A Problem concerning Pharaon

To determine generally the advantage of the banker with respect to the punter

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63. The principal rules of this game are, 1, that the banker deals with an entire deck composed of fifty-two cards, 2. That the banker draws all the cards in order, putting the ones to the right, & and the others to his left, by commencing at the right. 3. That to each hand, or to each deal, that is to say of two by two cards, the punter has the liberty to take one or many cards, & to venture over a certain sum. 4. That the banker wins the wager of the punter, when the card of the punter arrives in the right hand in an odd rank, & that he loses, when the card of the punter falls to the left hand and in an even rank. 5. That the banker takes the half of that which the punter has wagered on his card, when in the same deal the card of the punter comes twice, this which makes a portion of the advantage of the banker. And finally, that the last card which ought to be for the punter, is neither for him nor for the banker, this which is again an advantage for the banker.

It is evident that the conditions of this game are advantageous to the banker. The difficulty is to determine this advantage, because it changes, & according to the number of cards that are held by the banker, and also according as the card of the punter either has not passed, or has passed one or many times.

1. The card of the punter being only one time in the stock, the difference of the strength of the banker and the punter is founded on this, that among all the diverse possible permutations of the cards of the banker, there is of them a greater number which are winning, than there is of them which are losing, the

last card being considered as null, & in this case it is easy to notice that the advantage of the banker increases in measure as the number of cards of the banker diminish.

2. The card of the punter being twice in the stock, the advantage of the banker is drawn from the probability that is this, that the card of the punter will come twice in the same deal; because then the banker wins the half of the wager of the punter, except the sole case where the card of the punter would be in doublet in the last deal, this which would give to the banker the entire wager of the punter.

3. The card of the punter being either three or four times in the hand of the banker, the advantage of the banker is founded on the possibility that is this, that the card of the punter is found twice in the same deal, before it has come in pure gain or pure loss for the banker. Now this possibility increases or diminishes, & according as there are more or fewer cards in the hand of the banker, & according as the card of the punter is found there more or less times. Of all this it follows that in order to know the advantage of the banker with respect to the punters in all the different circumstances of this game, it is necessary to discover in all the different possible permutations of the cards that are held by the banker, & and under the supposition that the card of the punter is found, either one, or two, or three or four times, what are those which make him entirely win, what are those which give to him the half of the wager of the punter, what are those which make him lose, & finally what are the permutations which are neither winning nor losing.

In order to resolve this problem, it is apropos to begin with the simplest case, & next pass to the more compound cases, it is necessary to seek some uniform law, & some analogy which is able to serve to disentangle from all the possible cases, the permutations which are advantageous to the banker, those which are indifferent to him, & finally those which are unfavorable to him.

This way is not always the shortest; but as one employs it frequently with success, & as it presents itself first to mind, I follow it here in detail in order to render it familiar to the reader: he may certainly pass it if it is not to his taste. I will give next another method more sophisticated, more analytic, & of an infinitely more extensive use.

FIRST METHOD.

FIRST CASE.

On supposes that there remain four cards in the hand of the banker, & that the one of the punter is in it a certain number of times. The concern is to determine what is the strength of the banker & that of the punter: For example, if there is an ecu for the card of the punter, one demands what portion of the ecu the punter must give to the banker in order to buy the right of his withdraw, & to not incur the risk of the game; or, this which remains the same, what is in this case the disadvantage of the punter, in playing end to end against the banker.[1]

64. If one wants to express the four cards of the banker by the letters a, b, c, d, one will have all the different permutations of four cards represented in the following table.

$abcd$	$bacd$	$cabd$	$dabc$
$abdc$	$badc$	$cadb$	$dacb$
$acbd$	$bcad$	$cbad$	$dbac$
$acdb$	$bcda$	$cbda$	$dbca$
$adbc$	$bdac$	$cdab$	$dcab$
$adcb$	$bdca$	$cdba$	$dcba$

1. If one supposes that the card of the punter designated by the letter a, is one time in the four cards of the banker, & that the punter has wagered on his card a sum of money expressed by A, one will note in considering the preceding table, that there are twelve permutations which give $2A$ to the banker, six which make him lose or which give to him 0, & six which are indifferent to him.

Those which are the winning are:

$abcd$	$bcad$
$abdc$	$bdac$
$acbd$	$cbad$
$acdb$	$cdab$
$adbc$	$dbac$
$adcb$	$dcab$

[1] See the letter of Jean Bernoulli to Montmort dated 17 March 1710 and the reply by Montmort dated 15 November 1710.

Those which make him lose are:

$$\begin{matrix} bacd & cabd & dabc \\ badc & cadb & dacb \end{matrix}$$

Thus expressing the sought strength by the letter s, one will have

$$s = \frac{12 \times 2A + 6 \times 0 + 6 \times A}{24} = \frac{5}{4}A = A + \frac{1}{4}A.$$

2. If one supposes that the card of the punter is found twice among the four cards of the banker, & that the two letters a & b express those of the punter, one will find that of the twenty-four permutations of the table, there are twelve which give $2A$ to the banker:

$$\begin{matrix} acbd & bcad & cdba \\ acdb & bcda & cdab \\ adbc & bdac & dcba \\ adcb & bdca & dcab \end{matrix}$$

Four which give to him $\frac{3}{2}A$, that is to say, his ecu and the half of the one of the punter:

$$\begin{matrix} abcd & bacd \\ abdc & badc \end{matrix}$$

Eight which make him lose:

$$\begin{matrix} cabd & dabc \\ cadb & dacb \\ cbad & dbac \\ cbda & dbca \end{matrix}$$

Thus one will have $s = \frac{12\times 2A + 4\times\frac{3}{2}A + 8\times 0}{24} = \frac{5}{4}A = A + \frac{1}{4}A.$

3. If one supposes that the card of the punter is found three times among the four cards of the banker, & that the three letters $a, b, c,$ express those of the punter, one will find again the strength of the banker $= A + \frac{1}{4}A$; because there

are twelve permutations there which give to him $\frac{3}{2}A$.

$$\begin{array}{ccc} abcd & bacd & cabd \\ abdc & badc & cadb \\ acbd & bcad & cbad \\ acdb & bcda & cbda \end{array}$$

Six which give to him $2A$:

$$\begin{array}{ccc} adbc & bdac & cdab \\ adcb & bdca & cdba \end{array}$$

Six which make him lose:

$$\begin{array}{ccc} dabc & dbac & dcba \\ dacb & dbca & dcab \end{array}$$

One will have therefore $s = \frac{12\times\frac{3}{2}A+6\times 2A+6\times 0}{24} = A + \frac{1}{4}A$.

4. Finally it is evident that if the card of the punter is found four times in the four cards of the banker, the strength of the banker will be $= A + \frac{1}{2}A$.

COROLLARY I.

65. It seems by the solution in this first case, that if the wager of the punter is one ecu, he must give five sols which is the fourth of it to the banker, in order to buy the right of his withdraw, this is if his card is one time, or two times, or three times in the four cards of the banker.

COROLLARY II.

66. There would be an infinite labor to seek the other cases in the manner that one has resolved this one in searching in some tables the favorable & contrary permutations; because the number becomes immense in a greater number of cards; also I have put the preceding solution, only in order to make me more easily understood in the following.

In order to resolve the preceding case in a methodical manner, & in order to discover the chances by the sight of the mind, it is necessary to note, That if the card of the punter be one time in two cards, the strength of the banker will be $\frac{3}{2}A$: Because of two possible permutations of two letters, there is one of them which gives $2A$, & one which gives to him A; & if the card of the punter be in it more than one time, the strength of the banker will be $2A$, this which is evident.

It is necessary to observe next that the card of the punter being one time in four cards, if one places the twenty-four possible permutations of four letters on four columns, of which the first begin all by a, the second by b, the third by c, the fourth by d, the first column will give $2A$ to the banker in all its permutations.

And when partitioning each of the three other columns into three columns of two permutations, the one of these last three, namely the one where a occupies the second place, will give twice zero to the banker, & each of the two final others will give to the banker the same strength as he will have in the case that the banker holds two cards, the one of the punter being found one time there, that is to say $\frac{3}{2}A$, this which gives the strength of the banker, as hereafter

$$= \frac{1\times 6\times 2A + 3\times 2\times 2\times \frac{3}{2}A}{24} = \frac{30}{24}A = A + \frac{1}{4}A.$$

One will observe similarly that the card of the punter expressed by the letters a & b being twice in the four cards, if one imagines the twenty-four different permutations as the four cards may be received, put on four columns, as hereafter, the two columns which commence with the letters a & b, will contain each four permutations which will give $2A$ to the banker, & two permutations which will give $\frac{3}{2}A$ to him: Because in the one there are two permutations where a is followed by b, & in the other there are two permutations where b is followed by a; & partitioning each of the two other columns of six permutations into three others of two permutations, there will be two of these three which will give two times zero to the banker, a & b occupying the second place, & the third will give to the banker the same strength as he would have, if the card of the punter was found twice in the two cards; & consequently one would have again, according to this idea, the strength of the banker,

$$= \frac{2\times \overline{4\times 2A + 2\times \frac{3}{2}A} + 2\times 2\times 2A}{24} = A + \frac{1}{4}A.$$

One will notice again that the card of the punter expressed by the letters a, b, c, being three times in the four cards, the three columns which begin with the letters a, b, c, will contain each two permutations which will give $2A$ to the banker, & four permutations which will give to him $\frac{3}{2}A$, some two of the three letters a, b, c, being in sequence, & when partitioning the last column which commences with d into three columns of two permutations, each of the three will give twice zero to the banker, in a way that his strength will be again

$$\frac{3 \times \overline{2 \times 2A + 4 \times \frac{3}{2}A} + 1 \times 0}{24} = A + \frac{1}{4}A.$$

Finally it is evident that the card of the punter expressed by the letters $a, b, c, d,$ being four times in the four cards, the four columns which commence with the letters $a, b, c, d,$ will contain each six permutations, which will give to the banker $\frac{3}{2}A$, since all these different permutations will necessarily produce a doublet; whence it follows that the strength of the banker will be $A + \frac{1}{2}A$.

All this is founded on the order of the permutations, & it will be clarified by the application that I will make of it in the following.

COROLLARY III.

67. Whatever number of cards that are held by the banker, if that of the punter is found there only one time, the advantage of the banker will be expressed by a fraction which will be unity for the numerator, & for denominator the number of cards that are held by the banker: because six cards, for example, are able to be arranged in 720 different ways, it is clear that if one imagines all these different permutations put on six columns of one hundred twenty permutations each, in a way that in the first the letter a is everywhere in the first place, that in the second it is everywhere in the second place, that in the third it is everywhere in the third place, and thus in sequence, the first, the third & the fifth columns will give $2A$ to the banker in all their permutations; the second & the fourth will give to him zero, & the sixth will give to him A. One will have thus

$$s = \frac{3 \times 120 \times 2A + 2 \times 120 \times 0 + 1 \times 120 \times A}{720} = \frac{840}{720}A = A + \frac{1}{6}A.$$

And in general, if one names p the number of cards of the banker; m the number of all the possible permutations of these cards, one will have always the strength of the banker expressed by this formula

$$s = \frac{\frac{1}{2}p \times \frac{m}{p} \times 2A + \frac{m}{p} \times A}{m} = A + \frac{A}{p}.$$

SECOND CASE.

One supposes that six cards are held by the banker, & that those of the punter are in them a certain number of times. One requires what is the strength of the banker in all the variations of this second case.

68. Let be supposed that the card of the punter is found twice in the six cards.

If these six cards are represented by the six letters $a, b, c, d, e, f, g,$ in such a way that any two, for example, a & g express those of the punter.

One will remark, 1, that one is able to put the seven hundred twenty different permutations that six cards are able to receive on six columns, of which each will be composed of one hundred twenty perpendicular ranks; in such a way that the first column begins all with the letter a, the second with the letter b, the third with the letter c, & thus in sequence.

2. That the two columns which begin with a & with g, are each eighty-six perpendicular ranks, which give to the banker $2A$, and twenty-four which give to him $\frac{3}{2}A$: because each rank of these two columns give $2A$ to the banker, to the exception of those two where a is followed by g in the first, & where g is followed by a in the last. Now five letters are able to receive 120 different permutations, & each is found necessarily an equal number of times after a in the first column, & after g in the last, it is evident that it is necessary to divide 120 by 5, in order to have all the doublets in each of the two columns which commence either with a, or with g. This remark is important for the solution of this problem, & it is necessary to the solution of it in the following.

The greatest difficulty, this is to discover what the four other columns give to the banker. In order to disentangle it, it is necessary to remark first that each of these four columns give an equal strength to the banker (this which is evident,) & that thus it suffices to examine one. Let the column which commences with b, be the one which one wants to examine; & for greater facility, I partition it into five columns of twenty-four permutations each.

1	2	3	4	5
$bacdfg$	$bcadfg$	$bdacfg$	$bfacdg$	$bgacdf$
$bacdgf$	$bcadgf$	$bdacgf$	$bfacgd$	$bgacfd$
$bacfdg$	$bcafdg$	$bdafcg$	$fbadcg$	$bgadfc$
$bacgdf$	$bcagfd$	$bdagcf$	$bfagcd$	$bgafcd$
$bacgfd$	$bcagdf$	$bdagfc$	$bfagdc$	$bgafdc$
$badcfg$	$bcdafg$	$bdcafg$	$bfcadg$	$bgcadf$
$badcgf$	$bcdagf$	$bdcagf$	$bfcagd$	$bgcafd$
$badfcg$	$bcdfag$	$bdcfag$	$bfcdag$	$bgcdaf$
$badfgc$	$bcdfga$	$bdcfga$	$bfcdga$	$bgcdfa$
$badgfc$	$bcdgaf$	$bdcgaf$	$bfcgad$	$bgcfad$
$badgcf$	$bcdgfa$	$bdcgfa$	$bfcgda$	$bgcfda$
$bafcdg$	$bcfadg$	$bdfacg$	$bfdacg$	$bgdacf$
$bafcgd$	$bcfagd$	$bdfagc$	$bfdagc$	$bgdafc$
$bafdcg$	$bcfdag$	$bdfcag$	$bfdcag$	$bgdcaf$
$bafdgc$	$bcfdga$	$bdfcga$	$bfdcga$	$bgdcfa$
$bafgdc$	$bcfgad$	$bdfgac$	$bfdgac$	$bgdfac$
$bafgcd$	$bcfgda$	$bdfgca$	$bfdgca$	$bgdfca$
$bagcdf$	$bcgadf$	$bdgacf$	$bfgacd$	$bgfacd$
$bagcfd$	$bcgafd$	$bdgafc$	$bfgadc$	$bgfadc$
$bagdcf$	$bcgdaf$	$bdgcaf$	$bfgcad$	$bgfcad$
$bagdfc$	$bcgdfa$	$bdgcfa$	$bfgcda$	$bgfcda$
$bagfcd$	$bcgfcd$	$bdgfac$	$bfgdac$	$bgfdac$
$bagfdc$	$bcgfda$	$bdgfca$	$bfgdca$	$bgfdca$

It is easy to see, in consulting this table, that the first and the fifth columns give zero to the banker, since in the first the letter a, & in the fifth the letter g hold the second place there, & that each of the three other columns contain twelve permutations which give $2A$ to the banker, eight which give to him zero, & four which give to him $\frac{3}{2}A$, that is to say, that each of these three columns give the same chances that one has found for the banker in the preceding case, when one has supposed that he would hold four cards, among which those of the punter will be found twice, of which the reason is that the first two letters of the second, third & fourth columns of the table above are not those of the punter, there remain four letters, among which those which express the card of the punter are found twice: this which is reduced manifestly to the second article of the preceding case, where the card of the punter is found twice in four cards.

Thus the column which begins with the letter b will give to the banker $2 \times 24 \times 0 + 3 \times \overline{8 \times 0 + 12 \times 2A + 4 \times \frac{3}{2}A} = 90A$. Now the columns of 120 permutations which begin with c, with d, & with f, give the same value, & consequently in order to have all the favorable moves that the four columns give which begin neither with a, nor with g, it is necessary to multiply $90A$ by 4, this which makes $360A$; to which adding $2 \times \overline{96 \times 2A + 14 \times \frac{3}{2}A} = 456A$ for the favorable moves that the columns give which begin with a & with g, one will have $\frac{360A+456A}{720} = \frac{816}{720}A = A + \frac{2}{15}A$ for the strength of the banker in the proposed case.

COROLLARY

69. Whatever number of cards that are held by the banker, if those of the punter are encountered there twice, in order to find the strength of the banker, it is necessary to imagine all the possible permutations of the cards which he holds on as many columns as there are cards; & to note next that the two columns which begin with the letters which express the card of the punter, give each $2A$ to the banker, with the exception of the ranks, where one of the letters which expresses the card of the punter is next to the other, which permutation gives $\frac{3}{2}A$.

In order to find how many there are of those ranks in each of the two columns, it is necessary to divide all the permutations which compose them by the number of the cards less one; the exponent of this division will express the number of the permutations which give $\frac{3}{2}A$ in each of the two columns. In order to determine that which the other columns gives, one imagines each partitioned into as many columns less one than there are cards; & observing a permutation parallel to that of the two preceding tables, one will find that there are always two of these last columns which give zero to the banker, the two letters which express the card of the punter occupying the second place there; & that each of the others equal to this will give to the banker the same strength as he will have in the preceding case, that is to say in the case where the number of the cards of the banker being less by two, those of the punter would be twice there.

There is in the remarks of this corollary what consists the solution of the problem for the case where the card of the punter is found twice among the cards of the banker. I would have had difficulty to well make understood this method, without making application of it in some particular cases, & without my making use of the table which is found, *art. 68*.

GENERALLY.

70. Whatever number of cards that the banker holds, & whatever number of times that the card of the punter be among those of the banker, one will find always his strength in this way. 1. One will seek by the method of *art. 64*, the number of all the different possible permutations of the cards of the banker. 2. One will represent these cards by the letters $a, b, c, d, f,$ &c. & one will suppose that some at will designate those of the punter. 3. One will imagine all these different permutations distributed on as many columns as there will be cards; in such a way that the first begins all with the letter a, the second with the letter b, the third with the letter c, &c. 4. One will note that the columns which begin with the letters which designate the card of the punter, give $2A$ to the banker in all their permutations, with the exception of those where some two of among the letters which express the card of the punter, are found in sequence in the first & in the second place; this will give $\frac{3}{2}A$.

In order to find the number of those permutations in each of these columns, one will divide the number of permutations from which each column is composed by the number of the cards of the banker less one, & one will multiply the exponent by the number of times less one that the card of the punter is found in those of the banker; this product will give all the permutations of these columns, which give $\frac{3}{2}A$.

In regard to the other columns which begin with some letters different from those which express the card of the punter, it is necessary, in order to discover the favorable permutations there, to imagine them each partitioned & subdivided into as many columns less one, as there are cards, & to have regard to the order marked in the tables on pages 80 and 86; to observe that of those last columns there are always as many which give zero to the banker, as of times the card of the punter is found in those of the banker; & as each of the other small columns give to the banker the same strength as one has found in the case which has preceded; that is to say in the case where the number of cards of the banker being less by two, the card of the punter is found there an equal number of times.

Thus one will find among all the different possible permutations of the cards which are held by the banker, which are those which give to him either A, or $2A$, or $\frac{3}{2}A$, or zero; consequently one will have by this method the strength of the banker in all the possible cases: that which is was necessary to find.

In following the spirit of this method, if one names p the number of cards that are held by the banker, q the number of times that the card of the punter is in those of the banker, g the strength of the banker in a number of cards expressed by $p-2$, S the sought strength: one will have the strength of the banker expressed by this formula.

$$S = \frac{\overline{pq-qq} \times 2A + \overline{qq-q} \times \frac{3}{2}A + g \times \overline{p-q} \times \overline{p-q-1}}{p \times \overline{p-1}}$$

One is able to find by this formula the strength of the banker, whatever number of cards that he have within his hands, & whatever number of times that the card of the punter is contained there. But this formula has this very great inconvenience of giving the advantage of the banker for a certain number of cards designated by p, only when one knows already his advantage for a number of cards which are $p-2$. Thus this formula is only able to be useful in order to find all the different cases the one after the other, in beginning with the most simple. Thence here is one other, which gives without much calculation all the different cases in general, & each case in particular independently the one from the other.

SECOND METHOD.

71. Let B be the advantage of the banker at the first deal, when the punter comes to put a card into play, y his advantage at the second, z his advantage at the third, u his advantage at the fourth, &c. p & q signifying the same things as in

the preceding method, one find

$$B = \frac{q.q-1}{p.p-1} \times \frac{1}{2}A + \frac{p-q.p-q-1}{p.p-1}y,$$
$$y = \frac{q.q-1}{p-2.p-3} \times \frac{1}{2}A + \frac{p-q-2.p-q-3}{p-2.p-3}z,$$
$$z = \frac{q.q-1}{p-4.p-5} \times \frac{1}{2}A + \frac{p-q-4.p-q-5}{p-4.p-5}u,$$
$$u = \frac{q.q-1}{p-6.p-7} \times \frac{1}{2}A + \frac{p-q-6.p-q-7}{p-6.p-7}s, \text{ \&c.}$$

If one substitutes for y, z, u, s their values, one will have this indefinite formula

$$\begin{aligned} B = 1 &+ \frac{p-q.p-q-1}{p-2.p-3} + \frac{p-q.p-q-1.p-q-2.p-q-3}{p-2.p-3.p-4.p-5} \\ &+ \frac{p-q.p-q-1.p-q-2.p-q-3.p-q-4.p-q-5}{p-2.p-3.p-4.p-5.p-6.p-7} \\ &+ \frac{p-q.p-q-1.p-q-2.p-q-3.p-q-4.p-q-5.p-q-6.p-q-7}{p-2.p-3.p-4.p-5.p-6.p-7.p-8.p-9} \text{ \&c.} \end{aligned}$$

the whole multiplied by $\frac{q.q-1}{p.p-1} \times \frac{1}{2}A$. which gives the advantage of the banker, whatever be the value of p & of q. Therefore, for example, if $p = 12,$ & $q = 3$, the formula gives

$$B = \frac{10+8+6+4+2}{12.11.10} \times 3 \times 2 \times \frac{1}{2}A = \frac{3}{44}A;$$

& if $p = 12,$ & $q = 4$, the formula gives

$$B = \frac{45+28+15+6+1}{10.9} \times \frac{4.3.2.1}{12.11} \times \frac{1}{2}A = \frac{19}{198}A;$$

& if $p = 12$, & $q = 5$, the formula gives

$$B = \frac{120+56+20+4}{10.9.8} \times \frac{5.4.3.2.1}{12.11} \times \frac{1}{2}A = \frac{25}{198}A.$$

In examining this formula one finds,

1. That q being $=1$, it becomes $=0$; but that by the conditions of the game, because of the last card which is null, it is necessary to add $\frac{A}{p}$.

2. That q being $=2$, it becomes

$$1+1+1+1+1 \text{ \&c. } \times \frac{2\times 1}{p.p-1}\times\frac{1}{2}A.$$

But that by the conditions of the game it is necessary to multiply the last of these units for $\frac{2.1}{p.p-1}$ by A, & not by $\frac{1}{2}A$.

3. That q being $=3$, it becomes

$$\frac{p-2+p-4+p-6+p-10}{p-2}+\text{\&c. }\times\frac{3\times 2}{p.p-1}\times\frac{1}{2}A.$$

4. That q being $=4$, it becomes

$$\frac{p-2.p-3+p-4.p-5+p-6.p-7+p-8.p-9}{p-2.p-3}\text{\&c. }\times\frac{4\times 3}{p.p-1}\times\frac{1}{2}A, \text{ \&c.}$$

5. That q being $=5$, it becomes

$$\frac{p-2.p-3.p-4+p-4.p-5.p-6+p-6.p-7.p-8+p-8.p-9.p-10}{p-2.p-3.p-4}+\text{\&c.}\times\frac{5\times 4}{p.p-1}\times\frac{1}{2}A.$$

In such a way that in each sequence all the terms are composed of as many products as there are units in $q-2$.

6. That these products provide multiples of the figurate numbers of the order $q-1$, interposed two by two, which correspond to some natural even numbers, beginning with those which correspond to $p-2$: Whence it follows that in order to find the fraction which expresses the advantage of the banker, one is able to make this rule.

The denominator will contain as many products of the quantities $p.p-1.p-2.p-3$, &c. as there are units in q. In order to have the numerator it will be necessary to take in the arithmetic triangle, *art. 1*, a horizontal rank, of which the row number is $q-1$, to add into a sum all the terms of this rank taken two by two, beginning with that which corresponds to $p-2$; to multiply this sum by as many products of the natural numbers $1, 2, 3, 4, 5, 6,$ &c. as there are of units in q, & to multiply it again by $\frac{1}{2}A$, having regard to the two exceptions marked above, the one for the case $q=1$, the other for the case $q=2$.

In order to reduce this rule to some formulas which determine all at one strike, in substituting for p its value, the advantage of the banker for whatever number of cards that this be, when the card of the punter is found there a certain number of times expressed by q. It will suffice in the case of $q = 3$ to find the sum of an arithmetic progression; in this manner the first three cases where $q = 1 = 2 = 3$ are not difficult. In regard to the others, one has need of the general solution of the problem which follows.

To find the sum of a progression of which each term is formed of as many products of quantities which decrease from unity as there are units in $q - 2$.

72. If $q = 5$, for example, it is necessary to find the sum of this progression

$$p - 2.p - 3.p - 4 + p - 4.p - 5.p - 6 + p - 6.p - 7.p - 8 + p - 8.p - 9.p - 10 + \text{\&c.}$$

And if $q = 6$, it is necessary to find the sum of this series

$$p - 2.p - 3.p - 4.p - 5 + p - 4.p - 5.p - 6.p - 7 + p - 6.p - 7.p - 8.p - 9 + \text{\&c.}$$

& thus in sequence, in relation to the different values of q; this which is the same thing as to find generally the sum of the numbers interposed two by two in the arithmetic triangle.

One could come to the end of it easily enough by the general method of *art. 54.* I will employ here a more direct method, founded on a curious property of figurate numbers.

LEMMA.

If one takes an even number at will of figurate numbers of any order whatever, the sum of those which are found even, that is to say those which in Table 2, Art. 7*, are found in the second, fourth, sixth, eighth, &c. place, is equal to the excess of those which in the following rank are found corresponding to the evens of superior rank on those which are found corresponding to the odds.*

73. As this exposition may appear obscure, I am going to illustrate it by an example. Let be taken the first eight numbers of fourth order $1, 4, 10, 20, 35, 56, 84, 120$; I say that the sum of the four $4, 20, 56, 120,$ which are taken in the second, fourth, sixth, eighth place is equal to the excess of these $5.35.126.330$, which are taken in the 2nd, 4th, 6th & 8th place of the rank immediately inferior on these, $1, 15, 70, 210,$ which are in the 1st, 3rd, 5th & 7th place of this same rank.

One is able to demonstrate also in a few words this property.

Let the numbers of any order be represented by the letters a, b, c, d, e, f, g, h. The numbers of the order immediately inferior will be by the nature & the formation of these numbers $a,\ a+b,\ a+b+c,\ a+b+c+d,\ a+b+c+d+e,\ a+b+c+d+e+f,\ a+b+c+d+e+f+g,\ a+b+c+d+e+f+g+h$. Now it is evident that subtracting within this second rank the first from the second, there remains b; & that subtracting the third from the fourth, there remains d; & that subtracting the fifth from the sixth, there remains f; & that subtracting the seventh from the eighth, there remains h; & thus in sequence, by the necessity of the relationship which is among any order & the one which follows it immediately.

In order to make application of this Lemma,

Let be proposed to find the sum of the numbers of the second rank taken two by two $2+4+6+8+10$, in naming g the number of the terms comprising the evens & the odds: here is the sequence of the operation.

$$2+4+6+8+10\,\text{\&c.} = \frac{g.g+1}{1.2} - 1 - 3 - 5 - 7 - 9\ \text{\&c.}$$
$$= \frac{g.g+1}{1.2} + \frac{1}{2}g - 2 - 4 - 6 - 8 - 10\ \text{\&c.}$$

Therefore in transposing,

$$2 \times \overline{2+4+6+8+10} + \text{\&c.} = \frac{1}{2}g + \frac{g.g+1}{1.2}.$$

Therefore

$$2+4+6+8\,\text{\&c.} = \frac{1}{4}g + \frac{1}{2} \times \frac{g.g+1}{1.2}.$$

Let be again proposed to find the sum of those numbers of the third order $3+10+21+36+55$, &c.

One has

$$\begin{aligned} 3+10+21+36+55 \text{ \&c.} &= 2+4+6+8+10 \text{ \&c.} \\ &\quad +1+6+15+28+45 \text{ \&c.} \\ &= \frac{1}{4}g + \frac{1}{2} \times \frac{g.g+1}{1.2} + 1+6+15+28+45 \text{ \&c.} \\ &= \frac{g.g+1.g+2}{1.2.3} - 1-6-15-28-45 \text{ \&c.} \\ &= \frac{g.g+1.g+2}{1.2.3} + \frac{1}{2} \times \frac{g.g+1}{1.2} \\ &\quad + \frac{1}{4}g - 3-10-21-36-55 \text{ \&c.} \end{aligned}$$

Therefore in transposing & dividing by 2. One has

$$3+10+21+36+55 \text{ \&c.} = \frac{1}{2} \times \frac{g.g+1.g+2}{1.2.3} + \frac{1}{4} \times \frac{g.g+1}{1.2} + \frac{1}{8}g.$$

One will find in the same manner the sum of the numbers of the fourth order corresponding to the even natural numbers

$$\begin{aligned} 4+20+56+120+220 \text{ \&c.} = \frac{1}{2} &\times \frac{g.g+1.g+2.g+3}{1.2.3.4} \\ &+ \frac{1}{4} \times \frac{g.g+1.g+2}{1.2.3} + \frac{1}{8} \times \frac{g.g+1}{1.2} + \frac{1}{16}g. \end{aligned}$$

And the sum of the numbers of the fifth order

$$\begin{aligned} 5+35+126+330+715 \text{ \&c.} &= \frac{1}{2} \times \frac{g.g+1.g+2.g+3.g+4}{1.2.3.4.5} \\ &\quad + \frac{1}{4} \times \frac{g.g+1.g+2.g+3}{1.2.3.4} \\ &\quad + \frac{1}{8} \times \frac{g.g+1.g+2}{1.2.3} \\ &\quad + \frac{1}{16} \times \frac{g.g+1}{1.2} + \frac{1}{32} \times g, \end{aligned}$$

& thus the remainder, one sees sufficiently the order of these sequences.

Having put this as above, one will find by that which precedes, that the sum of this sequence $g+g-2+g-4+g-6+g-8+$ &c. is

$$\frac{1}{2}\frac{g\times g+1}{1.2}+\frac{1}{4}g.$$

And that the sum of this sequence $\frac{g.g-1}{1.2}+\frac{g-2.g-3}{1.2}+\frac{g-4.g-5}{1.2}+\frac{g-6.g-7}{1.2}+$ &c.

$$=\frac{1}{2}\times\frac{g+1.g.g-1}{1.2.3}+\frac{1}{4}\times\frac{g.g-1}{1.2}+\frac{1}{8}g-1+\frac{1}{8}.$$

One will find in the same manner that the sum of this sequence $\frac{g.g-1.g-2}{1.2.3}+\frac{g-2.g-3.g-4}{1.2.3}+\frac{g-4.g-5.g-6}{1.2.3}+\frac{g-6.g-7.g-8}{1.2.3},+$ &c. is

$$=\frac{1}{2}\times\frac{g+1.g.g-1.g-2}{1.2.3.4}+\frac{1}{4}\times\frac{g.g-1.g-2}{1.2.3}+\frac{1}{8}\times\frac{g-1.g-2}{1.2}+\frac{1}{16}\times g-2.$$

And again the sum of this sequence $\frac{g.g-1.g-2.g-3}{1.2.3.4}+\frac{g-2.g-3.g-4.g-5}{1.2.3.4}+\frac{g-4.g-5.g-6.g-7}{1.2.3.4}+\frac{g-6.g-7.g-8.g-9}{1.2.3.4.5}+$ &c.

$$=\frac{1}{2}\times\frac{g+1.g.g-1.g-2.g-3}{1.2.3.4.5}+\frac{1}{4}\times\frac{g.g-1.g-2.g-3}{1.2.3.4}+\frac{1}{8}\times\frac{g-1.g-2.g-3}{1.2.3}+\frac{1}{16}\times\frac{g-2.g-3}{1.2}+\frac{1}{32}\times g-3+\frac{1}{32}.$$

& thus all the others. One sees without difficulty the order of these sequences.

It is evident by the Lemma above that having the sum of the numbers interrupted two by two, & corresponding to the even natural numbers, one will have also the sum of those which correspond to the odds; since having the whole & one of the parts of the whole, one has the other part.

But in general, here is the rule in order to have the sum of the figurate numbers taken two by two, which correspond to the natural numbers either even or odd, & at the same time the demonstration of the rule.

If one requires the sum of the terms of any order m taken two by two, of which the first corresponds to g which designates here an even or an odd number at will. One knows that in the inferior horizontal rank, which is $m+1$, the term which corresponds to $g+1$ is equal to all the terms of the order m taken in sequence, from the perpendicular column where is found g, up to zero. Imagining therefore all the terms of the rank m, to begin with the term which corresponds to g, divided into two unequal parts, of which the greater is that which one seeks, & of which the first term corresponds to g, one has this greater part $= \frac{1}{2}$ of the term of order $m+1$, which corresponds to $g+1$ plus $\frac{1}{2}$ of the difference of the two unequal parts; & by the Lemma, this difference is the sum of all the terms of order $m-1$ taken two by two, to begin with the one which corresponds to $g-1$. In order to have the sum of these terms, in making as above, it is necessary to take $\frac{1}{2}$ of the term of order m which corresponds to g, $+\frac{1}{2}$ of the difference which is the sum of all the terms of order $m-2$ taken two by two, to begin always with that which is $g-2$, or corresponding to $g-2$, in continuing up to the last difference.

As this demonstration may seem a little abstract, I am going to try to clarify it by two examples which follow.

Let be proposed to find in the fourth rank, *Tab. I.* the sum of these terms $\ldots 120+56+20+4$ which correspond to the natural numbers $\ldots 10+8+6+4$. I note that this sequence is greater than the one $\ldots 84+35+10+1$, & that it surpasses it by the sum of these other numbers $\ldots 36+21+10+3$ which is the difference of them. Now one knows that the greater of the two quantities is equal to the half of the total, plus to the half of their difference. Imagining therefore the entire sum $\ldots 120+84+56+35+20+10+4+1=330$ partitioned into these two unequal parts. I say that the greater, which is the one of which one seeks the sum, is

$$= \frac{1}{2} \times 330 + \frac{1}{2} \times \overline{36+21+10+3};$$

& seeking in the same manner the sum of this sequence $36+21+10+3$, which surpasses that other $28+15+6+1$ by the sequence $8+6+4+2$.

One will find it

$$= \frac{1}{2} \times 120 + \frac{1}{2} \times \overline{8+6+4+2};$$

& seeking again the sum of this sequence $8+6+4+2$ greater than this one $7+5+3+1$ by the sequence $1+1+1+1$, one will find it

$$= \frac{1}{2} \times 36 + \frac{1}{2} \times \overline{1+1+1+1},$$

& seeking again the sum of this last sequence, one finds it

$$= \frac{1}{2} \times 8 + \frac{1}{2} \times 0,$$

because the difference is zero. One will have therefore the sought sum

$$\begin{aligned}
&= \frac{1}{2} \times 330 + \frac{1}{2} \times \overline{36 + 21 + 10 + 3} \\
&= \frac{1}{2} \times 330 + \frac{1}{4} \times 120 + \frac{1}{4} \times \overline{8 + 6 + 4 + 2} \\
&= \frac{1}{2} \times 330 + \frac{1}{4} \times 120 + \frac{1}{8} \times 36 + \frac{1}{8} \times \overline{1 + 1 + 1 + 1} \\
&= \frac{1}{2} \times 330 + \frac{1}{4} \times 120 + \frac{1}{8} \times 36 + \frac{1}{16} \times 8,
\end{aligned}$$

conforming to the formula given above

$$\frac{1}{2} \times \frac{g + 1.g.g - 1.g - 2}{1.2.3.4} + \frac{1}{4} \times \frac{g.g - 1.g - 2}{1.2.3} + \frac{1}{8} \times \frac{g - 1.g - 2}{1.2} + \frac{1}{16} \times g - 2.$$

Let there again be proposed to find in the fifth rank the sum of these terms $210 + 70 + 15 + 1$, & supposing for brevity of discourse the difference of this sequence to this other $126 + 35 + 5 + 0$, which is $84 + 35 + 10 + 1 = b$; & the difference of this sequence $84 + 35 + 10 + 1$ to this other $56 + 20 + 4 + 0$, which is $28 + 15 + 6 + 1 = c$. And the difference of this sequence $28 + 15 + 6 + 1$ to this other $21 + 10 + 3 + 0$, which is $7 + 5 + 3 + 1 = d$. And the difference of this sequence $7 + 5 + 3 + 1$ to this other $6 + 4 + 2 + 0$, which is $1 + 1 + 1 + 1 = e$; & the difference of this sequence $1 + 1 + 1 + 1$ to this other $1 + 1 + 1 + 0 = 1$, one will have the sought sum

$$\begin{aligned}
&= \frac{1}{2} \times 462 + \frac{1}{2} b \\
&= \frac{1}{2} \times 462 + \frac{1}{4} \times 210 + \frac{1}{4} e \\
&= \frac{1}{2} \times 462 + \frac{1}{4} \times 210 + \frac{1}{8} \times 84 + \frac{1}{8} d \\
&= \frac{1}{2} \times 462 + \frac{1}{4} \times 210 + \frac{1}{8} \times 84 + \frac{1}{16} \times 28 + \frac{1}{16} e \\
&= \frac{1}{2} \times 462 + \frac{1}{4} \times 210 + \frac{1}{8} \times 84 + \frac{1}{16} \times 28 + \frac{1}{32} \times 7 + \frac{1}{32} \times 1,
\end{aligned}$$

conforming to the formula given below

$$\frac{1}{2}\times\frac{g+1.g.g-1.g-2.g-3}{1.2.3.4.5}+\frac{1}{4}\times\frac{g.g-1.g-2.g-3}{1.2.3.4}+\frac{1}{8}\times\frac{g-1.g-2.g-3}{1.2.3}+\frac{1}{16}\times\frac{g-2.g-3}{1.2}+\frac{1}{32}\times g-3+\frac{1}{32}\times 1$$

It is à propos to observe that all the numbers of these sequences, which are divided by different powers of 2, are found in the transverse band, of which the row number is $g-m-2$, & that the first which is divided by the smallest power of 2, is found always in the perpendicular band which corresponds to $g+1$, that is to say, of which the row number is $g+2$.

THE REST OF THE SOLUTION

74. It is clear by all that which precedes, that in order to have some formulas which express the advantage of the banker in Pharaon for all the different values of q. It suffices to put everywhere $p-2$ in the place of g in the formulas of *art. 73*, to multiply them by as many natural numbers $1, 2, 3, 4, 5, 6,$ &c. as there are units in q, & again by $\frac{1}{2}A$, & to divisor them by $p.p-1.p-2.p-3,$ &c.

If one wishes to see, for example, the formula for the case of $q=3$, in putting $p-2$ in the place of g in the formula $\frac{1}{2}\times\frac{g.g+1}{1.2}+\frac{1}{4}g$, one will find for the sought formula

$$\frac{\frac{1}{2}\frac{p-2.p-1}{1.2}+\frac{1}{4}p-2}{p.p-1.p-2}\times 1.2.3.\times\frac{1}{2}A=\frac{3\times A}{4\times\overline{\overline{p-1}}},$$

& likewise if one wants to have the formula for the case of $q=4$, in putting $p-2$ in the place of g in the formula

$$\frac{1}{2}\times\frac{g+1.g.g-1}{1.2.3}+\frac{1}{4}\times\frac{g.g-1}{1.2}+\frac{1}{8}\times g-1+\frac{1}{8}\times 1,$$

one will have the sought formula

$$\frac{\frac{1}{2}\times\frac{p-1.p-2.p-3}{1.2.3}+\frac{1}{4}\times\frac{p-2.p-3}{1.2}+\frac{1}{8}\times p-3+\frac{1}{8}}{p.p-1.p-2.p-3}\times 1.2.3.4\times\frac{1}{2}A=\frac{2p-5\times A}{2\times\overline{p.p-4.p-3}}.$$

Here are the principal cases that I have put in formula

$$B \overset{1}{=} \frac{1}{p}$$
$$C \overset{2}{=} \frac{p+2}{2 \times \overline{pp-p}}$$
$$D \overset{3}{=} \frac{3}{4 \times \overline{\overline{p-1}}}$$
$$E \overset{4}{=} \frac{2p-5}{2 \times pp-4p+3}$$
$$F \overset{5}{=} \frac{5}{4} \times \frac{p-2}{pp-4p+3}$$
$$G \overset{6}{=} \frac{3}{4} \times \frac{2pp-13p+16}{p^3-9pp+23p-15}$$
$$H \overset{7}{=} \frac{7}{8} \times \frac{2pp-12p+13}{p^3-9pp+23p-15}$$
$$K \overset{8}{=} \frac{1}{2} \times \frac{4p^3-50pp+176p-151}{p^4-16p^3+86pp-176p+105}.$$

The first of these formulas expresses the advantage of the banker, when the card of the punter is found one time in the hand. The 2nd C expresses his advantage when it is found twice in it. The 3rd D expresses his advantage when it is found there three times, & thus the others.

If one wishes to have a sequence which gives the advantage of the banker, the values of p & of q being arbitrary, one will have in multiplying the first term of the sequences that I have given in *art. 73*, by as many products of natural numbers as there are units in q; or, this which is the same thing, by $q.q-1.q-2.q-3.\cdots q-q$, & again by $\frac{1}{2}A$; & dividing by as many products of the quantities $p.p-1.p-2.p-3.p-4$, &c. as there are units in q, one will have, I say, in reducing term by term, & canceling that which is found in common in the numerator & in the denominator, this general & very simple formula,

$$\frac{1}{4} \times \frac{q}{p} + \frac{1}{8} \times \frac{q.q-1}{p.p-1} + \frac{1}{16} \times \frac{q.q-1.q-2}{p.p-1.p-2} + \frac{1}{32} \times \frac{q.q-1.q-2.q-3}{p.p-1.p-2.p-3}$$
$$+ \frac{1}{64} \times \frac{q.q-1.q-2.q-3.q-4}{p.p-1.p-2.p-3.p-4} + \text{&c.}$$

in which it is necessary to note that when q is an odd number, it is necessary to take as many terms as $q-1$ expresses in units; but when q being an even number, it is necessary to take as many terms as there are units in q, & to multiply the last term by 2.

REMARK

75. One is able to observe that in the method that we have followed, we have always considered the terms taken two by two of which we seek the sum, as knowing the greater of two unequal parts which compose the total; & consequently it was always necessary to add to that the half of the difference. Now if one would wish to consider the sought sum as making the smaller part of the uninterrupted sequence, if would be necessary to consider this entire sequence augmented by terms which correspond to $g+1$, & thus in sequence of rank to rank; then one would have always the sum that one seeks equal to the half of the total, less the half of the difference; & this total would be found in the inferior horizontal rank, to that which corresponds to $p+2$; in continuing always in this manner instead of the terms of the transverse band of which the row number is $g-m+2$, one would have the terms of the perpendicular band $g+3$, with the signs plus & minus alternately. The remark reveals the base of the difference that one finds between my formula above, & that which follows, of which M. Nicolas Bernoulli[2] has made me part in the Letter of 26 February 1711, that one will find at the end of this book.

$$\frac{1}{4} \times \frac{q}{p-q+1} - \frac{1}{8} \times \frac{q.q-1}{p-q+1.p-q+2} + \frac{1}{16} \times \frac{q.q-1.q-2}{p-q+1.p-q+2.p-q+3} - \frac{1}{32} \times \frac{q.q-1.q-2.q-3}{p-q+1.p-q+2.p-q+3.p-q+4} + \&c.$$

This last seems preferable, in this that one employs only as many terms as there are units in $q-1$, instead as in mine, when q is an even number, it is necessary to take as many terms in the sequence as there are units in q; & in this case to multiply the last term by 2: this exception subtracts in some way from the uniformity of the formula. But this advantage is able to be compensated by the alternate signs & the q which is found in the denominator of M. Bernoulli, &

[2] See the remark by Nicolas Bernoulli.

principally because one operates there on greater numbers. In order to make the comparison, let be proposed to find the sum of those numbers, for example, which are of the fourth order, $120 + 56 + 20 + 4 = 200$. One has by my formula $\frac{1}{2} \times 330 + \frac{1}{4} \times 120 + \frac{1}{8} \times 36 + \frac{1}{16} \times 8$. And according to that of M. Bernoulli, $\frac{1}{2} \times 496 - \frac{1}{4} \times 220 + \frac{1}{8} \times 66 - \frac{1}{16} \times 12$.

REMARK II.

76. I have prepared two tables on the first four formulas, *art. 74*, in the intent to please the players, & to satisfy their curiosity. In order to understand the usage, it is necessary to know that in the first the number contained within the cell ☐ expresses the number of cards that the banker holds; & that the number which follows, either the cell in the first column, or two points in the other columns, expresses the number of times that the card of the punter is supposed to be found in the hand of the banker. The usage of the second table is to give some expressions less exact to the truth, but more simple & more intelligible to the players, of the fractions which in the first designate with precision the advantage of the banker. It is necessary to know in order to understand this table, that this mark $>$ signifies excess, & this other $<$ defect; in such a way that I intend by $> \frac{1}{4} < \frac{1}{3}$ a quantity greater than $\frac{1}{4}$, & smaller than $\frac{1}{3}$.

One is able to make, by relationship to the numbers of the first table, many rather curious observations. Here are the most important.

COROLLARY I.

77. In the first table the advantage of the banker is expressed in the first column by a fraction, of which the numerator is always unity, the denominator is the number of cards which are held by the banker.

In the second column this advantage is expressed by a fraction, of which the numerator is according to the sequence of natural numbers $1, 2, 3, 4,$ &c. the denominator has for difference among these terms the numbers 18, $26, 34, 42$, 50, 58, of which the difference is 8.

In the third column the numerator is always 3, the difference which rules in the denominator is 8.

In the fourth column the difference is always 4 in the numerator, the denominator has for difference among its terms the numbers 24, 40, 56, 72, 88, &c. of which the difference is 16.

One is able again to observe another uniformity singular enough among the last digits of the denominator of each term of a column.

In the first column the last digits of the denominator are according to this order $4,\ 6,\ 8,\ 0,\ 2\,|4,\ 6,\ 8,\ 0,\ 2\,|$ &c. In the second they are according to this order $2,\ 0,\ 6,\ 0,\ 2|2,\ 0,\ 6,\ 0,\ 2|$ &c. In the third they are according to this order $2,\ 0,\ 8,\ 6,\ 4|2,\ 0,\ 8,\ 6,\ 4|$ &c. In the fourth they are according to this order $6,\ 0,\ 0,\ 6,\ 8|6,\ 0,\ 0,\ 6,\ 8|$ &c. One will seek with pleasure the cause of this uniformity.

COROLLARY II

78. One may by means of these tables find all in one stroke how much a banker has in advantage on each card. One may likewise know how much each complete deal will have, in equal risk, to bring profit to the banker, if one remembers the number of cards which have been taken by the punters, of the diverse circumstances in which one has wagered on them in the game, & finally of the amount of money that one has ventured upon. One will find apparently that this advantage is very considerable. One would give to him fair limits in establishing that the doublets were indifferent for the banker & for the punter, or at least that they are worth solely the third or the fourth of the wager of the punter. Thus that which would remain advantage to the banker, would be sufficient for making preference to the players who understand their interest, the place of the banker to that of the punter, & it would not be considerable enough, in order that the punters would suffer by them much prejudice.

COROLLARY III.

79. So that the punter taking a card have the least disadvantage which is possible, it is necessary that he choose one which has passed twice; because there would be greater disadvantage for him, if he would take one card which has passed once; & greater disadvantage again, if he would take a card which has passed three times; & finally the worst choice that a punter is able to make, this is to take a card which has not passed yet.

Thus one will find, for example, that supposing A is equal to one pistole[3], the advantage of the banker which would be nineteen sols two deniers, in the supposition that the card of the punter was four times in twelve cards; & sixteen sols eight deniers, in the supposition that it was once there, is no more than

[3] 1 pistole = 200 sols, 1 sol = 12 denier.

thirteen sols seven deniers, when in these twelve cards that of the punter is found there three times, & ten sols seven deniers when it is there only twice.

One will note the same thing with respect to all other number of cards.

REMARK I.

80. The persons who have not examined at foundation the game of Pharaon & of Bassette, could find to criticize, that I have not spoken of the masses[4], of the parolis[5], of the paix, of the sept & the *va*[6], &c. because the majority of the players imagine that there is in everything that much mystery. I have known of them who believed to have good reasons to prefer to wager four Louis on a simple card to make the paroli of two Louis, or the sept & the *va* of one Louis. I have seen others of them who were persuaded that there would be much advantage to make frequently of the paix: nevertheless it is evident that, since the punter has the liberty to take a new card at each time that he loses or that he wins such as it is pleasing to him, he must not embarrass himself if this is either a sept & the *va*, or a paroli, or a paix, or a double paix, &c. Because to make the paroli of a Louis is nothing other than to wager two Louis on one card, after having won a Louis; & to make the sept & the *va* of a Louis is nothing other than to wager four Louis on one card, after having won three of them; & similarly to make the paix of one Louis is nothing other than to wager a Louis on one card, after having one Louis on that same card.

One has apparently invented the parolis, the sept & the *va*, &c. only in order to spare the banker the difficulty of paying those who have intent to wager on their cards the double of that which they have just won: nonetheless it would be more useful to the bankers to take this care, than to be exposed, as they make him, to that which one names Alpiou de Campagne[7].

For me I believe that the bankers have not abolished the usage making these points, of which the great number cause in the game a confusion which is often prejudicial to the banker, & and which favors the misdirections of the punters, it

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[4] The masse is a certain sum of money that one wagers in a game in order to lose or to win so much according to the established rules.

[5] A paroli is the double of that which one has gambled the first time.

[6] In a game of cards one calls a sept the card which is marked with seven pips. One says in the game of Bassette, of Pharaon, etc. Sept et la va, quinze et le va etc. in order to say, seven times, fifteen times the vade. This vade is the sum which is controlled by the players of which the first to go in the game is obligated to leave.

[7] This seems to be the act of knavery in which a player claims to score without his card having come into gain.

is that the bankers have well seen that the majority of men do not judge these things by reason, such a punter who would make without difficulty the sept & the *va* of one Louis, believing to risk only one Louis, could not resolve to wager four Louis on one simple card. Besides for the usual it is in the last cards, when the advantage of the banker is most considerable, that the punters are stung & make the parolis, the sept & the *va*, &c. that which shall compensate them with usage of the misdirections to which they are that way exposed, but of which they it is not moreover impossible to guarantee themselves with much application, & with the aid of a croupier.

REMARK II.

81. It was easy for the players themselves to understand that the advantage of the banker increases in proportion as the number of his cards diminish; but it was impossible to discover without analysis the law of this diminution, & this which is most important, to know how this advantage varies according as the card of the punter is found more or less times in the hand of the banker. The players have assuredly never been able to imagine that the advantage of the banker, in relation to one card which has not passed, is nearly double of that which he has on one card which has passed twice, & much less again than his advantage, in relation to one card which has passed three times, is to his advantage in relation to one card which has passed two times in a ratio greater than three to two. The players will find all this without difficulty, & possibly with some surprise in the tables here joined, they will see there, for example, that the advantage of the banker which would only be about twenty-four sols if the punter would wager six pistoles either to the first deal of the game, or on one card which would have passed twice when there would remain of them no more than twenty-eight in the hand of the banker (these two cases revert to very nearly the same thing) will be seven livres two sols, if the punter puts six pistoles on one card which has not yet passed, the stock being composed of no more than six cards, & that his advantage would be precisely six livres, if the card of the punter was in this last case passed three times. Thus all the knowledge of the game is reduced for the punter to observing the two rules which follow.

1. Take some cards only in the first deal, & venture on the game accordingly less as there are a greater number of deals passed.

2. Regard as the greatest evils those cards which have not passed at all yet, or which have passed three times, & prefer to all, those which have passed twice.

In following these two rules, the disadvantage of the punter will be the least that will be possible.

52	$1 = * \quad * \quad *$	$: 2 = * \quad * \quad *$	$: 3 = * \quad * \quad *$	$: 4 = a + \frac{2295086253}{115890841950}a$
50	$1 = * \quad * \quad *$	$: 2 = a + \frac{3117}{350350}a$	$: 3 = a + \frac{3}{196}a$	$: 4 = a + \frac{8208829}{349595300}a$
48	$1 = a + \frac{1}{48}a$	$: 2 = a + \frac{1787}{161304}a$	$: 3 = a + \frac{3}{188}a$	$: 4 = a + \frac{276199}{12842280}a$
46	$1 = a + \frac{1}{46}a$	$: 2 = a + \frac{3431}{296010}a$	$: 3 = a + \frac{3}{180}a$	$: 4 = a + \frac{11002}{489555}a$
44	$1 = a + \frac{1}{44}a$	$: 2 = a + \frac{822}{67639}a$	$: 3 = a + \frac{3}{172}a$	$: 4 = a + \frac{913}{38786}a$
42	$1 = a + \frac{1}{42}a$	$: 2 = a + \frac{3145}{246246}a$	$: 3 = a + \frac{3}{164}a$	$: 4 = a + \frac{79}{3198}a$
40	$1 = a + \frac{1}{40}a$	$: 2 = a + \frac{1501}{111540}a$	$: 3 = a + \frac{3}{156}a$	$: 4 = a + \frac{25}{962}a$
38	$1 = a + \frac{1}{38}a$	$: 2 = a + \frac{2849}{201068}a$	$: 3 = a + \frac{3}{148}a$	$: 4 = a + \frac{1349}{49210}a$
36	$1 = a + \frac{1}{36}a$	$: 2 = a + \frac{679}{45045}a$	$: 3 = a + \frac{3}{140}a$	$: 4 = a + \frac{1139}{39270}a$
34	$1 = a + \frac{1}{34}a$	$: 2 = a + \frac{2573}{160446}a$	$: 3 = a + \frac{3}{132}a$	$: 4 = a + \frac{357}{11594}a$
32	$1 = a + \frac{1}{32}a$	$: 2 = a + \frac{1215}{70928}a$	$: 3 = a + \frac{3}{124}a$	$: 4 = a + \frac{177}{5394}a$
30	$1 = a + \frac{1}{30}a$	$: 2 = a + \frac{2287}{124410}a$	$: 3 = a + \frac{3}{116}a$	$: 4 = a + \frac{55}{1566}a$
28	$1 = a + \frac{1}{28}a$	$: 2 = a + \frac{536}{27027}a$	$: 3 = a + \frac{3}{108}a$	$: 4 = a + \frac{221}{5850}a$
26	$1 = a + \frac{1}{26}a$	$: 2 = a + \frac{2001}{92950}a$	$: 3 = a + \frac{3}{100}a$	$: 4 = a + \frac{611}{14950}a$
24	$1 = a + \frac{1}{24}a$	$: 2 = a + \frac{929}{39468}a$	$: 3 = a + \frac{3}{92}a$	$: 4 = a + \frac{473}{10626}a$
22	$1 = a + \frac{1}{22}a$	$: 2 = a + \frac{1715}{66066}a$	$: 3 = a + \frac{3}{84}a$	$: 4 = a + \frac{143}{2926}a$
20	$1 = a + \frac{1}{20}a$	$: 2 = a + \frac{1572}{54340}a$	$: 3 = a + \frac{3}{76}a$	$: 4 = a + \frac{35}{646}a$
18	$1 = a + \frac{1}{18}a$	$: 2 = a + \frac{1429}{43758}a$	$: 3 = a + \frac{3}{68}a$	$: 4 = a + \frac{31}{510}a$
16	$1 = a + \frac{1}{16}a$	$: 2 = a + \frac{429}{11440}a$	$: 3 = a + \frac{3}{60}a$	$: 4 = a + \frac{9}{130}a$
14	$1 = a + \frac{1}{14}a$	$: 2 = a + \frac{44}{1001}a$	$: 3 = a + \frac{3}{52}a$	$: 4 = a + \frac{23}{286}a$
12	$1 = a + \frac{1}{12}a$	$: 2 = a + \frac{7}{132}a$	$: 3 = a + \frac{3}{44}a$	$: 4 = a + \frac{19}{198}a$
10	$1 = a + \frac{1}{10}a$	$: 2 = a + \frac{1}{15}a$	$: 3 = a + \frac{3}{36}a$	$: 4 = a + \frac{5}{42}a$
8	$1 = a + \frac{1}{8}a$	$: 2 = a + \frac{5}{56}a$	$: 3 = a + \frac{3}{28}a$	$: 4 = a + \frac{11}{70}a$
6	$1 = a + \frac{1}{6}a$	$: 2 = a + \frac{2}{15}a$	$: 3 = a + \frac{3}{20}a$	$: 4 = a + \frac{7}{30}a$
4	$1 = a + \frac{1}{4}a$	$: 2 = a + \frac{1}{4}a$	$: 3 = a + \frac{3}{12}a$	$: 4 = a + \frac{1}{2}a$

52	$1 = * \quad * \quad *$	$: 2 = * \quad * \quad *$	$: 3 = * \quad * \quad *$	$: 4 = a + \; > \frac{1}{51} < \frac{1}{50}$
50	$1 = * \quad * \quad *$	$: 2 = a + \; > \frac{1}{95} < \frac{1}{94}$	$: 3 = a + \; > \frac{1}{66} < \frac{1}{65}$	$: 4 = a + \; > \frac{1}{49} < \frac{1}{48}$
48	$1 = a + \frac{1}{48}a$	$: 2 = a + \; > \frac{1}{91} < \frac{1}{90}$	$: 3 = a + \; > \frac{1}{63} < \frac{1}{62}$	$: 4 = a + \; > \frac{1}{47} < \frac{1}{46}$
46	$1 = a + \frac{1}{46}a$	$: 2 = a + \; > \frac{1}{87} < \frac{1}{86}$	$: 3 = a + \frac{1}{60}$	$: 4 = a + \; > \frac{1}{45} < \frac{1}{44}$
44	$1 = a + \frac{1}{44}a$	$: 2 = a + \; > \frac{1}{83} < \frac{1}{82}$	$: 3 = a + \; > \frac{1}{58} < \frac{1}{57}$	$: 4 = a + \; > \frac{1}{43} < \frac{1}{42}$
42	$1 = a + \frac{1}{42}a$	$: 2 = a + \; > \frac{1}{79} < \frac{1}{78}$	$: 3 = a + \; > \frac{1}{55} < \frac{1}{54}$	$: 4 = a + \; > \frac{1}{41} < \frac{1}{40}$
40	$1 = a + \frac{1}{40}a$	$: 2 = a + \; > \frac{1}{75} < \frac{1}{74}$	$: 3 = a + \frac{1}{52}$	$: 4 = a + \; > \frac{1}{39} < \frac{1}{38}$
38	$1 = a + \frac{1}{38}a$	$: 2 = a + \; > \frac{1}{71} < \frac{1}{70}$	$: 3 = a + \; > \frac{1}{50} < \frac{1}{49}$	$: 4 = a + \; > \frac{1}{37} < \frac{1}{36}$
36	$1 = a + \frac{1}{36}a$	$: 2 = a + \; > \frac{1}{67} < \frac{1}{66}$	$: 3 = a + \; > \frac{1}{47} < \frac{1}{46}$	$: 4 = a + \; > \frac{1}{35} < \frac{1}{34}$
34	$1 = a + \frac{1}{34}a$	$: 2 = a + \; > \frac{1}{63} < \frac{1}{62}$	$: 3 = a + \frac{1}{44}$	$: 4 = a + \; > \frac{1}{33} < \frac{1}{32}$
32	$1 = a + \frac{1}{32}a$	$: 2 = a + \; > \frac{1}{59} < \frac{1}{58}$	$: 3 = a + \; > \frac{1}{42} < \frac{1}{41}$	$: 4 = a + \; > \frac{1}{31} < \frac{1}{30}$
30	$1 = a + \frac{1}{30}a$	$: 2 = a + \; > \frac{1}{55} < \frac{1}{54}$	$: 3 = a + \; > \frac{1}{39} < \frac{1}{48}$	$: 4 = a + \; > \frac{1}{29} < \frac{1}{28}$
28	$1 = a + \frac{1}{28}a$	$: 2 = a + \; > \frac{1}{51} < \frac{1}{50}$	$: 3 = a + \frac{1}{36}$	$: 4 = a + \; > \frac{1}{27} < \frac{1}{26}$
26	$1 = a + \frac{1}{26}a$	$: 2 = a + \; > \frac{1}{47} < \frac{1}{46}$	$: 3 = a + \; > \frac{1}{34} < \frac{1}{33}$	$: 4 = a + \; > \frac{1}{25} < \frac{1}{24}$
24	$1 = a + \frac{1}{24}a$	$: 2 = a + \; > \frac{1}{43} < \frac{1}{42}$	$: 3 = a + \; > \frac{1}{31} < \frac{1}{30}$	$: 4 = a + \; > \frac{1}{23} < \frac{1}{22}$
22	$1 = a + \frac{1}{22}a$	$: 2 = a + \; > \frac{1}{39} < \frac{1}{38}$	$: 3 = a + \frac{1}{28}$	$: 4 = a + \; > \frac{1}{21} < \frac{1}{20}$
20	$1 = a + \frac{1}{20}a$	$: 2 = a + \; > \frac{1}{35} < \frac{1}{34}$	$: 3 = a + \; > \frac{1}{26} < \frac{1}{25}$	$: 4 = a + \; > \frac{1}{19} < \frac{1}{18}$
18	$1 = a + \frac{1}{18}a$	$: 2 = a + \; > \frac{1}{31} < \frac{1}{30}$	$: 3 = a + \; > \frac{1}{23} < \frac{1}{22}$	$: 4 = a + \; > \frac{1}{17} < \frac{1}{16}$
16	$1 = a + \frac{1}{16}a$	$: 2 = a + \; > \frac{1}{27} < \frac{1}{26}$	$: 3 = a + \frac{1}{20}$	$: 4 = a + \; > \frac{1}{15} < \frac{1}{14}$
14	$1 = a + \frac{1}{14}a$	$: 2 = a + \; > \frac{1}{23} < \frac{1}{22}$	$: 3 = a + \; > \frac{1}{18} < \frac{1}{17}$	$: 4 = a + \; > \frac{1}{13} < \frac{1}{12}$
12	$1 = a + \frac{1}{12}a$	$: 2 = a + \; > \frac{1}{19} < \frac{1}{18}$	$: 3 = a + \; > \frac{1}{15} < \frac{1}{14}$	$: 4 = a + \; > \frac{1}{11} < \frac{1}{10}$
10	$1 = a + \frac{1}{10}a$	$: 2 = a + \frac{1}{15}$	$: 3 = a + \frac{1}{12}$	$: 4 = a + \; > \frac{1}{9} < \frac{1}{8}$
8	$1 = a + \frac{1}{8}a$	$: 2 = a + \; > \frac{1}{12} < \frac{1}{11}$	$: 3 = a + \; > \frac{1}{10} < \frac{1}{9}$	$: 4 = a + \; > \frac{1}{7} < \frac{1}{6}$
6	$1 = a + \frac{1}{6}a$	$: 2 = a + \; > \frac{1}{8} < \frac{1}{7}$	$: 3 = a + \; > \frac{1}{7} < \frac{1}{6}$	$: 4 = a + \; > \frac{1}{5} < \frac{1}{4}$
4	$1 = a + \frac{1}{4}a$	$: 2 = a + \frac{1}{4}$	$: 3 = a + \frac{1}{4}a$	$: 4 = a + \frac{1}{2}a$

Extract of a letter from M. (Jean) Bernoulli to M. de Montmort
From Basel this 17 March 1710 (pg. 284–287)
Page 8 (1st edition) on the game of Pharaon. In order to seek the strength of the banker who holds four cards within his hands, among which the card of the punter is once, you make a denumeration of all 24 permutations of four cards, in order to take of them the favorables to the banker, without making reflection that this is not appropriately the various permutations, but only the various situations of the card of the punter, among the others which make the diversity of the cases; thus instead of your 24 permutations, I have only these four variations to consider (I name a the card of the punter, & b each of the others)

$$\begin{array}{ll} 1.\, bbba & 3.\, babb \\ 2.\, bbab & 4.\, abbb \end{array}$$

Of these four variations the first is indifferent to the banker, the 2nd and 4th make him win, & the 3rd makes him lose, his strength will be therefore

$$= \frac{1 \times A + 2 \times 2A + 1 \times 0}{4} = \frac{5}{4}A = A + \frac{1}{4}A,$$

entirely as you have found. If the card of the punter is found twice among the cards of the banker, he will have these six variations, instead of 24 permutations,

$$\begin{array}{lll} 1.\, bbaa & 3.\, abba & 5.\, abab \\ 2.\, baba & 4.\, baab & 6.\, aabb \end{array}$$

The first, the third & the fifth make the banker win, the second & the fourth make him lose, & the sixth gives to him the half of the wager of the punter, & therefore the strength of the punter will be

$$= \frac{3 \times 2A + 2 \times 0 + 1 \times \frac{3}{2}A}{6} = \frac{5}{4}A = A + \frac{1}{4}A,$$

again as you. If the card of the punter is found three times among the cards of the banker, one sees clearly that he must have as many variations, as when the card of the punter is there only once; because he has only to make a permutation of the letters a to b, & b to a.

$$\begin{array}{ll} 1.\, baaa & 3.\, aaba \\ 2.\, abaa & 4.\, aaab \end{array}$$

Whence one draws anew the strength of the banker $= A + \frac{1}{4}A$. By this manner of distributing the cases, one sees without difficulty that whatever number of cards which are held by the banker expressed by p, if those of the punter are found there only once, the advantage of the banker will be $\frac{1}{p}A$. It is hardly otherwise if the card of the punter is found more than once among the cards of the banker; because instead of all the permutations which it would be necessary to examine, & of which the number is immense for a mediocre number of cards, here it is only necessary to consider the number of variations of the two letters a & b, which is always equal to the number of combinations that some things of which the number is the one of the card of the punter, are able to be taken differently in the number of all the cards; & then of those combinations of which the number is always much smaller than the one of all the permutations, if will be easy to choose those which make the banker win either entirely or in part, & thus to determine his strength: for example: We give to the banker six cards, among which we suppose that the card of the punter is twice. In this supposition I have only to examine as you make all the 720 permutations that the six cards are able to undergo, restricting myself to examine simply those fifteen possible variations that the letter a taken twice is able to make with the letter b taken four times.

$$\begin{array}{lllll} 1.\, bbbbaa & 4.\, babbba & 7.\, bbabab & 10.\, bbaabb & 13.\, baabbb \\ 2.\, bbbaba & 5.\, abbbba & 8.\, babbab & 11.\, bababb & 14.\, ababbb \\ 3.\, bbabba & 6.\, bbbaab & 9.\, abbbab & 12.\, abbabb & 15.\, abbbbb \end{array}$$

Among these fifteen variations one counts seven which give all to the banker, two which give to him his wager with the half of the wager of the punter, & the six others which make him lose; in such a way that the strength of the banker will be

$$= \frac{7 \times 2A + 2 \times \frac{3}{2}A + 6 \times 0}{15} = \frac{17}{15}A = A + \frac{2}{15}A,$$

conforming to that which you have found. Thus similarly if the card of the punter is three times among the six cards of the banker, there will be only 20 ways to vary the situation of two letters a & b taken each three times, which being untangled makes plain that the strength of the banker will be $= A + \frac{3}{20}A$. Following this principle, here are the general formulas that I have found for whatever number of cards that there are among the hands of the banker, & whatever number of times that the card of the punter is found there, without supposing known the strength of the banker in a number of cards expressed by

$p-2$, as you made in your general formula, that I have also found very easily; here is, I say, mine. Let $1.2.3.4 \ldots p-q=m$, $q+1.q+2.q+3 \ldots p=n$, $p-q+1.p-q+2.p-q+3 \ldots p=l$, I say that the advantage of the banker, if q is an even number, will be expressed by this sequence

$$\frac{m}{2n} \times 1+\frac{q-1.q}{1.2}+\frac{q-1.q.q+1.q+2}{1.2.3.4} + \frac{q-1.q.q+1 \ldots q+4}{1.2.3.4.5.6}+\cdots \frac{q-1.q.q+1 \ldots p-2}{1.2.3.4 \ldots p-q}.$$

Or else by this:

$$\frac{q-1.q}{2l} \times \overline{1.2.3 \ldots q-2}+\overline{3.4.5 \ldots q} + \overline{5.6.7 \ldots q+2}+\cdots \overline{p-q+1.p-q+2 \ldots p-2}.$$

But if q is an odd number, one will have the same advantage

$$\frac{\overline{q-1}.m}{2n} \times 1+\frac{q.q-1}{1.2.3}+\frac{q.q+1.q+2.q+3}{2.3.4.5}+\cdots \frac{q.q+1.q+2 \ldots p-2}{2.3.4 \ldots p-q},$$

or else

$$\frac{q+1.q}{2l} \times \overline{2.3.4 \ldots q-1}+\overline{4.5.6 \ldots q+1} + \overline{6.7.8 \ldots q+3}+\overline{p-q+1.p-q+2 \ldots p-2};$$

whence it is necessary to note that the numbers p & q are able to be anything, provided that q not be less great than 3. If you wish to take the difficulty, you may examine these general formulas if they accord with yours that you give for the particular cases, pg.. 24 & 25.

Extract from the remarks of M. (Nicolas) Bernoulli. (pg. 299)

Pg.. 23, 24, 25. In general, if $q>2$, the prerogative is the comprehensive expression

$$= \frac{1}{4} \times \frac{q}{p-q+1} - \frac{1}{8} \times \frac{q.q-1}{p-q+1.p-q+2}$$
$$+ \frac{1}{16} \times \frac{q.q-1.q-2}{p-q+1.p-q+2.p-q+3} - \dots$$
$$\text{all the way to } \pm \frac{1}{2^q} \times \frac{q.q-1.q-2\dots 2}{p-q+1.p-q+2\dots p-1}$$

in A.

Extract from a letter from M. de Montmort to M. Bernoulli
At Montmort 15 November 1710. (pg. 303–304)

Those which you give, Sir, in order to express the advantage of the banker

$$\frac{m}{2n} \times 1 + \frac{q-1.q}{1.2} + \frac{q-1.q.q+1.q+2}{1.2.3.4}$$
$$+ \frac{q-1.q.q+1\dots q+4}{1.2.3.4.5.6} + \dots \frac{q-1.q.q+1\dots p-2}{1.2.3.4\dots p-q},$$

if q is an even number. And

$$\frac{\overline{q-1}.m}{2n} \times 1 + \frac{q.q-1}{2.3} + \frac{q.q+1.q+2.q+3}{2.3.4.5} + \dots \frac{q.q+1.q+2\dots p-2}{2.3.4\dots p-q}$$

are very fine. I have one of them a long time since little different from yours, let it be that q is even or odd. Here it is:

$$\frac{1}{p \times p-1} + \frac{p-q.p-q-1}{p.p-1.p-2.p-3} + \frac{p-q.p-q-1.p-q-2.p-q-3}{p.p-1.p-2.p-3.p-4.p-5}$$
$$+ \frac{p-q.p-q-1.p-q-2.p-q-3.p-q-4.p-q-5}{p.p-1.p-2.p-3.p-4.p-5.p-6.p-7} + \text{ \&c.}$$

The whole multiplied by $q.q-1.\frac{1}{2}A$.