

DIVERSE PROBLEMS concerning the game OF TREIZE

Pierre Renard de Montmort

Extracted from the
Essay D'analyse sur les jeux de hazard

2nd Edition of 1713, pp. 130–143

EXPLICATION OF THE GAME.

98. The players draw first for who will have the hand.¹ We suppose that this is Pierre, & that the number of the players is as such as one would wish. Pierre having an entire deck composed of fifty-two cards shuffled at discretion, draws them one after the other, naming & pronouncing one when he draws the first card, two when he draws the second, three when he draws the third, & thus in sequence up to the thirteenth which is a King. Now if in all this sequence of cards he has drawn none of them according to the rank that he has named them, he pays that which each of the players has wagered in the game, & gives the hand to the one who follows him at the right.

But if it happens to him in the sequence of thirteen cards, to draw the card which he names, for example, to draw one ace at the time which he names one, or a two at the time which he names two, or a three at the time which he names three, &c. he takes all that which is in the game, & restarts as before, naming one, next two, &c.

It is able to happen that Pierre having won many times, & restarting with one, has not enough cards in his hand in order to go up to thirteen, now he must, when the deck falls short to him, to shuffle the cards, to give to cut, & next to

¹ The player who holds the cards is the dealer.

draw from the entire deck the number of cards which is necessary to him in order to continue the game, by commencing with the one where he is stopped in the preceding hand. For example, if drawing the last card from them he has named seven, he must in drawing the first card from the entire deck, after one has cut, to name eight, & next nine, &c. up to thirteen, unless he rather not win, in which case he would restart, naming first one, next two, & the rest as it happens in the explanation. Whence it seems that Pierre is able to make many hands in sequence, & likewise he is able to continue the game indefinitely.

PROBLEM
PROPOSITION V.

Pierre has a certain number of different cards which are not repeated at all, & which are shuffled at discretion: he bets against Paul that if he draws then in sequence, & if he names them according to the order of the cards, beginning of them either with the highest, or with the lowest, there will happen to him at least one time to draw the one that he will name. For example, Pierre having in his hand four cards, namely an ace, a deuce, a three & a four shuffled at discretion, bets that drawing them in sequence, & naming one when he will draw the first, two when he will draw the second, three when he will draw the third, there will happen to him either to draw one ace when he will name one, or to draw a deuce when he will name two, or to draw a three when he will name three, or to draw a four when he will name four. Let be imagined the same thing for all other number of cards. One asks what is the strength or the expectation of Pierre for whatever number of cards that this may be from two up to thirteen.

99. Let the cards with which Pierre makes the wager, be represented by the letters $a, b, c, d,$ &c. If one names m the number of cards which he holds, & n the number which expresses all the possible arrangements of these cards, the fraction $\frac{n}{m}$ will express how many different times each letter will occupy each of the positions. Now it is necessary to note that these letters are not encountered always in their place advantageously for the banker; for example, a, b, c only give a winning move in the one which has the hand, although each of these three letters be in its place there; And similarly b, a, c, d give only a winning move to Pierre, although each of the letters c & d be in its place there. The difficulty of this problem consists therefore in untangling how many times each letter is in its place advantageously for Pierre, & how many times it is useless to him.

FIRST CASE.

Pierre holds an ace & a deuce, & bets against Paul, that having shuffled these two cards, & naming one when he will draw the first, & two when he will name the second, there will happen to him either to draw an ace for the first card, or to draw a deuce for the second card. The money of the game is expressed by A .

100. Two cards are able to be arranged only in two different ways: the one makes Pierre win, the other makes him lose: therefore his strength will be $\frac{A+0}{2} = \frac{1}{2}A$.

SECOND CASE.

Pierre holds three cards.

101. Let there be three cards represented by the letters a, b, c : one will observe that of the six different arrangements that these three letters are able to admit, there are two of them where a is in the first place; that there is one of them where b is in the second place; a being not at all in the first, & one where c is in the third place, a not at all in the first, & b not at all in the second; whence it follows that one will have $S = \frac{2}{3}A$; & consequently that the strength of Pierre is to that of Paul, as two is to one.

THIRD CASE.

Pierre has four cards.

102. Let the four cards be represented by the letters a, b, c, d : one will observe that of the twenty-four different permutations that these four letters are able to admit, there are six of them where a occupies the first place; that there are four of them where b is in the second, a not being in the first; three where c is in the third, a not being in the first, & b not being in the second; finally two where d is in the fourth, a not being in the first, b not being in the second, & c not being in the third; whence it follows that one will have the strength of Pierre

$$= S = \frac{6 + 4 + 3 + 2}{24}A = \frac{15}{24}A = \frac{5}{8}A;$$

& consequently that the strength of Pierre is to the strength of Paul as five to three.

FOURTH CASE.

Pierre holds five cards.

103. Let the five cards be represented by the letters a, b, c, d, f : one will observe that of the 120 different permutations that five letters are able to admit, there are twenty-four where a occupies the first place, eighteen where b occupies the second, a not occupying the first; fourteen where c is in the third place, a not being in the first place, nor b in the second; eleven where d is in the fourth place, a not being in the first, nor b in the second, nor c in the third; finally nine permutations where f is in the fifth place, a not being in the first, nor b in the second, nor c in the third, nor d in the fourth; whence it follows that one will have the strength of Pierre

$$= S = \frac{24 + 18 + 14 + 11 + 9}{120}A = \frac{76}{120}A = \frac{19}{30}A;$$

& consequently that the strength of Pierre is to the strength of Paul as nineteen is to eleven.

GENERALLY

104.² If one names S the strength that one seeks, the number of cards that Pierre holds being expressed by p ; g the strength of Pierre, the number of cards being $p - 1$; d his strength, the number of cards that he holds being $p - 2$, one will have

$$S = \frac{g \times \overline{p - 1} + d}{p}.$$

This formula will give all the cases, so that one sees them resolved in the table

² See derivation by Nicolas Bernoulli in the Appendix.

adjoined here.

Table

If $p = 1$, one will have $S = A$.
If $p = 2$, one will have $S = \frac{1}{2}A$.
If $p = 3$, one will have $S = \frac{2}{3}A = \frac{1}{2}A + \frac{1}{6}A$.
If $p = 4$, one will have $S = \frac{5}{8}A = \frac{1}{2}A + \frac{1}{8}A$.
If $p = 5$, one will have $S = \frac{19}{30}A = \frac{1}{2}A + \frac{2}{15}A$.
If $p = 6$, one will have $S = \frac{91}{144}A = \frac{1}{2}A + \frac{19}{144}A$.
If $p = 7$, one will have $S = \frac{531}{840}A = \frac{1}{2}A + \frac{111}{840}A$.
If $p = 8$, one will have $S = \frac{3641}{5760}A = \frac{1}{2}A + \frac{761}{5760}A$.
If $p = 9$, one will have $S = \frac{28673}{45360}A = \frac{1}{2}A + \frac{5993}{45360}A$.
If $p = 10$, one will have $S = \frac{28319}{44800}A = \frac{1}{2}A + \frac{5919}{44800}A$.
If $p = 11$, one will have $S = \frac{2523223}{3991680}A = \frac{1}{2}A + \frac{527383}{3991580}A$.
If $p = 12$, one will have $S = \frac{302786759}{479001600}A = \frac{1}{2}A + \frac{63285959}{479001600}A$.
If $p = 13$, one will have $S = \frac{109339663}{172972800}A = \frac{1}{2}A + \frac{22853263}{172972800}A$.

This formula will give the same advantage to Pierre, if one would suppose that he had there a greater number of cards of different kind.

REMARK I.

105.³ The preceding solution furnishes a singular usage of the figurate numbers, because I find on examining the formula, that the strength of Pierre is expressed by an infinite sequence of terms which have alternately + & - , & such that the numerator is the sequence of numbers which compose in the Table, *art. 1*,⁴ the perpendicular column which corresponds to p , beginning at p , & the denominator the sequence of products $p \times p - 1 \times p - 2 \times p - 3 \times p - 4 \times p - 5$, &c. in such a way that these products which are found in the numerator & in the denominator destroying themselves, there remains for expression of the

³ See the extract of the letter from Jean Bernoulli 17 March 1710, the reply by Montmort 15 November 1710, and the discussion by Nicolas Bernoulli in the Appendix.

⁴ See Pascal's triangle in the Appendix.

strength of Pierre this very simple series

$$\frac{1}{1} - \frac{1}{1.2} + \frac{1}{1.2.3} - \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5} - \frac{1}{1.2.3.4.5.6} + \&c.$$

If one forms a logarithm⁵ of which the subtangent be unity, & if one takes two ordinates, of which the one is unity, & the other is extended to this first by a quantity equal to the subtangent, the excess of the constant ordinate on the last will be equal to this series.

In order to demonstrate it let the general formula of the subtangent be

$$s = \pm \frac{ydx}{dy},$$

the subtangent being named s , the abscissa x , the ordinate y . One will suppose y equal to a series of powers of x affected by indeterminate coefficients, for example,

$$= 1 + ax + bxx + cx^3 + dx^4 + \&c.$$

& taking on all sides the difference, dividing next by dx , & multiplying by s , one will find

$$\begin{aligned} \pm \frac{sdy}{dx} &= y = 1 + ax + bxx + cx^3 + dx^4 + \&c. \\ &= \pm as \pm 2bsx \pm 3csxx \pm 4dsx^3 + \&c. \end{aligned}$$

If one compares the homologous terms of these two series, & if one draws from this comparison the value of the coefficients a, b, c, d , one will have

$$y = 1 \pm \frac{x}{s} \pm \frac{1xx}{1.2ss} \pm \frac{1x^3}{1.2.3s^3} \pm \frac{1x^4}{1.2.3.4s^4} \pm \&c.$$

this which shows that if one determine, y , to be the ordinate of a logarithm of which the constant subtangent be $= 1$, one will have the ordinate which corresponds to x taken on the side that the ordinate decreases,

$$= 1 - \frac{x}{1} + \frac{xx}{1.2} - \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} - \&c.$$

one is able to see this demonstration in the Actes of Leipzig for the year 1693, p.

⁵ The usage is curious. It is the exponential function $y = e^x$ which has subtangent unity.

179, where the celebrated Mr. Leibnitz resolves this problem: *A logarithm being given, to find the number which corresponds to it.* Now it is clear that if in this series one supposes $x = 1$, that is to say equal to the subtangent or to the constant ordinate, & if one subtracts this series from unity, it will become the series of the present problem.

One is able again to demonstrate it more simply in this manner. Let be imagined a logarithm of which the subtangent is unity; one will take on this curve a constant ordinate $= 1$, & another smaller ordinate $= 1 - y$, one will name x the abscissa contained between the two ordinates, one will have $dx = \frac{dy}{1-y}$, and

$$x = y + \frac{1}{2}yy + \frac{1}{3}y^3 + \frac{1}{4}y^4 + \&c.$$

& by the method for the reversion of series,

$$y = x - \frac{xx}{1.2} + \frac{x^3}{1.2.3} - \frac{x^4}{1.2.3.4} + \frac{x^5}{1.2.3.4.5} - \&c.$$

this which, in supposing $x = 1$, becomes

$$= 1 - \frac{1}{1.2} + \frac{1}{1.2.3} - \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5} - \&c. \quad \text{Q.E.D.}$$

One is able to observe that the series

$$B \quad \frac{1}{1} - \frac{1}{1.2} + \frac{1}{1.2.3} - \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5} - \frac{1}{1.2.3.4.5.6} + \&c.$$

is equal to each of the three C, D, F ⁶ which follow, which under some very different forms do not give up having the same value; in such a way that all that which agrees to the series B agrees to them also.

$$C \quad \frac{1}{1.2} + \frac{4}{1.2.3} + \frac{9}{1.2.3.4} + \frac{16}{1.2.3.4.5} + \frac{25}{1.2.3.4.5.6} + \frac{36}{1.2.3.4.5.6.7} + \&c.$$

$$- 2 \times \frac{1}{2} + \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5.6} + \frac{1}{1.2.3.4.5.6.7.8}$$

$$+ \frac{1}{1.2.3.4.5.6.7.8.9.10} + \&c.$$

⁶ Series B, D, F all sum to $1 - \frac{1}{e}$. Montmort errors with series C for it sums to $1 + \frac{1}{e}$.

$$D \quad \frac{1}{2} + \frac{3}{1.2.3.4} + \frac{5}{1.2.3.4.5.6} + \frac{7}{1.2.3.4.5.6.7.8} + \frac{9}{1.2.3.4.5.6.7.8.9.10} + \&c.$$

$$F \quad \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5} + \frac{1}{1.2.3.4.5.6} + \frac{1}{1.2.3.4.5.6.7} + \&c.$$

$$- \frac{1}{3.4} - \frac{1}{3.4.5.6} - \frac{1}{3.4.5.6.7.8} - \frac{1}{3.4.5.6.7.8.9.10} - \frac{1}{3.4.5.6.7.8.9.10.11.12} - \&c.$$

One could make many curious remarks on the relation of these series; but that would digress us from our subject, & would lead us too far.

REMARK II

106. The two formulas of *art. 104 & 105* inform how much the one who holds the cards has to risk in order to win with any card that it be; but it does not at all distinguish how much he has to risk for each card that he draws from the first to the last. One sees well that this number of chances always diminishes, & that there are, for example, more chances to win with the ace than with the deuce, & with the three than with the four, &c. But one does not draw easily from that which precedes the law of this diminution, one will find it in this table.

1 = 1									
0	1 = 1								
1	1	2 = 4							
2	3	4	6 = 15						
9	11	14	18	24 = 76					
44	53	64	78	96	120 = 455				
265	309	362	426	504	600	720 = 3186			
1854	2119	2428	2790	3216	3720	4320	5040 = 25487		

This table shows that with five cards, for example, an ace, a deuce, a three, a four & a five, Pierre has twenty-four ways to win with an ace; eighteen to win with a deuce having not at all won with an ace; fourteen to win with the three, having won neither with the ace nor with the deuce; eleven to win with the four, having won neither with the ace, nor with the deuce, nor with the three, & finally that there are only nine ways to win with the five, having won neither with the ace, nor with the deuce, nor with the three, nor with the four.

Each rank of this table is formed on the preceding in a very easy manner. In order to make it understood, we suppose again that there were five cards. One

sees first that there are twenty-four ways to win with the ace. This is evident, since the ace was determined to be in the first place, the four other cards are able to be arranged in all the possible ways; & in general it is clear that the number of the cards being p , the number of chances in order to win with the ace is expressed by as many products of the natural numbers 1, 2, 3, 4, 5, &c. as there are units in $p - 1$. Thus put, $24 - 6 = 18$ gives me the chances in order to win with the deuce, $18 - 4 = 14$ gives me the chances in order to win with the three, $14 - 3 = 11$ gives me the chances in order to win with the four; & finally $11 - 2 = 9$ gives me the chances in order to win with the five.

It is the same for all other number of cards, & generally each number of the table is equal to the difference of that which is to its right & that one has already found, to the one which is immediately above.

One is able yet to find a steady order in the numbers 1, 1, 4, 15, 76, 455, &c. which expresses all the ways to win with whatever number of cards: this order is visible in the following table.

$$\begin{aligned} \overline{0 \times 1} + 1 &= 1 \\ \overline{1 \times 2} - 1 &= 1 \\ \overline{1 \times 3} + 1 &= 4 \\ \overline{4 \times 4} - 1 &= 15 \\ \overline{15 \times 5} + 1 &= 76 \\ \overline{76 \times 6} - 1 &= 455 \\ \overline{455 \times 7} + 1 &= 3186 \\ \overline{3186 \times 8} - 1 &= 25487 \end{aligned}$$

These numbers 1, 1, 4, 15, 76, &c. express how many chances there are in order that some one among the p cards is found ordered in its place; that is to say, for example, the 3 in the 3rd, or the 4 in the 4th, or the 5 in the 5th, &c.

COROLLARY I

107. Let p be the number of cards, q the number of chances that Pierre has in order to win when the number of cards is $p - 1$. The number of chances favorable to Pierre is expressed in this very simple formula $pg \pm 1$; namely $+$ when p is an odd number, & $-$ when it is even.

COROLLARY II.

108. The numbers 0, 1, 2, 9, 44, 265, &c. which comprise the first perpendicular band of the table which is in the page preceding, <Section 106> expresses the number of chances that there are of them in order that each card is not in its place.

PROPOSITION VI.
PROBLEM

Pierre holds a certain number p of cards of a suit, for example, all the color of diamond, in naming first ace, next deuce, next three up to King, Paul will give to him a pistole for each card that he will bring to its rank: One asks how many chances Pierre has in order to win either one, or two, or three or four, &c. pistoles.

SOLUTION.

109. The formula

$$\begin{aligned}
 &1 \times 1 + p \times 0 + \frac{p \cdot p - 1}{1 \cdot 2} \times \overline{0 + 1} + \frac{p \cdot p - 1 \cdot p - 2}{1 \cdot 2 \cdot 3} \times 0 \\
 &\quad - 1 + 3 + \frac{p \cdot p - 1 \cdot p - 2 \cdot p - 3}{1 \cdot 2 \cdot 3 \cdot 4} \times \overline{0 + 1 - 4 + 4 \cdot 3} \\
 &\quad + \frac{p \cdot p - 1 \cdot p - 2 \cdot p - 3 \cdot p - 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \times \overline{0 - 1 + 5 - 5 \cdot 4 + 5 \cdot 4 \cdot 3} \\
 &\quad + \frac{p \cdot p - 1 \cdot p - 2 \cdot p - 3 \cdot p - 4 \cdot p - 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \times \overline{0 + 1 - 6 + 6 \cdot 5 - 6 \cdot 5 \cdot 4 + 6 \cdot 5 \cdot 4 \cdot 3} + \&c.
 \end{aligned}$$

will give the number of sought chances.

The order of this series is easy to understand, & one is able to continue it indefinitely. The first term expresses how many chances there are in order that each card is found in its place. The sum of the first two expresses how many chances there are in order that there is found of them at least $p - 1$ in their rank; the sum of the first three express how many chances there are in order that there is found of them at least $p - 2$ in their rank.

In applying this formula to the case of thirteen cards, I find that on the 6227020800 different ways of which thirteen things are able to be arranged,

there are of them so that all are found in their places		1
So that there be twelve of them,		0
So that there be eleven of them,		78
So that there be ten of them,		572
So that there be nine of them,	precisely;	6435
So that there be eight of them,		56628
So that there be seven of them,		454740
So that there be six of them,		3181464
So that there be five of them,		19090071
So that there be four of them,	precisely	95449640
So that there be three of them,		381798846
So that there be two of them,		1145396460
So that there be one of them,		<u>2290792933</u>
So that there be one at least of them		3936227868

And consequently if Paul is obliged to give to Pierre a pistole for each card that he will bring to its place, one will have the advantage of Pierre by multiplying the first of these numbers by 13, the 2nd by 12, the 3rd by 11, &c.

DEMONSTRATION.

110. The law of these numbers 1, 0, 0 + 1, 0 - 1 + 3, 0 + 1 - 4 + 4.3, 0 - 1 + 5 - 5.4 + 5.4.3, &c. is drawn easily from the formula

$$B = \begin{matrix} \text{odd} \\ \text{even} \end{matrix} \pm 1 \mp p \pm p.p - 1 \mp p.p - 1.p - 2 \pm p.p - 1.p - 2.p - 3. \mp \&c.$$

because this series expresses the number of arrangements where any one of the cards is found ordered in its place, employing of it the upper signs when p is an odd number, and those lower when p is an even number. This other series

$$p.p - 1.p - 2.p - 3.p - 4, \&c.$$

which expresses all the various possible arrangements, less B , will express the number of the arrangements where no card will be found in its place, & will give the numbers above for all the values of p . Now if one names q the number of cards that one supposes must not at all be found ordered in their place, the number q must be multiplied by the one which expresses in how many ways q is able to be taken in p , since being checked when there will be q of them there which will not be found at all ordered in their place, it is indeterminate which among the p cards will not be at all in their rank. Now by *art.* 5, the formulas

$$p, \quad \frac{p \cdot p - 1}{1 \cdot 2}, \quad \frac{p \cdot p - 1 \cdot p - 2}{1 \cdot 2 \cdot 3}, \quad \frac{p \cdot p - 1 \cdot p - 2 \cdot p - 3}{1 \cdot 2 \cdot 3 \cdot 4}, \quad \&c.$$

express in how many different ways p cards are able to be taken either one by one, or two by two, or three by three, or four by four, &c. Therefore, &c.

PROPOSITION VII.

The same things being supposed as in the preceding problem, one asks the advantage of Pierre.

SOLUTION.

111. His advantage is always equal to the unit whatever number of cards that he has. This seems a paradox, however the demonstration of it is easy. Because it is evident that Pierre having any number p of cards expressed by the letters $a, b, c, d, e, f, \&c.$ If one imagines these letters ordered on p columns of 1, 2, 3, 4, 5, ... $p - 1$ permutations, such that the one begins with b , the second with c , the third with d , the fourth with e , the fifth with f , &c. The column which begins with b will give

$$2 \times 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \dots p - 1 \times A,$$

& each of the others will give

$$\overline{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \dots p - 1 \cdot A} - \overline{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \dots p - 2} \times A.$$

And more simply still, It is clear that there are $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \dots p - 1$ permutations where b is found in its place, & that there are as many of them where c will be found in its place; and thus the others.

And consequently naming C the number of all the different possible permutations, & D the number of chances that there are in order that no card is found in its rank, A the wager of Paul, B the wager of Pierre, the advantage of Pierre is expressed by $\frac{CA - DB}{C}$, this which shows that B must be $\frac{C}{D}A$ in order that the game be fair, & that in the case of thirteen cards & of $B = A$, the advantage of Pierre is

$$\frac{6227020800A - 2290792932A}{6227020800} = 61.6f\ 5d. \frac{6439}{720722}$$

in supposing that A expresses one pistole, & that Pierre pays it to Paul, when drawing the thirteen cards none are brought to its rank.

PROPOSITION VIII.

Pierre plays against Paul in the same conditions as in the problem of Proposition 5, except that one will suppose here that Paul is obliged to keep the game, & to wager always the same sum when he has lost, until Pierre manages to draw until the last card, without naming any of them in its place. One supposes also that Pierre always restarts in naming ace. One asks what is the advantage of Pierre.

FIRST CASE.

Pierre holds an ace & a deuce.

112.⁷ I suppose that Pierre & Paul each wager & will wager each time in the game a certain sum that I call a . I express the two cards by two letters, namely the ace by the letter a , & the 2 by the letter b . Thus put I examine that which the two different permutations ab , ba give to Pierre. Now I see that the permutation ba makes Pierre lose, & that the other permutation ab puts him in a situation that I see in truth is very favorable to him, but which is unknown to me; since Pierre, in order to finish, is obliged to shuffle the cards, & to restart. Now in restarting it is equally able to happen to him, either to lose that which he would have already won, if the cards are found arranged such as the permutation ab representing it; or to win anew, with the right to restart, if the cards are disposed such as the permutation ba representing it; because in this disposition he will win with b , having to name a deuce; & next by a , having to name an ace; & there will be still the right to continue the game, after having shuffled the cards anew.

Therefore naming B the sought advantage of Pierre, x his advantage when he has brought for first card an ace, one has

⁷ Correction made by Jean Bernoulli in the letter of 17 March 1710.

$$B = \frac{1}{2} \times \overline{a+x} + \frac{1}{2} \times -a,$$

&

$$x = \frac{1}{2} \times \overline{2a+B} + \frac{1}{2} \times -a:$$

whence one obtains $B = \frac{1}{3}a$.

SECOND CASE.

Pierre holds three cards, an ace, a deuce & a three.

113. One has six arrangements.

$$\begin{array}{cccccc} a+x & abc & a+B & bac & -a & cab \\ 2a+B & acb & -a & bca & 2a+B & cba \end{array}$$

I call x the advantage of Pierre, when in replaying, after having shuffled the cards, he names three.

In order to determine it I make this 2nd table.

$$\begin{array}{cccccc} -a & abc & -a & bac & 2a+y & cab \\ -a & acb & -a & bca & a+x & cba \end{array}$$

I call y the advantage of Pierre, when in replaying, after having shuffled the cards, he names two.

In order to determine it I make this third table.

$$\begin{array}{cccccc} -a & abc & 2a+y & bac & -a & cab \\ a+y & acb & a+x & bca & -a & cba \end{array}$$

Comparing these equalities I obtain $B = a + \frac{16}{57}a$, $x = -\frac{3}{19}a$, $y = \frac{4}{19}a$.

THIRD CASE.

Pierre holds four cards, an ace, a deuce, a three & a four.

114.⁸ In following the same route as before, one will find the advantage of Pierre
 $= \frac{130225}{172279}a$.

⁸ See the first exercise on the game of Treize.

This method is already quite lengthy for four cards, & becomes impractical for a greater number: it is necessary to be content with what is available, until one has found a better of it.

Extract of the letter of M. (Jean) Bernoulli to M. de Montmort
From Basel this 17 March 1710 (pg. 290)

Page 59, l. 26, 1st edition. (See section 105).

The series that you gave here in order to determine the strength of Pierre holding the hand in the game of Treize is very good & very interesting, one obtains it easily from the general formula on *page 58*. (Section 104)

I have also found this formula, with one other which has furnished me the same series, but without changing the signs, & which supposes the strength from the preceding numbers of cards known as you show it. Let S be the strength of Pierre that one seeks the number of cards that Pierre holds being expressed by n ; t the strength of Pierre the number of cards being $n - 1$; s his strength the number of cards being $n - 2$; r the strength, when the number of cards is $n - 3$; & thus in sequence; one will have

$$S = \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \cdots \frac{1}{1.2.3\dots n}$$

$$- \frac{t}{1} - \frac{s}{1.2} - \frac{r}{1.2.3} - \cdots \frac{o}{1.2.3\dots n};$$

this is able to pass for a theorem, your series being more appropriate in order to find first the value of S .

Page 63, l. 13, 1st edition. (See section 112.)

You make $x = \frac{1}{2} \times \overline{4A + S} + \frac{1}{2} \times A$; but you mislead yourself, if is necessary to make $x = \frac{1}{2} \times \overline{4A + S - A} + \frac{1}{2} \times A$; & thus the advantage of Pierre is $\frac{1}{3}A$, & not $\frac{2}{3}A$.

Remark of M. (Nicolas) Bernoulli
adjoined to the letter of Mr. Jean Bernoulli to Mr. Montmort (pg. 301–302)

Page 58 on the Game of Treize, 1st edition. (See section 104)

Let the cards which Peter holds be designated by the letters a, b, c, d, e , &c. of which the number is n , the number of all possible cases will be $= 1.2.3\dots n$, the number of cases when a is in the first place

$$= 1.2.3 \dots n - 1;$$

the number of cases when b is in the second, but a not in the first

$$= 1.2.3 \dots n - 1 - 1.2.3 \dots n - 2;$$

the number of cases when c is in the third place, yet neither a in the first nor b in the second

$$= 1.2.3 \dots n - 1 - 2 \times 1.2.3 \dots n - 2 + 1.2.3 \dots n - 3;$$

the number of cases when d is in the fourth, none indeed of the preceding in its place

$$= 1.2.3 \dots n - 1 - 3 \times 1.2.3 \dots n - 2 + 3 \times 1.2.3 \dots n - 3 - 1.2.3 \times n - 4;$$

and generally, the number of cases, in which it is able to happen when the letter which is at rank m , but none of the preceding is in its place,

$$\begin{aligned} &= 1.2.3 \dots n - 1 - \frac{m-1}{1} \times 1.2.3 \dots n - 2 \\ &\quad + \frac{m-1.m-2}{1.2} \times 1.2.3 \dots n - 3 - \frac{m-1.m-2.m-3}{1.2.3} \times 1.2.3 \times n - 4 \\ &\quad + \dots \text{ up to } \pm \frac{m-1.m-2 \dots m-m+1}{1.2.3 \dots m-1} \times 1.2.3 \dots n - m \end{aligned}$$

hence the risk of the player who in this letter finally, which is at rank m , wishes to win, is

$$\begin{aligned} &\frac{1}{n} - \frac{m-1}{1} \times \frac{1}{n.n-1} + \frac{m-1.m-2}{1.2} \times \frac{1}{n.n-1.n-2} \\ &\quad - \frac{m-1.m-2.m-3}{1.2.3} \times \frac{1}{n.n-1.n-2.n-3} + \dots \\ &\quad \text{up to } \pm \frac{m-1.m-2 \dots m-m+1}{1.2 \dots m-1} \times \frac{1}{n.n-1 \dots n-m+1}, \end{aligned}$$

& the risk of the player who at least in the case of some m of the letters wishes to win = the sum of all the possible preceding values of the series being put for m successively 1.2.3 &c. that is

$$\begin{aligned} \frac{m}{n} &- \frac{m.m-1}{1.2} \times \frac{1}{n.n-1} + \frac{m.m-1.m-2}{1.2.3} \times \frac{1}{n.n-1.n-2} \\ &- \frac{m.m-1.m-2.m-3}{1.2.3.4} \times \frac{1}{n.n-1.n-2.n-3} + \dots \\ &\text{up to } \pm \frac{m.m-1.m-2\dots m-m+1}{1.2\dots m-1} \times \frac{1}{n.n-1\dots n-m+1}, \end{aligned}$$

I put $m = n$ the risk of the player is

$$= 1 - \frac{1}{1.2} + \frac{1}{1.2.3} - \frac{1}{1.2.3.4} + \dots \text{ up to } \pm \frac{1}{1.2.3\dots n}.$$

In another way. (See section 105)

Either a is in first place, or it is not; if a is in first place, thereupon the risk is $= 1$, if it is not, thereupon he has as many chances to obtain 1, which were held if the number of letters were $n - 1$, with this excepted case, in which it happens, when this letter, of which a entered the position, again is in first place, for these do not surrender 1 to him, but merely that expectation, which he had if the number of letters were $n - 2$; however there are as many cases when this happens, as they admit variations of $n - 2$ letters, certainly $1.2.3\dots n - 2$; hence putting the strength of him when the number of letters is $n - 2 = d$, & g for the strength when the number of letters is $n - 1$, there will be by the existing number of letters $= n - 1$, out of the entire cases $1.2.3\dots n - 1$, $1.2.3\dots n - 1 \times g$ winning cases (for he has the whole deposit or 1 to the value of the expectation the same ratio as the number of all cases to the number of winning cases) hence the expectation which he has if a not be in its place is

$$= \frac{1.2.3\dots n-1 \times g - 1.2.3 \times n-2 + 1.2.3\dots n-2d}{1.2.3\dots n-1} = \frac{\overline{n-1} \times g - 1 + d}{n-1},$$

since therefore out of n cases precisely one is when a is in first place, & $n - 1$ cases when it is not, the obtained risk will be

$$= \frac{1 \times 1 + n-1 \frac{\overline{n-1} \times g - 1 + d}{n-1}}{n} = \frac{n-1 \times g + d}{n}.$$

Hence it appears the difference between the sought strength & the one which he has, if the number of letters is $n - 1$, to be $= \frac{-g+d}{n}$ = difference between this same strength & the one, which he has if the number of letters is $n - 2$, but supposing negative & dividing by the number of letters n , whence with the

existing number of letters 0 & 1, furthermore the risk is 0 & 1, will be the difference between the strength if the number of letters is 2, & between the preceding strength, when certainly the number of letters is less by unity, $= -\frac{1}{2}$; if the number of letters be 3, $= +\frac{1}{2.3}$; if 4, $= -\frac{1}{2.3.4}$; if 5, $= +\frac{1}{2.3.4.5}$, & generally if the number of letters be $n = \pm \frac{1}{2.3.4\dots n}$, and even the total risk

$$= 1 - \frac{1}{2} + \frac{1}{2.3} - \frac{1}{2.3.4} + \dots \text{ up to } \pm \frac{1}{2.3.4\dots n}.$$

Extract of the letter in reply from M. de Montmort to M. (Jean) Bernoulli
At Montmort 15 November 1710 (pg. 304)

Page 59, 1st edition. (See section 105)

I am very comfortable that you approve the series

$$\frac{1}{1} - \frac{1}{1.2} + \frac{1}{1.2.3} - \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5} + \&c.$$

I have found well some curious things on this matter. I have found, for example, that the advantage of the one who holds the cards on the wager of the players which I call A , is

$$\frac{69056823787189897}{241347817621535625}A.$$

I would make you part of my method, if I did not fear to be too long, I humor myself that it would be to your taste.

Page 62, 1st edition. (See section 112). It is true that there is an error in this place; however I excuse myself this inattention, & I prefer to have faltered in this place which is simple than in the essential of some method, that which I would not excuse so easily. I thank you for having warned me of it, & I will correct myself in the new edition. I have calculated the following case for four cards, & I have found that A expressing the money of the game, the strength of the one who holds the cards is

$$\frac{56908325}{75285923}A.$$

PROBLEMS TO SOLVE

FIRST PROBLEM

ON THE GAME OF TREIZE

To determine generally what is in the game the advantage of the one who holds the cards. One will find the explanation of the rules of this game on pages 130 & 131. See the letters of 10 April & of 19 November 1711. (page 278)

Extract of the letter from Jean Bernoulli to M. de Montmort
At Basel this 17 March 1710 (page 298)

The four problems that you propose at the end of your treatise are interesting; but the first seems to me insoluble because of the length of the calculation that it would demand, & that the human lifespan would not suffice to accomplish it:

Extract of the letter of Nicolas Bernoulli to M. de Montmort
From Basel this 26 February 1711 (pages 308–309)

I have not yet attempted the general solution of the problem on the game of Treize, because it seems to me almost impossible; this is also why I was greatly astonished by that which you say, that you have found $\frac{69056823787189897}{241347817621535625}A$ for the advantage of the one who holds the cards; but in examining the thing a little more closely, I had the thought, that you perhaps have resolved generally this problem only in the supposition, that the one who holds the cards having won or lost, the game would conclude; that which confirms to me in this thought, is that I have found for this hypothesis a general formula, which applied to the particular case of 52 cards, gives for the advantage of the one who holds the cards this fraction $\frac{99177450342464537}{336245122781568000}A$ which is a little greater than yours, but which has for denominator a number composed of nearly the same factors as the one of yours, this which makes me believe that you have made an error of calculation in the application of your formula: here is mine of which I come to speak.

$$S = \frac{1}{1} - \frac{n-p}{1.2 \times n-1} + \frac{n-2p}{1.2.3 \times n-2} - \frac{n-3p}{1.2.3.4 \times n-3} + \&c.$$

up to the a term which is = 0; by p I intend the number of times that each different card is repeated, & by n the number of all cards. I have also calculated the case for 4 cards, of which you speak, & I have found $\frac{130225}{172279} = \frac{56908325}{75285923}$ as you; but it is apropos to observe here, that according to the rules of this game there, it is not necessary to suppose that the game is complete, when the one who has the hand just loses, because then he is obligated to cede the hand to another, & the game continues; this is why the advantage of the one who holds the cards being diminished by the disadvantage that he had in losing the hand, will be in the aforesaid case only $\frac{130225}{344558}$ the half of that which had been found. If one assumes that there were many players against the one who has the hand, & that their number is = n , his advantage will be $\frac{130225}{344558} \times n$, & the one of the other players $\frac{130225}{344558} \times$ either $n - 2$, or $n - 4$, or $n - 6$, &c. according to the rank that each occupys by relation to the right of the one who holds the card. This remark extends itself on all of the players in which the hand passes from one to the other; also in your first case of Lansquenet I have found that the advantage of Pierre is only $\frac{161}{1006}A$, the disadvantage of Paul — $\frac{163}{4024}A$, & the one of Jacques $\frac{481}{4024}A$.

Extract of the letter of M. de Montmort to Nicolas Bernoulli
At Montmort 10 April 1711.

(page 315) It is necessary, Sir, that you have badly copied your general formula for Treize, because I am not able to find in it my count: here is mine. Let n be the number of cards, p the number of times that each different card is repeated, let also $\frac{n}{p} = m$ & $n - m = q$, one will have the sought strength

$$\begin{aligned}
 &= p^m - mp^{m-1} \times q + 1 + \frac{m.m-1}{1.2} p^{m-2} \times q + 1.q + 2 \\
 &\quad - \frac{m.m-1.m-2}{1.2.3} p^{m-3} \times q + 1.q + 2.q + 3 \\
 &\quad + \frac{m.m-1.m-2.m-3}{1.2.3.4} p^{m-4} \times q + 1.q + 2.q + 3.q + 4 - \&c.
 \end{aligned}$$

the whole divided by as many products of the numbers $n - 1.n - 2.n - 3.n - 4$ &c. as there are units in $\frac{n}{p}$.

Note, 1. That it is necessary to take as many terms of this sequence as m expressed in units. 2. That it is necessary to change all the signs of this sequence when m is an even number.

Also I find that the strength of the one who holds the cards at the beginning of the game is $\frac{310404641408725522}{241347817621535625}$, and his advantage is $\frac{69056823787189897}{241347817621535625}$. I do not believe that there is an error in this calculation; but surely there is none at all in the method.

(pages 317–318) You say, Sir, that you have calculated the case of four cards, *page 64*, <See section 114> & that you have found as me $\frac{130225}{172279}$; but you add that according to the rules of this game it is not necessary to suppose that the game be complete, when the one who has the hand just loses; because now, say you, he is obligated to cede the hand to another. This is why the advantage of the one who holds the cards being diminished by the disadvantage that he has in losing the hand, will be only $\frac{130225}{344558}$ the half of that which had been found, & you add next; if one supposes that there are many players against the one who has the hand, & that their number be n , his advantage will be $\frac{130225}{344558} \times n$, & the one of the other players $\frac{130225}{344558} \times n - 2$, $\frac{130225}{344558} \times n - 4$ &c. according to the rank that each occupies. You extend next this remark on Lansquenet, & and it seems that your series of notices to apply to all sorts of games. For me I believe myself to have some reasons to think otherwise: I am going to expose them to you. Firstly, in regard to Treize, it is certain that the one who quits the hand is not at all obligated to continue to play, & moreover, he is not obligated to wager the same sum in the game; on the contrary it happens that in this game those who noticed, as it is easy to discover by practice, that the advantage is for the one who holds the cards, they keep everything when they hold the hand, & wager little money in the game when they do not have the hand. There is yet to remark that in this game the wagers increase or diminish without ceasing likewise the number of players, & that in Lansquenet the number of players is able to decrease from one hand to the other of the same player. In such a way that in my note one is able to say nothing useful & certain on these games, when in taking the part to determine at each move the advantage or the disadvantage of the one who holds the hand with respect to a determined number of wagers of the players. If I make to enter in Lansquenet the consideration of the expectation that the one who holds the cards has to make the hand: this had been only by elegance, because in the fund that one is only just when in supposing that the number of players will be always the same amount as when Pierre will have the hand, this which is uncertain. It suffices it seems to me in order to be educated, as perfectly as it is possible, of the chances of these games, for example of Lansquenet, knowing

that by ratio to such number of players & of wagers there is as much advantage & disadvantage for each of the players, according to the different places that they occupy.

Extract of the letter of Mr. Nicolas Bernoulli to Mr. de Montmort
At Basel this 10 November 1711.

You have reason to say that you have not found your count in my formula for Treize, because an error is slipped there; it is necessary to put

$$S = \frac{1}{1} - \frac{n-p}{1.2.n-1} + \frac{n-p.n-2p}{1.2.3.n-1.n-2} - \frac{n-p.n-2p.n-3p}{1.2.3.4.n-1.n-2.n-3} + \&c.$$

instead of

$$S = \frac{1}{1} - \frac{n-p}{1.2.n-1} + \frac{n-2p}{1.2.3.n-2} - \frac{n-3p}{1.2.3.4.n-3} + \&c.$$

This error, to that which I myself can remember, comes from that which by making the calculation I have put on the table on these last factors of the terms of each fraction, in order to indicate the law of the progression which there is among the terms of this series; whence it happens that next no more remembering the true solution, I have allowed to escape the other factors. You will see that this formula thus corrected will agree exactly with yours. The number

$$\frac{69056823787189897}{241347817621535625}$$

which you give for the case $n = 52$ & $p = 4$ is not yet correct, it is necessary according to your formula & mine

$$\frac{69056823706869897}{241347817621535625} = \frac{7672980411874433}{26816424180170625}.$$

The method of which I am being served in order to find this formula is the same as that of which I was being served once in my Latin Remarks for the resolution of the particular case of $p = 1$.

Extract of the letter of Nicolas Bernoulli to M. de Montmort
At Basel this 10 November 1711 (pages 327–328)

I am surprised, Sir, to see your objections against my remarks on the games in which the hand turns from one to another; it seems to me that you are much wrong to oppose me some things which are also as much against you as against me; because if you are in a state to suppose, for example, at Lansquenet, that the number of players & of the wagers are always the same, & that the game continues as long as Pierre will have the hand, why would there not be permitted to me to suppose again the same thing, the same after Pierre will have lost the hand? You say that one is able to say nothing useful & certain on these games, because the number of wagers & of the players are always able to vary there: this is true, & this is also the reason why one must make a certain hypothesis to which one can take oneself in the calculation. I have therefore made this hypothesis, namely that one continues to play when one just loses the hand, because it is more natural & more conforming to that which happens ordinarily, than yours which supposes that the game continues as long as Pierre will have the hand, this which is a condition which is being scarcely practiced among the players, especially when they know that there is advantage to have the hand. But you oppose me still when, by example at Treize, the one who quits the hand is no longer obligated to continue the game, to which I respond that an honest man must be held obligated to it, although one is not expressly agreed to that; because it is certain that ordinarily one begins the game with the plan to make a great enough number of games, & not to end immediately after the first move, this which engages the players tacitly to continue the game during a certain time. It will not be permissible to quit the game after having had the advantage of the hand, at least one does not wish to pass in order for a man who thinks rather of grabbing the money of the others than to amuse them. You see by this, Sir, that you would not have done badly to take into consideration, not only the advantage that one has in conserving the hand; but also the disadvantage that one has in losing it.

Extract of the letter of M. de Montmort to Mr. Nicolas Bernoulli
At Paris 1 March 1712 (pages 344)

Your formula for Treize is quite correct. I myself have well doubted that the error of the preceding could result only from some inadvertance in transcription. The idea that I myself have made of your infallibility in Geometry has not permitted to suspect that you had been able to deceive yourself at the base of a method.

Table of M. Pascal for the combinations. (Art. 1, pg. 2)

1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	2	3	4	5	6	7	8	9	10	11	12	13
		1	3	6	10	15	21	28	36	45	55	66	78
			1	4	10	20	35	56	84	120	165	220	286
				1	5	15	35	70	126	210	330	495	715
					1	6	21	56	126	252	462	792	1287
						1	7	28	84	210	462	924	1716
							1	8	36	120	330	792	1716
								1	9	45	165	495	1287
									1	10	55	220	715
										1	11	66	286
											1	12	78
												1	13
													1

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