{(1)} \Delta y_x + A^{(2)} \Delta^2 y_x + \cdots$$

One sees thus that the same equation, which holds between T and Z, holds between their characteristics δ and Δ, provided that, in the development of this equation according to the powers of δ and of Δ, one substitutes, instead of any power δ^r, $\delta^r y_x$; instead of a power $\Delta r'$, $\Delta r' y_x$; instead of a product such as $\delta^r \Delta r'$, $\delta^r \Delta r' y_x$; and that one multiplies by y_x the terms independent of δ and Δ. Thus, by supposing T equal to $\frac{1}{t} - 1$, $Z = \frac{1}{t^i} - 1$, δy_x will be the finite difference of y_x, x varying from unity; Δy_x will be the finite difference of y_x, x varying from i; one has next

$$Z = (1 + T)^i - 1,$$

and, consequently,

$$Z^n = [(1 + T)^i - 1]^n;$$

this which gives

$$\Delta^n = [(1 + \delta)^i - 1]^n,$$

provided that after the development one places y_x after the powers of the characteristics. This equation will hold furthermore by making n negative, but then the differences are changed into integrals. The consideration of the generating functions show thus, in the most natural and most simple manner, the analogy of the powers and of the differences. One is able to consider this theory as the calculus of characteristics.

If one has $0 = \delta y_x$, one will have an equation in the finite differences: UT becomes then a polynomial which contains only some powers of t smaller than the highest of t in T. We designate by Q the polynomial in t the most general of this nature; one will have

$$U = \frac{Q}{T}.$$

The coefficient of t^x in the development of U will be the integral y_x of the equation $0 = \delta y_x$; by this reason, I name U generating function of this equation.

If one imagines U a function of two variables t and t', the coefficient of the product $t^x t^{x'}$, in the development of U, will be a function of x and of x' that I designate by $y_{x,x'}$; T being a function developed in the same variables t and t', the product UT will be the generating function of a derivative of $y_{x,x'}$, that I will designate by $\delta y_{x,x'}$; and it is easy to conclude from it that UT^n will be the generating function of $\delta^n y_{x,x'}$.

If one has $0 = \delta y_{x,x'}$, one will have an equation in the partial finite differences. We represent this equation by the following

$$\begin{aligned} 0 = a y_{x,x'} &+ b y_{x,x'+1} + c y_{x,x'+2} + \cdots, \\ &+ a' y_{x+1,x'} + b' y_{x+1,x'+1} + \cdots, \\ &+ a'' y_{x+1,x'} + \cdots \\ &+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots; \end{aligned}$$

it is easy to see that the generating function of the proposed equation will be

$$\frac{A + Bt' + Ct'^2 + \cdots + Ht'^{n'-1} + A' + B't + C't^2 + \cdots + H't^{n-1}}{\left\{\begin{aligned} a t^n t'^{n'} &+ b t^n t'^{n'-1} + a t^n t'^{n'-2} + \cdots \\ &+ a' t^{n-1} t'^{n'} + b' t^{n-1} t'^{n'-1} + \cdots \\ &+ a'' t^{n-2} t'^{n'} + \cdots \\ &+ \cdots\cdots\cdots\cdots\cdots, \end{aligned}\right\}}$$

n and n' being the greatest increases of x and of x', in the proposed equation in partial differences; A, B, C, $\ldots$, H are some arbitrary functions of t; A', B', C', $\ldots$, H' are some arbitrary functions of t'. One will determine all these functions by means of the generating functions of

$$\begin{gathered} y_{0,x'},\ y_{1,x'},\ y_{2,x'},\ \ldots,\ y_{n-1,x'}, \\ y_{x,0},\ y_{x,1},\ y_{x,2},\ \ldots,\ y_{x,n'-1}. \end{gathered}$$

One of the principal advantages of this manner to integrate the equations in partial differences consists in this that, the algebraic analysis furnishing diverse ways to develop the functions, one is able to choose the one which agrees best to the proposed question. The solution of the following problems, by the count de Laplace, my son, and the considerations that he has joined will spread a new day on the calculus of generating functions.

2. A player A draws from an urn, containing some white and black balls, one ball which he returns after the trial, with the probability p to bring forth a white ball and the probability q to extract from it a black; a second player B draws next, from another urn, a ball which he returns equally after the drawing, with the probabilities p' of a white ball and q' of a black. These two players continue thus to extract alternately, each from their respective urn, one ball which they have always need to return. If one of the players brings forth a white ball, he counts a

point; if, on the contrary, he makes a black ball exit, he counts nothing, and the turn of the player passes simply to the other. The players having settled, by the conditions of their game, the number of points that each must attain first in order to win the set, and having commenced to play, there lacks yet to player A the number x points in order to win, and x' to player B; and the turn to play belongs to player A. One demands, in this position, what is the probability of each player to win the set.

Let $z_{x,x'}$ be the probability of second player B, and we represent by $Y_{x,x'}$ his probability, if he were the first to play. Player A, by beginning, is able to bring forth a white ball, and the probability of B becomes $Y_{x-1,x'}$; or the first player makes a black ball exit, and then counts nothing, and the probability of the second is changed into $Y_{x,x'}$; but the probability of the first case is p, that of the second q; one will have therefore the equation

$$z_{x,x'} = pY_{x-1,x'} + qY_{x,x'};$$

by a similar reasoning, one will have further that here

$$Y_{x,x'} = p'z_{x,x'-1} + q'z_{x,x'};$$

whence one draws

$$Y_{x-1,x'} = p'z_{x-1,x'-1} + q'z_{x-1,x'},$$

and consequently

$$z_{x,x'} = p(p'z_{x-1,x'-1} + q'z_{x-1,x'}) + q(p'z_{x,x'-1} + q'z_{x,x'}) \qquad (^1)$$

or

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[1] One arrives again to this equation in partial differences by considering the set of two successive drawings of A and B as one trial, and by examining the different cases which are able to present themselves after this trial played; now they are in number of four: 1° either the two players bring forth each one white ball, an event of which the probability is pp'; then the probability $z_{x,x'}$ will be changed into that here $z_{x-1,x'-1}$; 2° or the first player extracts a white ball and the second a black; under this hypothesis, which has for probability pq', $z_{x,x'}$ will become $z_{x-1,x'}$; 3° or on the contrary the first player makes a black ball exit and the second a white; under this hypothesis, which has for probability $p'q$, $z_{x,x'}$ will become $z_{x,x'-1}$; 4° or finally each player draws a black ball, an event of which the probability is qq', and then the probability $z_{x,x'}$ remains the same. One will have therefore, by the known principles of

$$z_{x,x'} = \frac{pq'}{1-qq'} z_{x-1,x'} + \frac{p'q}{1-qq'} z_{x,x'-1} + \frac{pp'}{1-qq'} z_{x-1,x'-1},$$

and by making

$$\frac{pq'}{1-qq'} = m, \quad \frac{p'q}{1-qq'} = m', \quad \frac{pp'}{1-qq'} = n,$$

it will become

$$z_{x,x'} = m z_{x-1,x'} + m' z_{x,x'-1} + n z_{x-1,x'-1}.$$

The generating function of $z_{x,x'}$, in this equation in partial differences, is

$$\frac{A + A'}{1 - mt - m't' - ntt'},$$

A being an arbitrary function of t, and A' another arbitrary function of t'; I observe first that by attributing to the function A' the term independent of t in the function A, the generating function above is able to be set under this form

$$\frac{A_1 t + A'_1}{1 - mt - m't' - ntt'},$$

A_1 and A'_1 being new arbitrary functions of t and of t' that it is the question to determine. Now, if one pays attention that $z_{0,x'}$ is null, whatever be x', the probability of player A is changed then to certitude, one sees that the coefficient of t^0 in the development of the generating function with respect to the powers of t must be null, and one will have

$$\frac{A'_1}{1 - m't'} = 0 \qquad \text{or} \qquad A'_1 = 0.$$

Moreover, $z_{x,0}$ is null when x is zero, and equal to unity when x is either 1 or 2, or 3, ..., since then the probability of player B is changed into certitude; the

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probabilities, the equation

$$z_{x,x'} = pp' z_{x-1,x'-1} + pq' z_{x-1,x'} + p'q z_{x,x'-1} + qq' z_{x,x'}.$$

On obtains the generating function of $z_{x,x'}$, in this equation in partial differences, by applying to this case the general rule which has just been exposed.

generating function of $z_{x,0}$ is therefore $\frac{t}{1-t}$; it is the coefficient of ${t'}^0$ in the development of the generating function according to the powers of t'; one will have therefore

$$\frac{A_1 t}{1-mt} = \frac{t}{1-t};$$

this which gives

$$A_1 t = \frac{t(1-mt)}{1-t};$$

consequently the generating function of $z_{x,x'}$ is

$$(a) \qquad \frac{t(1-mt)}{(1-t)(1-mt-m't'-ntt')};$$

by putting it under this form

$$\frac{t}{1-t}\frac{1}{1-\left(\frac{m'+nt}{1-mt}\right)t'}$$

and the development with respect to the powers of t', one has

$$\frac{t}{1-t}\left[1+\left(\frac{m'+nt}{1-mt}\right)t'+\left(\frac{m'+nt}{1-mt}\right)^2{t'}^2+\left(\frac{m'+nt}{1-mt}\right)^3{t'}^3+\cdots\right].$$

The coefficient of ${t'}^{x'}$ in this series is

$$\frac{t}{1-t}\left(\frac{m'+nt}{1-mt}\right)^{x'},$$

and the one of t^x in the development of this last function will be the expression of $z_{x,x'}$. Now, if one reduces first the expression $t\left(\frac{m'+nt}{1-mt}\right)^{x'}$ into a series ordered according to the powers of t, and if one multiplies it next by the development of $\frac{1}{1-t}$, it is easy to see that the coefficient of t^x in this product is that which the series becomes by making $t=1$ in it and stopping oneself at the power x of t; and one will find, for the value of this coefficient or of $z_{x,x'}$,

$$z_{x,x'} = m'^{x'} \left\{ \begin{array}{l} 1 + \frac{x'}{1}\frac{n}{m'} + \frac{x'(x'-1)}{1.2}\frac{n^2}{m'^2} + \frac{x'(x'-1)(x'-2)}{1.2.3}\frac{n^3}{m'^3} + \cdots + \frac{x'(x'-1)\cdots(x'-x+2)}{1.2.\ldots(x-1)}\frac{n^{x-1}}{m'^{x-1}} \\ + \frac{x'}{1} m \left[1 + \frac{x'}{1}\frac{n}{m'} + \frac{x'(x'-1)}{1.2}\frac{n^2}{m'^2} + \cdots + \frac{x'(x'-1)\cdots(x'-x+3)}{1.2.\ldots(x-2)}\frac{n^{x-2}}{m'^{x-2}} \right] \\ + \frac{x'(x'-1)}{1.2} m^2 \left[1 + \frac{x'}{1}\frac{n}{m'} + \cdots + \frac{x'(x'-1)\cdots(x'-x+4)}{1.2.\ldots(x-3)}\frac{n^{x-3}}{m'^{x-3}} \right] \\ + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ + \frac{x'(x'+1)\cdots(x'+x-2)}{1.2.\ldots(x-1)} m^{x-1} \end{array} \right\}$$

By designating by $y_{x,x'}$ the probability of player A, one will be led, by the same reasonings, to a similar equation in the partial differences,

$$y_{x,x'} = my_{x-1,x'} + m'y_{x,x'-1} + ny_{x-1,x'-1},$$

which gives similarly for the variable $y_{x,x'}$ a generating function of the form

$$\frac{A_1 t + A_1'}{1 - mt - m't' - ntt'},$$

A_1 and A_1' being, as above, some arbitrary functions of t and of t' what one will determine by the same considerations. In effect the generating function of $y_{0,x'}$ is $\frac{1}{1-t'}$, that of $y_{x,0}$ is unity: one will form therefore the equations

$$\frac{A_1'}{1 - m't'} = \frac{1}{1 - t'};$$

whence one draws

$$A_1' = \frac{1 - m't'}{1 - t'}$$

and

$$\frac{A_1 t + 1}{1 - mt} = 1;$$

whence one concludes

$$A_1 t = -mt.$$

The generating function of $y_{x,x'}$ will be therefore

(*b*)
$$\frac{\frac{1-m't'}{1-t'}-mt}{1-mt-m't'-ntt'},$$

which, developed according to the powers of t and of t', will give, by the coefficient of $t^x t'^{x'}$, the expression of $y_{x,x'}$ which will be of a form similar to that of $z_{x,x'}$, although a little more complicated.

By adding the two generating functions (*a*) and (*b*), their sum is reduced to that here

$$\frac{1}{(1-t)(1-t')},$$

in which the coefficient of $t^x t'^{x'}$ is unity; thus one has

$$y_{x,x'}+z_{x,x'}=1;$$

and effectively, the set must be necessarily won by one of the players, because both are certain to be able to extract each from their urn the determined numbers of white balls.

Now, we suppose $p=0$ and consequently $q=1$, one has

$$m=0, \quad m'=1 \quad \text{and} \quad n=0;$$

then the expression of $z_{x,x'}$ becomes unity; this which is evident, since the player B, not having more chances to lose, must always end by winning.

If, to the contrary, one supposes $p=1$ and $q=0$, that is to say if the first player A counts a point before each drawing of player B, then

$$m=q', \quad m'=0 \quad \text{and} \quad n=p';$$

x' being greater than x or equal, the expression $z_{x,x'}$ is reduced to zero; and, in effect, it is evidently impossible that, in this case, player B is able to win the set; but, when x is greater than x', the value of $z_{x,x'}$ takes this form

$$z_{x,x'}=p'^{x'}\left[1+\frac{x'}{1}q'+\frac{x'(x'+1)}{1.2}q'^2+\cdots+\frac{x'(x'+1)\cdots(x-2)}{1.2\ldots(x-x'-1)}q'^{x-x'-1}\right].$$

Under this assumption, player B is able to win only as much as he will bring forth x' white balls before $x-x'$ black balls; otherwise, he is anticipated by player A who counts a point at each trial: this expression of $z_{x,x'}$ is therefore the probability that player B will have drawn x' white balls before having extracted from it $x - x'$ blacks, and, consequently, the probability to win, if he made the wager with player A, who would count then a point with the exit of each black ball while he counts one of them at the exit of a white, to attain x' points before his adversary has $x - x'$ of them; this which is the *problem of points*. ([2])

If one examines with attention the form of the general expression which gives $z_{x,x'}$, one will recognize that this problem is able yet to be resolved, and even with simplicity, by means of the theory of combinations: in effect, let a be the number of white balls contained in the urn of player A, and b the one of the blacks; a' the number of white balls of player B, and b' the one of the blacks; by considering, as one has already done, the collection of two successive drawings of A and B as one trial,

aa' will be the number of combinations in which the players bring forth each one white ball;

ab' the one of the combinations which will give one white ball to A and one black to B;

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[2] The generating function of $z_{x,x'}$ is reduced in this case to

$$\frac{t(1-q't)}{(1-t)(1-q't-p'tt')},$$

and the equation in the corresponding partial differences will be

$$z_{x,x'} = q' z_{x-1,x'} + p' z_{x-1,x'-1},$$

in which $z_{x,x'}$ is a function of x and of x' which we will designate by $\phi(x,x')$; if one makes $x - x' = s$, one will have

$$\phi(x,x') = \phi(s+x',x'),$$

and, if one represents by $z_{s,x'}$ this last function, there results from it

$$z_{x,x'} = z_{s,x'}, \quad z_{x-1,x'} = z_{s-1,x'}, \quad z_{x-1,x'-1} = z_{s,x'-1};$$

and the equation in the partial differences is changed into that here

$$z_{s,x'} = q' z_{s-1,x'} + p' z_{s,x'-1},$$

an equation to which the problem of points would lead directly under the conditions enunciated above. By paying attention that, in consequence of this transformation, $z_{s,0} = 1$ and $z_{0,x'} = 0$, and that $z_{0,0}$ is not able to take place, it is easy to see that the generating function of $z_{s,x'}$ will be

$$\frac{t(1-q't)}{(1-t)(1-q't-p't')},$$

in the development of which the coefficient of $t^s t'^{x'}$ will be the expression of $z_{s,x'}$.

$a'b$ the one of the combinations which will give, to the contrary, one black ball to A and one white to B;

bb' the one of the combinations in which one and the other player draw a black ball;

And the sum $aa' + ab' + a'b + bb'$ will form the collection of all the combinations which are able to take place in a trial. The combinations where the players bring forth each one black ball bring no change to their position, we are able to set aside, and then we occupy ourselves only with the trials where there will be brought forth at least one white ball. It is clear that in $x + x'$ similar trials one of the players has necessarily won, and the set must be decided: now the number of all the equally possible combinations, following which these $x + x'$ trials are able to be presented, will be

$$(aa' + ab' + a'b)^{x+x'};$$

the question is reduced therefore to choose in all these combinations those which make player B win, that is to say those in which this player will have x' white balls before player A has brought forth x of them. In order to fix the ideas, we suppose x' greater than x; one is able to form the following hypotheses: either player B will have won at the xth trial, that is to say by drawing without interruption a white ball at each trial, and then the number of the preceding combinations which are returned to this case is evidently

$$a'^{x'}\left[b^{x'} + \frac{x'}{1}ab^{x'-1} + \frac{x'(x'-1)}{1.2}a^2b^{x'-2} + \cdots \right.$$
$$\left. + \frac{x'(x'-1)\cdots(x'-x+2)}{1.2\ldots(x-1)}a^{x-1}b^{x'-x+1}\right](aa' + ab' + a'b)^x;$$

and by dividing it by $(aa' + ab' + a'b)^{x+x'}$, the total number of combinations, one will have, for the probability of this hypothesis,

$$\frac{a'^{x'}b^{x'}}{(aa' + ab' + a'b)^{x'}}\left[1 + \frac{x'}{1}\frac{a}{b} + \frac{x'(x'-1)}{1.2}\frac{a^2}{b^2} + \cdots + \frac{x'(x'-1)\cdots(x'-x+2)}{1.2\ldots(x-1)}\frac{a^{x-1}}{b^{x-1}}\right];$$

or the player B will have won at the $(x'+1)$st trial, that is to say by having drawn only a single black ball, for example at the commencement, and then the number of combinations favorable to this event is

$$b'a'^{x'}\left[b^{x'}+\frac{x'}{1}ab^{x'-1}+\frac{x'(x'-1)}{1.2}a^2b^{x'-2}+\cdots\right.$$
$$\left.+\frac{x'(x'-1)\cdots(x'-x+3)}{1.2\ldots(x-2)}a^{x-2}b^{x'-x+2}\right](aa'+ab'+a'b)^{x-1};$$

but this number is the same, if the black ball is brought forth in the first trial or in the second, $\ldots$, or in the x^{th} trial; it is necessary therefore to multiply it by x' in order to have all the combinations relative to this hypothesis, of which the probability is, by this means,

$$\frac{x'}{1}\frac{ab'a'^{x'}b^{x'}}{(aa'+ab'+a'b)^{x'+1}}\left[1+\frac{x'}{1}\frac{a}{b}+\frac{x'(x'-1)}{1.2}\frac{a^2}{b^2}+\cdots+\frac{x'(x'-1)\cdots(x'-x+3)}{1.2\ldots(x-2)}\frac{a^{x-2}}{b^{x-2}}\right];$$

or player B will have won at the $(x'+2)^{\text{nd}}$ trial, and one will see in the same manner that the probability of this hypothesis will be

$$\frac{x'(x'+1)}{1.2}\frac{a^2b'^2a'^{x'}b^{x'}}{(aa'+ab'+a'b)^{x'+2}}\left[1+\frac{x'}{1}\frac{a}{b}+\cdots+\frac{x'(x'-1)\cdots(x'-x+4)}{1.2\ldots(x-3)}\frac{a^{x-3}}{b^{x-3}}\right];$$

By continuing thus, one will have the probabilities of all the successive hypotheses which are able to present themselves under the supposition of the gain of the set by player B, until that where he will win only at the $(x'+x-1)^{\text{st}}$ trial, an event of which the probability will be

$$\frac{x'(x'+1)\cdots(x'+x-2)}{1.2\ldots(x-1)}\frac{a^{x-1}b'^{x-1}a'^{x'}b^{x'}}{(aa'+ab'+a'b)^{x'+x-1}};$$

and effectively, in this case, it is not able to have trials where the players bring forth at the same time a white ball.

The sum of all these probabilities will give evidently that of player B in order to win the set.

If one pays attention that

$$\frac{ab'}{aa'+ab'+a'b}=m,\quad \frac{a'b}{aa'+ab'+a'b}=m',\quad\text{and}\quad \frac{q}{b}=\frac{n}{m'},$$

one recovers the expression of $z_{x,x'}$.

We imagine presently that there are in the urns some white balls carrying the n$^{\mathrm{o}}$ 1, and other balls, of the same color, which carry the n$^{\mathrm{o}}$ 2; each ball diminishing its numeral, by its exit, the number of points which are lacking yet to the player to which it is favorable. The problem is no longer susceptible to be resolved generally by means of combinations, instead that the calculation of the generating functions will continue to furnish a general expression of which the development will contain the complete solution of the question and will be able, in certain cases, to be executed by some laws easy to know, as we will have occasion to see.

Let p be the probability player A to extract a ball labeled 1, p_1 that to extract a ball labeled 2, and q that to bring forth a black ball; p', p_1' and q' the corresponding probabilities for player B; and let always $z_{x,x'}$ be the probability of this last player in order to win the set. By following the same march as above, one will be led to the equation in partial differences

$$\begin{aligned} z_{x,x'} &= mz_{x-1,x'} + m_1 z_{x-2,x'} + m' z_{x,x'-1} + m_1' z_{x,x'-2} \\ &\quad + nz_{x-1,x'-1} + n_1 z_{x-2,x'-1} + n' z_{x-1,x'-2} + n_1' z_{x-2,x'-2} \end{aligned}$$

in which one makes

$$\begin{gathered} \frac{pq'}{1-qq'} = m, \quad \frac{p_1 q'}{1-qq'} = m_1, \quad \frac{p'q}{1-qq'} = m', \quad \frac{p_1' q}{1-qq'} = m_1', \\ \frac{pp'}{1-qq'} = n, \quad \frac{p_1 p'}{1-qq'} = n_1, \quad \frac{pp_1'}{1-qq'} = n', \quad \frac{p_1 p_1'}{1-qq'} = n_1'; \end{gathered}$$

the generating function of the variable $z_{x,x'}$ given by this equation, will be

$$(c) \qquad \frac{A + Bt' + A' + B't}{1 - mt - m_1 t^2 - m't' - m_1' t'^2 - ntt' - n_1 t^2 t' - n' tt'^2 - n_1' t^2 t'^2},$$

A and B being some arbitrary functions of t, A' and B' some arbitrary functions of t', which will be determined by means of the generating functions of

$$z_{0,x'}, \quad z_{x,0}, \quad z_{1,x'}, \quad z_{x,1}$$

which are themselves it by the conditions of the game.

One finds, as previously, that the generating function of $z_{0,z'}$ is zero and that of $z_{x,0}$, $\frac{t}{1-t}$.

From the general equation, one deduces the equation in finite differences

$$z_{1,x'} = m'z_{1,x'-1} + m'_1 z_{1,x'-2},$$

which holds for all the values of x' from $x' = 2$ inclusively, and which gives consequently, for the generating function of $z_{1,x'}$,

$$\frac{a + bt'}{1 - m't' - m'_1 t'^2},$$

a and b being some constants that one determines by means of the values of $z_{1,0}$ and $z_{1,1}$; and as $z_{1,0}$ is equal to unity, $z_{1,1}$ is equal to $m' + m'_1$, and is at the same time the coefficient of t' in the development of the generating function; there results from it

$$a = 1 \quad \text{and} \quad b = m'_1;$$

the generating function of $z_{1,x'}$ is therefore

$$\frac{1 + m'_1 t'}{1 - m't' - m'_1 t'^2}.$$

Now, if in the preceding equation one puts $1 - y_{x,x'}$ in the place of $z_{x,x'}$, $y_{x,x'}$ being always the probability of the first player A, it is reformed in the same manner with respect to this last variable, and one will deduce from it the equation in the finite differences

$$y_{x,1} = my_{x-1,1} + m_1 y_{x-2,1}.$$

But one will see at the same time that it begins to hold only when x surpasses 2; because, x being 2, one will have

$$y_{2,1} = my_{1,1} + m_1 y_{0,1} + n_1 + n'_1.$$

It is necessary therefore to employ it only to depart from $x = 3$, and then the generating function of $y_{x,1}$ is of the form

$$\frac{a + bt + ct^2}{1 - mt - m_1 t^2},$$

a, b and c being some constants that one will determine, as previously, by means of the values of $y_{1,0}$, $y_{1,1}$ and $y_{1,2}$; now $y_{1,0}$ is unity; $y_{1,1}$ is equal to $1 - m' - m'_1$, and is the coefficient of t in the development of the generating function; $y_{2,1}$ has for value, as we have just seen,

$$m(1-m'-m'_1)+m_1+n_1+n'_1;$$

this is the coefficient of t^2 in the development of the function. One will conclude from it

$$a=1, \quad b=1-m-m'-m'_1, \quad \text{and} \quad c=n_1+n'_1,$$

and the generating function of $y_{x,1}$ will be therefore

$$\frac{1+(1-m-m'-m'_1)t+(n_1+n'_1)t^2}{1-mt-m_1t^2};$$

consequently that of $z_{x,1}$ is

$$\frac{1}{1-t}-\frac{1+(1-m-m'-m'_1)t+(n_1+n'_1)t^2}{1-mt-m_1t^2} \\ =\frac{(m'+m'_1)t+(n_1+n'_1)t^2+(n_1+n'_1)t^3}{(1-t)(1-mt-m_1t^2)}.$$

We resume actually the generating function (*c*); one is able always to restore it to this form

$$\frac{A_1t+B_1t^2t'+A'_1+B'_1tt'}{1-mt-m_1t^2-m't'-m'_1t'^2-ntt'-n_1t^2t'-n'tt'^2-n'_1t^2t'^2},$$

A_1 and B_1 being the arbitrary functions of t, A'_1 and B'_1 the arbitrary functions of t'; which one determines easily, by equating first the coefficient of t^0 in the development of this function to the generating function of $z_{0,x'}$ or zero, next the one of ${t'}^0$ to the generating function of $z_{x,0}$ or $\frac{t}{1-t}$, since the one of t to the generating function of $z_{1,x'}$, and finally the one of t' to the generating function of $z_{x,1}$, this which will give successively

$$A'_1=0, \quad A_1=\frac{1-mt-m_1t^2}{1-t}, \quad B'_1=m'_1, \quad B_1=\frac{m'_1+n'+n'_1t}{1-t},$$

and, consequently, for the generating function of $z_{x,x'}$,

$$(d) \quad \frac{(1-mt-m_1t^2)t+m'_1tt'+n't^2t'+n'_1t^3t'}{(1-t)(1-mt-m_1t^2-m't'-m'_1t'^2-ntt'-n_1t^2t'-n'tt'^2-n'_1t^2t'^2)}.$$

If one supposes p and p' null, then one has

$$m=0,\quad m'=0,\quad n=0,\quad n_1=0,\quad \text{and}\quad n'=0,$$

and the function (*d*) takes this form

$$\frac{tt'(m_1'+n_1't^2)}{(1-t)(1-m_1t^2)\left[1-\left(\frac{m_1'+n_1't^2}{1-m_1t^2}\right)t'^2\right]}+\frac{t}{(1-t)\left[1-\left(\frac{m_1'+n_1't^2}{1-m_1t^2}\right)t'^2\right]},$$

under which it is susceptible of the same developments as the function (*a*). There is to remark that one will resume the same coefficient for

$$t^{2r}t'^{2r'},\quad t^{2r-1}t'^{2r'},\quad t^{2r}t'^{2r'-1},\quad t^{2r-1}t'^{2r'-1};$$

this which is seen *a priori*, by paying attention that the players count always two points at each white ball that they make exit.

We suppose that player A has only some balls labeled 1 and 2, and that the other player has only some white balls marked 1, or which count to him only one point on exiting; then

$$p_1'=0$$

and, hence,

$$m_1'=0,\quad n'=0,\quad n_1'=0;$$

the function (*d*) becomes

$$\frac{t(1-mt-m_1t^2)}{(1-t)(1-mt-m_1t^2-m't'-ntt'-n_1t^2t')}=\frac{t}{1-t}\,\frac{1}{1-\left[\frac{m'+(n+n_1t)t}{1-(m+m_1t)t}\right]t'};$$

by developing it according to the powers of t', the coefficient of $t'^{x'}$ will be

$$\frac{t[m'+(n+n_1t)t]^{x'}}{(1-t)[1-(m+m_1t)t]^{x'}},$$

an expression that the concern is to develop with respect to the powers of t in order to have the coefficient of t^x; now this coefficient will be the sum of all the coefficients of the powers of t inferior or equal to t^{x-1}, in the development of the expression

$$\frac{[m' + (n + n_1 t)t]^{x'}}{[1 - (m + m_1 t)t]^{x'}},$$

which, by omitting the terms where the powers of t outside the binomials are superior to t^{x-1}, is able to be put under this form

$$m'^{x'}\left\{\begin{array}{l}
1 + \dfrac{x'}{1}\left(\dfrac{n+n_1t}{m'}\right)t + \dfrac{x'(x'-1)}{1.2}\left(\dfrac{n+n_1t}{m'}\right)^2 t^2 + \cdots + \dfrac{x'(x'-1)\cdots(x'-x+2)}{1.2\ldots(x-1)}\left(\dfrac{n+n_1t}{m'}\right)^{x-1} t^{x-1} \\
+ \dfrac{x'}{1}(m+m_1t)t\left[1 + \dfrac{x'}{1}\left(\dfrac{n+n_1t}{m'}\right)t + \cdots + \dfrac{x'(x'-1)\cdots(x'-x+3)}{1.2\ldots(x-2)}\left(\dfrac{n+n_1t}{m'}\right)^{x-2} t^{x-2}\right] \\
+ \dfrac{x'(x'+1)}{1.2}(m+m_1t)^2t^2\left[1 + \cdots + \dfrac{x'(x'-1)\cdots(x'-x+4)}{1.2\ldots(x-3)}\left(\dfrac{n+n_1t}{m'}\right)^{x-3} t^{x-3}\right] \\
+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
+ \dfrac{x'(x'-1)\cdots(x'-x+2)}{1.2\ldots(x-1)}(m+m_1t)^{x-1}t^{x-1}.
\end{array}\right\}$$

If one rejects further from this series all the powers of t superior to t^{x-1}, which will result from the developments of the binomials, and if, in that which remains, one makes $t = 1$, one will have the expression of $z_{x,x'}$.

We examine further the case where player A would be certain to extract at each trial a ball which would count to that player one point, that is to say where one would have

$$p = 1, \quad p_1 = 0, \quad q = 0,$$

and consequently

$$\begin{array}{llll}
m = q', & m_1 = 0, & m' = 0, & m'_1 = 0, \\
n = p', & n_1 = 0, & n' = p'_1, & n'_1 = 0.
\end{array}$$

The generating function of $z_{x,x'}$ or the function (*d*) would be reduced to

$$\frac{t(1 - q't) + p'_1 t^2 t'}{(1 - t)(1 - q't - p'tt' - p'_1 tt'^2)},$$

and that of $y_{x,x'}$ would be, hence,

$$\begin{aligned}\frac{1}{(1-t)(1-t')} - \frac{t(1-q't)+p'_1t^2t'}{(1-t)(1-q't-p'tt'-p'_1tt'^2)},\\ = \frac{1}{1-t'} + \frac{tt'}{(1-t)(1-q't-p'tt'-p'_1tt'^2)}.\end{aligned}$$

In this last expression, the first term represents the generating function of $y_{0,x'}$, which is equal to unity whatever be x', and the second will give, by developing it with respect to the powers of t and of t', all the other values of $y_{x,x'}$; now the coefficient of t^x will be

$$\frac{t'[q'+(p'+p'_1t')t']^{x-1}}{2-t'};$$

whence it results that, if one rejects from the development of the series

$$q'^{x-1}\left[t' + \frac{(x-1)}{1}\left(\frac{p'+p'_1t'}{q'}\right)t'^2 + \frac{(x-1)(x-2)}{1.2}\left(\frac{p'+p'_1t'}{q'}\right)^2 t'^3 + \cdots\right]$$

all the powers of t' superior to $t'^{x'}$, and if one made in that which remains $t'=1$, one will have, by supposing x' even and equal to $2r+2$, the coefficient of $t^xt'^{x'}$, or

$$y_{x,x'} = q'^{x-1}\left\{\begin{array}{l} 1 + \frac{(x-1)}{1}\left(\frac{p'+p'_1}{q'}\right) + \frac{(x-1)(x-2)}{1.2}\left(\frac{p'+p'_1}{q'}\right)^2 + \cdots + \frac{(x-1)(x-2)\cdots(x-r)}{1.2\ldots r}\left(\frac{p'+p'_1}{q'}\right)^r \\ + \frac{(x-1)(x-2)\cdots(x-r-1)}{1.2\ldots(r+1)}\frac{p'^{r+1}}{q'^{r+1}}\left[1 + \frac{(r+1)}{1}\frac{p'_1}{p'} + \frac{(r+1)r}{1.2}\frac{p'^2_1}{p'^2} + \cdots + \frac{(r+1)r\cdots2}{1.2\ldots r}\frac{p'^r_1}{p'^r}\right] \\ + \frac{(x-1)(x-2)\cdots(x-r-2)}{1.2\ldots(r+2)}\frac{p'^{r+2}}{q'^{r+2}}\left[1 + \frac{(r+2)}{1}\frac{p'_1}{p'} + \cdots + \frac{(r+2)(r+1)r\cdots4}{1.2\ldots(r+1)}\frac{p'^{r-1}_1}{p'^{r-1}}\right] \\ + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ + \frac{(x-1)(x-2)\cdots(x-2r-1)}{1.2\ldots(2r+1)}\frac{p'^{2r+1}}{q'^{2r+1}} \end{array}\right\}$$

and, in the case of x' odd or equal to $2r+1$,

$$y_{x,x'} = {q'}^{x-1} \left\{ \begin{array}{l} 1 + \dfrac{(x-1)}{1}\left(\dfrac{p'+p'_1}{q'}\right) + \dfrac{(x-1)(x-2)}{1.2}\left(\dfrac{p'+p'_1}{q'}\right)^2 + \cdots + \dfrac{(x-1)(x-2)\cdots(x-r)}{1.2\ldots r}\left(\dfrac{p'+p'_1}{q'}\right)^r \\ + \dfrac{(x-1)(x-2)\cdots(x-r-1)}{1.2\ldots(r+1)}\dfrac{{p'}^{r+1}}{{q'}^{r+1}}\left[1 + \dfrac{(r+1)}{1}\dfrac{p'_1}{p'} + \dfrac{(r+1)r}{1.2}\dfrac{{p'_1}^2}{{p'}^2} + \cdots + \dfrac{(r+1)r\cdots 3}{1.2\ldots(r-1)}\dfrac{{p'_1}^{r-1}}{{p'}^{r-1}}\right] \\ + \dfrac{(x-1)(x-2)\cdots(x-r-2)}{1.2\ldots(r+2)}\dfrac{{p'}^{r+2}}{{q'}^{r+2}}\left[1 + \dfrac{(r+2)}{1}\dfrac{p'_1}{p'} + \cdots + \dfrac{(r+2)(r+1)r\cdots 5}{1.2\ldots(r-2)}\dfrac{{p'_1}^{r-2}}{{p'}^{r-2}}\right] \\ + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ + \dfrac{(x-1)(x-2)\cdots(x-2r)}{1.2\ldots 2r}\dfrac{{p'}^{2r}}{{q'}^{2r}} \end{array} \right\}$$

It is clear that player B is able to expect to only as long as x is greater than $r+1$, or that x' equal $2r+2$ or $2r+1$; and effectively, beyond this supposition, the preceding values of $y_{x,x'}$ become all equals to unity.

We will make also remark that player A has necessarily won the set when player B will have drawn $x-r-1$ black balls before having attained x' points; but this last player is able yet to have lost before having drawn the totality of this number of black balls, that which makes that this question is not at all susceptible to return into that which is treated in the analytic Theory, after from the *problem of points*, as previously a similar supposition has led us to this last problem.

3. The problem of points having been the object of the researches of two great geometers of the XVII^th century ([3]), and to some extent the first of this kind subject to some analytic methods, one will be perhaps curious to see how this same problem is deduced again, as corollary, of another question of probability, of which the solution will offer besides a new application of the method of generating functions.

One draws successively from an urn, which contains a determined quantity of white and black balls, a ball that one does not return after the trial, and one demands, after a certain number of known drawings, what is the probability to complete the drawing of such given number of white balls before that of such other number, given equally of black balls.

Let a and a' be the numbers of white and black balls contained originally in the urn, n the number of white balls that one is proposed to attain before having extracted another number n' of black balls; and we suppose that after having

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[3] Pascal and Fermat.

drawn successively from the urn a ball without returning it, one has brought forth $n-x$ white balls and $n'-x'$ black balls, x and x' being then the number of white and black balls that there remain to make exit in order to decide the question. We represent by $y_{x,x'}$ the probability to bring forth in the following drawings x white balls before x' black balls, or to attain the totality of n white balls before having extracted n' blacks; one will have, according to the known rules of probabilities, the equation

$$y_{x,x'}=\frac{a-n+x}{a+a'-n-n'+x+x'}y_{x-1,x'}+\frac{a'-n'+x'}{a+a'-n-n'+x+x'}y_{x,x'-1}.$$

We make

$$a-n+x=s,\quad a'-n'+x'=s'\quad\text{and}\quad y_{x,x'}=u_{s,s'};$$

the preceding equation becomes

$$u_{s,s'}=\frac{s}{s+s'}u_{s-1,s'}+\frac{s'}{s+s'}u_{s,s'-1},$$

and, by supposing

$$u=\frac{1.2.3\ldots s.1.2.3\ldots s'}{1.2.3\ldots(s+s')}z_{s,s'},$$

it is restored to this form

$$z_{s,s'}=z_{s-1,s'}+z_{s,s'-1},$$

an equation in the partial differences with constant coefficients, which must hold for all the entire and positive values of s and of s', by departing from $s=a-n$ and from $s'=a'-n'$, and gives consequently for the generating function of $z_{s,s'}$

$$t^{a-n}t'^{a'-n'}\frac{A+A'}{1-t-t'},$$

A being an arbitrary function of t, and A' an arbitrary function of t'. One is able always to transform this expression into that here

$$t^{a-n}t'^{a'-n'}\frac{A_1+A_1't'}{1-t-t'},$$

in which A_1 and A_1' are new arbitrary functions of t and of t'. In order to determine them, we will observe that, $y_{0,0}$ not being able to hold and $y_{x,0}$ being equal to zero, whatever be the entire and positive values of x, one will have

$$0 = u_{s,a'-n'} = \frac{1.2.3\ldots s.1.2.3\ldots(a'-n')}{1.2.3\ldots(a'-n'+s)} z_{s,a'-n'};$$

consequently the generating function of $z_{s,a'-n'}$ will be null, this which gives

$$t^{a-n} {t'}^{a'-n'} \frac{A_1}{1-t} = 0, \quad \text{and hence} \quad A_1 = 0.$$

Moreover, $y_{0,x'}$ being equal to unity for all the values of x' from $x' = 1$, one will have similarly

$$1 = u_{a-n,s'} = \frac{1.2.3\ldots(a-n).1.2.3\ldots s'}{1.2.3\ldots(a-n+s')} z_{a-n,s'};$$

whence one draws, for the value of $z_{a-n,s'}$ or the coefficient of $t^{a-n}{t'}^{s'}$ in the development of its generating function,

$$z_{a-n,s'} = \frac{(a-n+1)(a-n+2)\cdots(a-n+s')}{1.2.3\ldots s'},$$

this which gives

$$\begin{aligned} t^{a-n}{t'}^{a'-n'} \frac{A'_1 t'}{1-t'} &= t^{a-n}{t'}^{a'-n'} \frac{(a-n+1)\cdots(a+a'-n-n'+1)}{1.2.3\ldots(a'-n'+1)} \\ &\quad \times \left[\begin{aligned} & t' + \frac{(a+a'-n-n'+2){t'}^2}{a'-n'+2} + \cdots \\ & \quad + \frac{(a+a'-n-n'+2)\cdots(a+a'-n-n'+x'){t'}^{x'}}{(a'-n'+2)\cdots(a'-n'+x')} + \cdots \end{aligned} \right] \end{aligned}$$

The second member of this equation multiplied by $\frac{1}{1-\frac{t}{1-t'}}$ will be therefore the generating function of $z_{s,s'}$; by developing it with respect to the powers of t and next with respect to those of t', it is easy to see that the coefficient of t^s or of t^{a-n+x} is

$$\begin{aligned}&t'^{a'-n'}\frac{(a-n+1)\cdots(a+a'-n-n'+1)}{1.2.3\ldots(a'-n'+1)}\\ &\quad\times\left[t'+\frac{(a+a'-n-n'+2)}{a'-n'+2}t'^2+\cdots\right]\\ &\quad\times\left[1+\frac{x}{1}t'+\frac{x(x+1)}{1.2}t'^2+\cdots+\frac{x(x+1)\cdots(x+x'-2)}{1.2\ldots(x'-1)}t'^{x'-1}+\cdots\right],\end{aligned}$$

and the one of $t'^{s'}$, or of $t'^{a'-n'+x'}$ in this last expression, or $z_{s,s'}$, is equal to

$$\begin{aligned}&\frac{(a-n+1)\cdots(a+a'-n-n'+1)}{1.2.3\ldots(a'-n'+1)}\\ &\times\left[\begin{aligned}&\frac{x(x+1)\cdots(x+x'-2)}{1.2\ldots(x'-1)}+\frac{a+a'-n-n'+2}{a'-n'+2}\,\frac{x(x+1)\cdots(x+x'-3)}{1.2\ldots(x'-2)}+\cdots\\ &\qquad+\frac{(a+a'-n-n'+2)\cdots(a+a'-n-n'+x')}{(a'-n'+2)\cdots(a'-n'+x')}\end{aligned}\right].\end{aligned}$$

Now, by multiplying this value of $z_{s,s'}$ by

$$\frac{1.2.3\ldots(a'-n'+x')}{(a'-n'+x+1)\cdots(a+a'-n-n'+x+x')},$$

one will have, after all the reductions, for the expression of $y_{x,x'}$,

$$\begin{aligned}y_{x,x'}=&\frac{(a-n+x)\cdots(a-n+1)}{(a+a'-n-n'+x+x')\cdots(a+a'-n-n'+x+1)}\\ &\times\left[\begin{aligned}&1+\frac{x}{1}\frac{a'-n'+x'}{a+a'-n-n'+x'}+\frac{x(x+1)}{1.2}\frac{(a'-n'+x')(a'-n'+x'-1)}{(a+a'-n-n'+x')\cdots(a+a'-n-n'+x+2)}+\cdots\\ &+\frac{x(x+1)\cdots(x+x'-2)}{1.2\ldots(x'-1)}\frac{(a'-n'+x')\cdots(a'-n'+2)}{(a+a'-n-n'+x')(a+a'-n-n-1)}\end{aligned}\right]\end{aligned}$$

We imagine actually $a-n$ and $a'-n'$ in the ratio of p to q, so that one has $a-n=pk$ and $a'-n'=qk$, and we imagine that k becomes a very great number or infinity; it is clear that the probability of the exit of a white ball or of a black ball in the successive drawings will become constant and will be $\frac{p}{p+q}$ for a white ball and $\frac{q}{p+q}$ for a black, and the probability $y_{x,x'}$ will be reduced to this expression

$$y_{x,x'} = \left(\frac{p}{p+q}\right)^x \left[1 + \frac{x}{1}\frac{q}{p+q} + \frac{x(x+1)}{1.2}\left(\frac{q}{p+q}\right)^2 + \cdots \right. \\ \left. + \frac{x(x+1)\cdots(x+x'-2)}{1.2\ldots(x'-1)}\left(\frac{q}{p+q}\right)^{x'-1}\right];$$

such is the formula to which the *problem of points* leads, and effectively we return to the conditions of this problem by the supposition of k infinite.

If one supposes n equal to a and n' equal to a', $y_{x,x'}$ will express then the probability of the exit of all the white balls remaining in the urn before all the blacks had been depleted, and its expression will be changed into that here

$$\frac{1.2.3\ldots x}{(x+x')\cdots(x'+1)}\left[1 + \frac{x}{1} + \frac{x(x+1)}{1.2} + \cdots + \frac{x(x+1)\cdots(x+x'-2)}{1.2\ldots(x'-1)}\right],$$

which is reduced itself to

$$\frac{x'}{x+x'}.$$

The probability of extraction from the urn the totality of the white balls before that of the blacks is therefore in probability contrary in inverse ratio of the number of white balls to the one of the blacks.

One arrives to this last result, in an extremely simple manner, by means of combinations; in effect, the probability of the exit of all the balls from the urn, in any order, by color, will be

$$\frac{x(x-1)\cdots 2.1\, x'(x'-1)\cdots 2.1}{(x+x')(x+x'-1)\cdots 3.2.1} = \frac{1.2.3\cdots x'}{(x+1)\cdots(x+x')}.$$

But, in order that the white balls exit in totality first, it is necessary necessarily that a ball of the color black exit last: by combining $x'-1$ with $x'-1$ the $x+x'+1$ ranks of exit which are found before the last, one will form as many different rankings for the balls of the color black, and as many orders of exit by color, which will comprehend all those where one black ball exits in last place; now the number of these combinations is

$$\frac{(x+x'-1)(x+x'-2)\cdots(x+1)}{1.2\ldots(x'-1)},$$

and by multiplying them by the probability common to each order of exit by color, one will have the sought probability equal to

$$\frac{1.2.3\cdots x'}{(x+1)\cdots(x+x')}\frac{(x+1)\cdots(x+x'-1)}{1.2.3\ldots(x'-1)}=\frac{x'}{x+x'}.$$

Remarks on generating functions.

4. Let u be a generating function in one or many variables; each equation between this function and its variables, linear with respect to u, rational with respect to the variables, will subsist still if one passes from the generating functions to the coefficients, among these same coefficients, and will give place to an equation in the partial differences; but if, in this equation in partial differences, one passes again from the coefficients to the generating functions, one will no longer arrive to an equation rigorously exact, at least if one restores at the same time the functions of the variables which have been able to vanish in the first passage. Thus, in one of the questions that we have treated above, the equation in the partial differences

$$z_{x,x'}=mz_{x-1,x'}+m'z_{x,x'-1}+nz_{x-1,x'-1}$$

will give, by going up again simply from the coefficients to the generating functions, that here

$$u=mut+m'ut'+nutt',$$

which is not at all exact; because it is easy to see that, according to the conditions of the problem, it would be necessary to add to the second member the generating function of $z_{x,0}$ less this same function multiplied by m. This function of t, which it is necessary to restore in the second member of the equation in order to complete it, is precisely the arbitrary function that we have had to determine in the solution of this question. In general, the functions to add in order to have still one equation in the passage from the coefficients to the generating functions are the same as the arbitrary functions which form the numerator of the generating function integral before it is developed.

For lack of having regard to these functions, one is able to fall into some grave errors, by serving oneself in this manner in order to integrate the equations in the partial differences. For this same reason, the march followed in the solution of problems n$^{\text{os}}$ 8 and 10 of Book II of the *Théorie analytique des Probabilités* is not is by no means rigorous, and seems to implicate contradiction in this that it established a liaison among the variables which are and must be always independent. Without entering into the particular considerations which have been able to make it pass here, and that it is easy to know, we will show that the method of integration exposed at the beginning of this *Supplément* is applied equally to these questions, and resolves them with no less simplicity.

In the problem of n$^\text{o}$ 8, one is proposed to determine the lot of a number n of players A, B, C, $\ldots$ of which p, q, r, $\ldots$ represent the respective probabilities, that is to say their probabilities to win a trial when, in order to win the set, there lacks x trials to player A, x' trials to player B, x'' trials to player C, etc. By naming $y_{x,x',x'',\ldots}$ the probability of player A to win the set, one has the equation in partial differences

$$y_{x,x',x'',\ldots} = py_{x-1,x',x'',\ldots} + 1y_{x,x'-1,x'',\ldots} + ry_{x,x',x''-1,\ldots} + \cdots,$$

which gives for $y_{x,x',x'',\ldots}$ this generating function

$$\frac{P + Q + R + \cdots}{1 - pt - qt' - rt'' - \cdots},$$

in which P, Q, R, $\ldots$ are as many arbitrary functions of the variables t, t', t'', $\ldots$ as there are of these variables, by comprehending not at all t in the first, t' in the second, t'' in the third, etc. Now, this function is able to be set under this form

$$\frac{P' + Q't + R'tt' + S'tt't''\cdots}{1 - pt - qt' - rt'' - st'''\cdots},$$

P', Q', R', $\ldots$ being, as above, of the arbitrary functions, the first of all the variables with the exception of t, the second of all the variables of it excepting t', the third equally of all the variables except t'', and thus consecutively. In order to determine them, we will observe that, in $y_{x,x',x'',\ldots}$, two of the indices x, x', x'', $\ldots$ or a greater number are not able to be nulls at the same time, since the set ceases when one of the players has attained his points; moreover, $y_{0,x',x'',\ldots}$ is equal to unity, whatever be x', x'', $\ldots$; the generating function of this expression, or that which gives unity for the coefficient of any product whatsoever $t'^{x'}t''^{x''}t'''^{x'''}\ldots$, is

$$\frac{t'}{1-t'}\frac{t''}{1-t''}\frac{t'''}{1-t'''}\cdots;$$

consequently, one will have

$$P' = \frac{t'}{1-t'}\frac{t''}{1-t''}\frac{t'''}{1-t'''}\cdots(1 - qt' - rt'' - st''' - \cdots).$$

Each value of $y_{x,x',\,x'',\dots}$ in which another index than x is null being equal to zero, the corresponding generating function becomes null also; one will have therefore successively

$$Q' = 0, \quad R' = 0, \quad S' = 0, \quad \dots$$

Hence, the generating function of $y_{x,x',\,x'',\dots}$ will be

$$\frac{t'}{1-t'}\frac{t''}{1-t''}\cdots\frac{1 - qt' - rt'' - \cdots}{1 - pt - qt' - rt'' - \cdots},$$

and the coefficient of t^x, in the development of this function with respect to the powers of t,

$$\frac{t'}{1-t'}\frac{t''}{1-t''}\cdots\frac{p^x}{(1 - qt' - rt'' - \cdots)^x};$$

whence it is easy to draw the coefficient of $t'^{x'}t''^{x''}\dots$, or

$$y_{x,x',\,x'',\dots} = p^x\left\{\begin{array}{l} 1 + \frac{x}{1}(q + r + \cdots) \\ \quad + \frac{x(x+1)}{1.2}(q + r + \cdots)^2 \\ \quad + \frac{x(x+1)(x+2)}{1.2.3}(q + r + \cdots)^3 \\ \quad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right\},$$

in having need to reject the terms in which the power of q surpasses $x' - 1$, those in which the power of r surpasses $x'' - 1$,…

In the problem of n° 10, one considers two players A and B of whom the skills are p and q, and of whom the first has a tokens and the second b tokens; and one supposes that at each trial, the one who loses gives a token to his adversary, and that the set finishes only when one of the players will have lost all his tokens. One demands the probability that one of the players, A for example, will win the set before or at the nth trial.

By representing by $y_{x,x'}$ the probability of this player in order to win the set when he has x tokens and when he has no more than x' trials to play in order to attain the n trials, one will arrive, by the first principles of the probabilities, to the equation in the partial differences

$$r_{x,x'} = py_{x+1,x'-1} + qy_{x-1,y'-1},$$

which gives, for the generating function of $y_{x,x'}$,

$$\frac{A + A' + B't}{qt^2t' - t + pt'},$$

A being an arbitrary function of t, A' and B' two arbitrary functions of t'. In order to determine them more commodiously, we will transform this generating function into that here

$$\frac{A_1 t + A_1' + B_1' tt'}{qt^2t' - t + pt'},$$

in which A_1, A_1' and B_1' are, as above, some arbitrary functions of t and of t'. Now $\frac{A_1'}{pt'}$ is the coefficient of t^0 in the development of the function with respect to the powers of t, or the generating function of $y_{0,x'}$; but, by the conditions of the problem, $y_{0,x'}$ is null whatever be x'; consequently its generating function is also it; A_1' is therefore equal to zero.

The coefficient of t'^0, in the development of the generating function with respect to t', is $-A_1$, this which is at the same time the generating function of $y_{x,0}$, a quantity which is null so long as x is less than the sum of the tokens or $a + b$, and which becomes unity when $x = a + b$; A_1 is therefore a function of t which has for factor t^{a+b}, and of which one is able to take no account in the numerator of the generating function, because it must give only some powers of t superior to t^{a+b}, and we have seen of it only to have a generating function composed of the powers inferior to t, since x is able to be extended only from $x = 0$ to $x = a + b$.

The generating function of $y_{x,x'}$, thus limited between these values, is reduced therefore to

$$\frac{B_1' tt'}{qt^2t' - t + pt'},$$

which one is able to put easily under this form

$$(\Pi) \qquad \left\{ \begin{aligned} & \frac{B'_1}{p} t \frac{1}{\left(1-\frac{\frac{1}{t'}+\sqrt{\frac{1}{t'^2}-4pq}}{2p}t\right)\left(1-\frac{\frac{1}{t'}-\sqrt{\frac{1}{t'^2}-4pq}}{2p}t\right)} \\ & = \frac{B'_1}{p} \frac{t}{\sqrt{\frac{1}{t'^2}-4pq}} \left\{ \frac{\frac{1}{t'}+\sqrt{\frac{1}{t'^2}-4pq}}{1-\frac{\frac{1}{t'}+\sqrt{\frac{1}{t'^2}-4pq}}{2p}t} - \frac{\frac{1}{t'}-\sqrt{\frac{1}{t'^2}-4pq}}{1-\frac{\frac{1}{t'}-\sqrt{\frac{1}{t'^2}-4pq}}{2p}t} \right\}; \end{aligned} \right.$$

whence one draws, for the coefficient of t^{a+b}, the expression

$$\frac{B'_1}{p} \frac{1}{(2p)^{a+b-1}} \frac{\left(\frac{1}{t'}+\sqrt{\frac{1}{t'^2}-4pq}\right)^{a+b} - \left(\frac{1}{t'}-\sqrt{\frac{1}{t'^2}-4pq}\right)^{a+b}}{2\sqrt{\frac{1}{t'^2}-4pq}}.$$

But this coefficient is the generating function of $y_{a+b,x'}$, a quantity which is equal to unity; because it is certain that player A has won the set when he has won all the tokens of B: moreover, x' must be here zero or an even number, since the number of trials in which A is able to win the set is equal to b plus an even number; and, in effect, he must win all the tokens of B, and again win again each token that he has lost, this which requires two trials. The series

$$y_{a+b,0}{t'}^0 + y_{a+b,2}{t'}^2 + y_{a+b,4}{t'}^4 + \cdots,$$

which represents the coefficient of t^{a+b}, is therefore equal to $\frac{1}{1-t'^2}$, and one concluded from it

$$\frac{B'_1}{p} \frac{(2p)^{a+b-1}}{1-t'^2} \frac{2\sqrt{\frac{1}{t'^2}-4pq}}{\left(\frac{1}{t'}+\sqrt{\frac{1}{t'^2}-4pq}\right)^{a+b} - \left(\frac{1}{t'}-\sqrt{\frac{1}{t'^2}-4pq}\right)^{a+b}}.$$

Now the coefficient of t^a, drawn from the development of the function (Π), always with respect to the powers of t, will be

$$\frac{B'_1}{p} \frac{1}{(2p)^{a-1}} \frac{\left(\frac{1}{t'}+\sqrt{\frac{1}{t'^2}-4pq}\right)^{a} - \left(\frac{1}{t'}-\sqrt{\frac{1}{t'^2}-4pq}\right)^{a}}{2\sqrt{\frac{1}{t'^2}-4pq}},$$

and by substituting for $\frac{B'_1}{p}$ its value, one will have this coefficient or the generating function of $y_{x,x'}$ equal to

$$\frac{2^b p^b}{1-t'^2}\frac{\left(\frac{1}{t'}+\sqrt{\frac{1}{t'^2}-4pq}\right)^a-\left(\frac{1}{t'}-\sqrt{\frac{1}{t'^2}-4pq}\right)^a}{\left(\frac{1}{t'}+\sqrt{\frac{1}{t'^2}-4pq}\right)^{a+b}-\left(\frac{1}{t'}-\sqrt{\frac{1}{t'^2}-4pq}\right)^{a+b}}$$

or

$$\frac{2^b p^b t'^b}{1-t'^2}\frac{\left(1+\sqrt{1-4pqt'^2}\right)^a-\left(1-\sqrt{1-4pqt'^2}\right)^a}{\left(1+\sqrt{1-4pqt'^2}\right)^{a+b}-\left(1-\sqrt{1-4pqt'^2}\right)^{a+b}},$$

this which is formula (*o*) of the *Théorie analytique*.