CORRESPONDENCE ON THE PROBLEM OF POINTS

BLAISE PASCAL AND PIERRE FERMAT
ŒUVRES DE FERMAT VOLUME 2, PP. 288–314, 1894

The following letters are printed in the *Œuvres de Fermat* [1, Vol. II, pp. 288–314] which has been used as source. With that edition of Fermat's works the sequence of the correspondence was established. The letters as found in the 1880 *Œuvres Complètes de Pascal* [4, Vol. III, pp. 220–236] are incorrectly ordered, placing Letter XXIX between Letters LXXIV and LXXV. Subsequent editions of the works of Pascal correct this error. These include, for example, [2] and especially the edition of Mesnard [3].

Sequence of correspondence	
Letter LXIX	Letter from Fermat to Pascal, reply to lost introductory letter.
Letter LXX	Letter from Pascal to Fermat, reply lost.
Letter LXXI	Letter from Fermat to Carcavi.
Letter LXXII	Letter from Pascal to Fermat.
Letter LXXIII	Letter from Fermat to Pascal, Pascal does not reply.
Letter LXXIV	Letter from Fermat to Pascal, response to LXXII
Letter LXXV	Letter from Pascal to Fermat, response to LXXIV.

Letter LXIX. Fermat to Pascal

Reply to the lost introductory letter. Date unknown.

1654

SIR,

If I undertake to make a point with a single die in eight throws; and if, we agree, after the money is in the game, that I will not make the first throw, it is necessary, by my principle, that I take from the game $\frac{1}{6}$ of the total in order to be disinterested, by reason of the said first throw.

That if further we agree after this that I will not make the second throw, I must, for my compensation, take the 6^{th} of the remaining, which is $\frac{5}{36}$ of the total.

And if after this we agree that I will not make the third throw, I must, for my compensation, take the 6^{th} of the remaining, which is $\frac{25}{216}$ of the total.

And if after this we agree again that I will not make the fourth throw, I must take the 6^{th} of the remaining, which is $\frac{125}{1296}$ of the total, and I agree with you that this is the value of the fourth throw, assuming that one has already negotiated the previous.

But you propose to me in the last example of your letter (I quote your own terms) that if I undertake to find the six in eight throws and if I have played three without encountering it, and if my opponent proposes to me to not play my fourth throw and he wishes to buy out my interest because I would be able to encounter it, $\frac{125}{1296}$ of the entire sum of our stakes will belong to me.

Translated by Richard J. Pulskamp, Department of Mathematics and Computer Science, Xavier University, Cincinnati OH. This document created October 21, 2018. I thank Glenn Shafer for several corrections to an earlier version.

This however is not true according to my theory. For, in this case, the first three throws having acquired nothing to the one who holds the die, the total sum remaining in the game, the one who holds the die and who agrees not to make the fourth throw, must take for his compensation $\frac{1}{6}$ of the total.

And if he had made four throws without finding the point sought and if we agreed that he would not throw the fifth, he would have likewise for his compensation $\frac{1}{6}$ of the total. For the entire sum remaining in the game, it follows not only from the principle, but it is likewise from common sense that each throw must give an equal advantage.

I beg you therefore that I know if we are conformed in the principle, thus as I believe, or if we differ only in application.

I am, with all my heart, etc.,

FERMAT

Extract of Letter LXX from Pascal to Fermat

Wednesday, 29 July 1654
Fermat's reply to this letter is lost.

SIR,

1. Impatience holds me as well as you and, although I am still in bed, I cannot prevent myself telling you that I received yesterday evening, from M. de Carcavi, your letter on divisions, which I admire more than I can say to you. I have not the leisure to extend myself, but, in a word, you have solved the two divisions[1] of the dice and of the points with perfect fairness; I am entirely satisfied with it for I no longer doubt now that I was not correct, according to the admirable agreement where I find myself with you.

I admire even more your method for the points than that of the dice: I have seen many people obtain that for the dice, as M. le Chevalier de Méré, who is the one who has posed these questions to me, and also M. de Roberval: but M. de Méré never could find the true value for the points nor a method to arrive to it, so that I found myself alone who knew this proportion.

2. Your method is very sure and is that which is first come to thought in this research; but, because the labor of combinations is excessive, I have found an abridgment and properly another method much shorter and and neater, which I would like to be able to tell you here in a few words: for I would like henceforth to open my heart to you, if it were able, so much I have joy to see our agreement. I see well that truth is the same in Toulouse as in Paris.

Here is as near as I do in order to to know the value of each of the divisions, when two players play, for example, to *three* games, and each person has staked 32 pistoles into the game:

Let us put that the first had *two* and the other *one*; now they play one game, of which the condition is such that, if the first wins it, he wins all the money which is in the game, namely 64 pistoles; if the other wins it, they are *two* games to *two* games, and consequently, if they wish to separate themselves, it is necessary that they each take back their stake, namely 32 pistoles each.

[1] *division*, that is *Parti* signifies here the division of the stake between the players in the case where the game is abandoned. The *division of the dice*, *parti des dés* appears to have been simply demanded in the case where the one who holds the dice has wagered to bring forth a given point in an agreed number of throws. The *division of the points*, *parti des parties* refers to the stake in a prematurely terminated game.

Therefore consider, Sir, if the first wins, 64 belongs to him; if he loses 32 belongs to him. Therefore, if they do not wish to risk this game and to separate themselves without playing it, the first must say: "I am certain to have 32 pistoles, because even the loss gives them to me; but for the other 32, perhaps I will have them, perhaps you will have them, the chance is equal. Therefore we divide these 32 pistoles in half and give me, beyond those, my 32 which are certain for me." He will then have 48 pistoles and the other 16.

Suppose now that the first had *two* games and the other *nothing*, and they begin to play a game. The condition of this game is such that, if the first wins it, he takes all the money, 64 pistoles; if the other wins it, they are returned to the preceding case, in which the first will have *two* games and the other *one*.

Now, we have already shown that in this case 48 pistoles belong to the one who has the *two* games: therefore, if they wish to no longer play this game, he must say thus: "If I win it, I will win all, which is 64; if I lose it, 48 legitimately belong to me; therefore give me the 48 which are certain to me, even in the case if I lose, and let us divide the 16 others in half, since there is as many chances that you win them as me." Thus he will have 48 and 8, which are 56 pistoles.

Suppose finally that the first has only *one* game and the other *nothing*. You will see, Sir, that, if they begin a new game, the condition is such that, if the first wins it, he will have *two* games to *nothing*, and hence, by the preceding case, 56 belong to him; if he loses it, they are game to game: therefore 32 pistoles belong to him. Therefore he must say: "If you do not wish to play longer, give me 32 pistoles which are certain to me, and let us divide the rest of 56 in half. From the 56 remove 32, 24 remain; divide therefore 24 in half, take 12 of them and to me 12, which, with 32, make 44."

Now, by this method, you see, by the simple subtractions, that, for the first game, 12 pistoles of the money of the other belong to him; for the second, another 12; and for the last, 8.

Now, to no longer make mystery, since you see quite well, all to reveal and that I would make only in order to see if I would not deceive myself, the value (I understand his value of the money of the other only) of the last game of *two* is double that of the ⟨last⟩ game of *three* and quadruple of the last game of *four* and eight times the last game of *five*, etc.

3. But the proportion of the first games is not so easy to find: it is therefore thus, for I wish to hide nothing, and here is the problem of which I made so much fuss, as indeed it pleases me greatly.

Being given as many games as one will wish, find the value of the first.

Let the given number of games be, for example, 8. Take the first *eight* even numbers and the first *eight* odd numbers, namely:

$$2,\quad 4,\quad 6,\quad 8,\quad 10,\quad 12,\quad 14,\quad 16$$

and

$$1,\quad 3,\quad 5,\quad 7,\quad 9,\quad 11,\quad 13,\quad 15.$$

Multiply the even numbers in this way: the first by the second, the product by the third, the product by the fourth, the product by the fifth, etc.; multiply the odd numbers in the same way: the first by the second, the product by the third, etc.

The last product of the evens is the *denominator* and the last product of the odds is the *numerator* of the fraction which expresses the value of the first game of *8*: that is that, if each one stakes the number of pistoles expressed by the product of the even numbers, there would belong of it out of the money of the the other the number expressed by the product of the odds.

This is demonstrated, but with much trouble, by combinations such as you have imagined them, and I have not been able to demonstrate it by that other method which I have just mentioned to you, but only by that of combinations. And here are the propositions which lead to it, which are properly some arithmetical propositions bearing on combinations, of which I have enough beautiful properties:

4. If from any number of letters, for example, 8:

$$A, \quad B, \quad C, \quad D, \quad E, \quad F, \quad G, \quad H,$$

you take all the possible combinations of 4 letters and next all the possible combinations of 5 letters, and next of 6, of 7 and of 8 etc., and if thus you take all possible combinations starting from the multitude which is half of the total to the total, I say that, if you join together half the combination of 4 with each of the superior combinations, the sum will be the number such a quantity of the quaternary progression beginning with the binary, which is the half of the multitude.

For example, and I will say it in Latin, for French is worth nothing.

If, of as many letters as we please, for example eight:

$$A, \quad B, \quad C, \quad D, \quad E, \quad F, \quad G, \quad H,$$

let all combinations of four, five, six, etc. be added, all the way to 8, I say, if you add half of the combinations of four, namely 35 (half 70), with all the combinations of five, namely 56, plus all the combinations of six, namely 28, plus all the combinations of seven, namely 8, plus all the combinations of eight, namely 1, it becomes the fourth number of the fourth progression of which the origin is 2: I say the fourth number, because 4 is half of eight.

For the numbers of the fourth progression, whose origin is 2, are those:

2, 8, 32, 128, 512, etc.

of which 2 is the first, 8 second, 32 third and 128 fourth, to which 128 equals

$+35$	*half the combinations of 4 letters*
$+56$	*the combinations of 5 letters*
$+28$	*the combinations of 6 letters*
$+8$	*the combinations of 7 letters*
$+1$	*the combination of 8 letters*

5. Here is the first proposition which is purely arithmetical; the other concerns the theory of games and is such:

It is necessary to say in the first place,: if one has has *one* game of 5, for example, and that thus he lacks 4, the game will be infallibly decided in 8, which is double of 4.

The value of the first game of 5 out of the money of the other, is the fraction which has for numerator the half of the combination of 4 out of 8 (I take 4 because it is equal to the number of games which are lacking, and 8 because it is double of 4) and for denominator this same numerator plus all the superior combinations.

Thus, if I have *one* game out of 5, $\frac{35}{128}$ belongs to me out of the money of my opponent, ; that is, if he has staked 128 pistoles, I take from them 35 and leave to him the rest, 93.

Now this fraction $\frac{35}{128}$ is the same as this here $\frac{105}{384}$, which is made by the multiplication of the evens for the denominator and the multiplication of the odds for the numerator.

You will undoubtedly see all this well, if you take a little trouble: that is why I find useless for you to undertake more.

6. I send you nevertheless one of my former Tables; I have not the leisure to copy it, I will remake it.

You will see there, as always, that the value of the first game is equal to that of the second, that which is found easily by combinations.

You will see likewise that the numbers in the first line always increase; those in the second likewise; those in the third likewise.

But next those in the fourth decrease, those in the fifth, etc. That which is strange.

Out of my opponent's 256 pistoles I get, for the

	If each one stakes 256 in					
	6 games	5 games	4 games	3 games	2 games	1 game
1st. game	63	70	80	96	128	256
2nd. game	63	70	80	96	128	
3rd. game	56	60	64	64		
4th. game	42	40	32			
5th. game	24	16				
6th. game	8					

From my opponent's 256 pistoles I get, for

	If each one stakes 256 in					
	6 games	5 games	4 games	3 games	2 games	1 game
the first game	63	70	80	96	128	256
the first 2 games	126	140	160	192	256	
the first 3 games	182	200	224	256		
the first 4 games	224	240	256			
the first 5 games	248	256				
the first 6 games	256					

7. I have not time to send you the demonstration of a difficulty which greatly astonished M. de Méré, for he has a very good mind, but he is not a geometer (this is, as you know, a great defect) and he does not even understand that a mathematical line is divisible to infinity and believes very well to understand that it is composed of points in finite number, and never have I been able to pull him from it. If you could do that, one would render him perfect.[2]

He told me therefore that he had found a fallacy in the numbers for this reason:

If one undertakes to make a *six* with one die, there is advantage to undertake it in 4, as 671 to 625.

If one undertakes to make *sonnez*[3] with two dice, there is a disadvantage to undertake it in 24.

And nevertheless 24 is to 36 (which is the number of faces of two dice) as 4 to 6 (which is the number of faces of one die).

Here is what was his great scandal which made his say haughtily that the propositions were not consistent and that Arithmetic contradicted itself: but you will quite easily the reason by the principles where you are.

[2]This comment is clarified by Letter XIX from de Méré to Pascal.

[3]A pair of sixes.

I will put in order all this that I have done, when I will have finished the geometrical Treatises where I have been working for some time.

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PASCAL

Extract of Letter LXXI from Fermat to Carcavi

Sunday, 9 August 1654

SIR,

1. I have been delighted to have had the thoughts conformed to those of M. Pascal, for I admire infinitely his genius and I believe him very capable of coming to the end of all that which he will undertake. The friendship that he offers me is so dear to me and so considerable that I must have no difficulty in making some use of it in the publishing of my Treatises.

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FERMAT

at Toulouse, this 9 August 1654.

Letter LXXII from Pascal to Fermat

Monday, 24 August 1654

SIR,

1. I could not offer to you all my thoughts touching on the divisions for several players in the last post, and likewise I have some repugnance to do it, for fear that by this, that admirable accord, which was between us and which was so dear to me, begins to be refuted, for I believe that we are of different opinions on this subject. I wish to offer to you all my reasonings, and you will do me the favor to correct me, if I err, or to affirm me, if I have met well. I ask this of you earnestly and sincerely, for I will hold myself for certain only when you will be on my side.

When there are only *two* players, your method, which proceeds by combinations, is very sure; but when there are *three*, I believe I have a demonstration that it is not correct, if it is only you proceed in some other manner that I do not understand. But the method that I have offered to you and of which I serve myself especially is common to all the conditions imaginable in all sorts of games, instead that of the combinations (which serves me only in the particular encounters where it is shorter than the general) is good only for those sole occasions and not in the others.

I am sure that I will make myself understood, but a little discourse will be necessary for me and a little patience by you.

2. Here is how you precede when there are *two* players:

If two players, playing several games, find themselves in that state that the first lacks *two* games and the second *three*, in order to find the division, it is necessary, say you, to see in how many games the game will be decided absolutely.

It is easy to suppose that this will be in *four* games, whence you conclude that it is necessary to see in how many ways four games are arranged between two players and to see in how many ways there are combinations making the first win and in how many for the second and to divide the money according to this proportion. I would have had difficulty to understand this discourse, if I had not known it by myself already; also you had written with this thought. Therefore, to see in how many ways four games are combined between two players, it is necessary to imagine that they play with a die with two faces (since there are only two players) as in heads and tails, and that they cast four of these dice (because they play to four games); and now one it is necessary to see in how many ways these dice have different states. This is easy to calculate; they are able to have *sixteen* which is the second degree of *four*, that is the square. For we figure that one of the faces is marked a, favorable to the first player, and the other b, favorable to the second; thus these four dice are able to be turned up on one of the following sixteen states:

a	a	a	a	a	a	a	a	b	b	b	b	b	b	b	b
a	a	a	a	b	b	b	b	a	a	a	a	b	b	b	b
a	a	b	b	a	a	b	b	a	a	b	b	a	a	b	b
a	b	a	b	a	b	a	b	a	b	a	b	a	b	a	b
1	1	1	1	1	1	1	2	1	1	1	2	1	2	2	2

and since the first player lacks two games, all the faces which have two a make him win; therefore there are 11 of them for him; and since the second lacks three games, all the faces where there are three b are able to make him win; therefore there are 5 of them. Therefore it is necessary that they divide the sum as 11 to 5.

Here is your method when there are *two* players; on which you say that, if there is more, it will not be difficult to find the division by the same method.

3. Upon this, Sir, I have to tell you that this division for two players, founded on combinations, is very correct and very good; but that, if there are more than two players, it will not always be correct, and I will say to you the reason for this difference.

I communicated your method to our colleagues, upon which M. de Roberval made this objection to me:

What is wrong is that one takes the art to make the division under the supposition that one plays to *four* games, seeing that, when the one lacks *two* games and the other *three*, it is not of necessity that one plays *four* games, being able to happen that one will play only *two* or *three*, or in truth, perhaps *four*.

And although he could not see why one would claim to make the fair division under a make-believe condition that one will play *four* games, seeing that the natural condition of the game is that one will throw dice no longer if one of the players will win, and that at least, if that were not false, that would not be demonstrated, so that he had some suspicion that we had made a paralogism.

I responded to him that I myself did not rely so much on the method of combinations, which truly is not in its place on this occasion, as on my other universal method, by which nothing escapes and which bears its demonstration by itself, which finds the same division precisely as that of the combinations; and moreover I will demonstrate to him the truth of the division between two players by the combinations in this way:

Is it not true that, if two players, finding themselves in the hypothetical state that one lacks *two* games and the other *three*, agree now by private contract that one plays *four* complete games, that is that one casts the four dice with two faces at the same time, is it not true, I say, that, if they have deliberated to play the four games, the division must be, such as we have said, according to the multitude of states favorable to each?

He agreed with this and that indeed it is demonstrated, but he denied that the same thing subsisted in not being compelled to play the *four* games. I said to him therefore thus:

Is it not clear that the same players, not being compelled to play the four games, but wishing to quit the game as soon as one had attained his number, is able without loss or gain to be compelled to play the four entire games and that this convention changes in no manner their condition? For, if the first wins the first two games out of *four* and that thus he has won, would he refuse to play yet two games, seeing that, if he wins them, he has not won more, and if he loses them, he has not won less? because these two that the other has won do not suffice for him, since three are necessary for him, and thus there is not enough in four games in order to make that both are able to attain the number which they lack.

Certainly it is easy to consider that it is absolutely equal and indifferent to the one and to the other to play to the natural condition in their game, which is to end as soon as one will have his count, or to play the entire four games: therefore these two conditions are equal and indifferent, the advantage must be entirely equal to the one and to the other. Now, it is correct when they are obliged to play four games, as I have shown: therefore it is correct also in the other case.

Here is how I demonstrated it, and if you take care, this demonstration is based on the equality of the two conditions, true and imagined, in regard to two players; and that in the one and in the other one same will always win, and if he wins or loses in the one, he will win or lose in the other, and never will both have their count.

4. We follow the same point for *three* players and let us suppose that the first lacks *one* game, that the second lacks *two* and the third *two*. In order to make the division, according to the same method of combinations, it is necessary to seek first in how many games the game will be decided, as we have done when there were two players: this will be in *three* for they would not know how to play *three* games but that the decision is necessarily arrived.

It is necessary to see now in how many ways three games are combined among three players, and in how many ways they are favorable to the one, how many to the other, and how many to the last, and according to that proportion, to distribute the money likewise as one has done under the hypothesis of two players.

In order to see how many combinations there are in all, this is easy: it is the third power of 3, that is its cube 27. For, if one throws *three* dice at the same time (since it is necessary to play three games), which each have *three* faces (since there are three players), the one marked a favorable to the first, the other b for the second, the other c for the third, it is clear that these three dice cast together are able to turn up in 27 different states, namely:

a	a	a	a	a	a	a	a	a	b	b	b	b	b	b	b	b	b	c	c	c	c	c	c	c	c	c
a	a	a	b	b	b	c	c	c	a	a	a	b	b	b	c	c	c	a	a	a	b	b	b	c	c	c
a	b	c	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c	a	b	c
1	1	1	1	1	1	1	1	1	1	1	1	1			1			1	1	1	1			1		
				2						2		2	2	2		2						2				
								3									3			3			3	3	3	3

Now the first lacks only *one* game: therefore all the states where there is an a are for to him: therefore there are 19 of them.

The second lacks *two* games: therefore all the states where there are two b's are for him: therefore there are 7 of them.

The third lacks *two* games: therefore all the states where there are two c's are for him: therefore there are 7 of them.

If thence one concluded that it would be necessary to give to each according to the proportion of 19, 7, 7, one would be deceived most grossly and I could not hold believe that you would do thus: for there are some faces favorable to the first and to the second together, such as abb, for the first player finds one a which is necessary to him, and the second two b's which are lacking to him; and thus acc is for the first and and the third.

Therefore it is not necessary to count these faces which are common to two as being worth the entire sum to each, but only the half. For, if the state acc happened, the first and the third would have the same right to the sum, each having made his count, therefore they would divide the money in half; but if the state aab arrives, the first wins alone. It is necessary to make the calculation thus:

There are 13 states which give the whole to the first and 6 which give the half to him and 8 which are worth nothing to him: therefore, if the entire sum is one pistole, there are 13 faces which are each worth to him one pistole, there are 6 faces which are each worth to him $\frac{1}{2}$ pistole and 8 which are worth nothing.

Thus, in case of division, it is necessary to multiply

	13	by one pistole, which makes		13
	6	by one half, which makes		3
	8	by zero, which makes		0
Sum	27		Sum	16

and to divide the sum of the values, 16, by the sum of the states, 27, which makes the fraction $\frac{16}{27}$; which is that which belongs to the first in the case of division, namely 16 pistoles of 27.

The part for the second and for the third player will be found likewise:

There are	4	states which are worth 1 pistole to him: multiply		4
There are	3	states which are worth $\frac{1}{2}$ pistole to him: multiply		$1\frac{1}{2}$
And	20	states are worth nothing to him		0
Sum	27		Sum	$5\frac{1}{2}$

Thus, $5\frac{1}{2}$ pistoles out of 27 belong to the second player, and the as much to the third, these three sums, $5\frac{1}{2}$, $5\frac{1}{2}$ and 16 being added, make 27.

5. Here is, it seems to me, in what manner it would be necessary to make the divisions by the combinations according to your method, if it is only you have some other thing on this subject that I am unable to know. But, if I am not deceived, this division is not fair.

The reason for this is that one supposes a false thing, which is that one plays in *three* games infallibly, instead that the natural condition of this game here is that one plays only until one of the players has attained the number of games which are lacking to him, in which case the game ceases.

It is not that it is able to happen that one of them plays three games, but it is able to happen also that one will play only one or two, and nothing of necessity.

But whence comes, one will say, that it is not permitted to make in this encounter the same make-believe supposition as when there were two players? Here is the reason for it:

Under the true condition of these three players, there is only one who is able to win, for the condition is that, as soon as one has won, the game ceases. But, under the make-believe condition, two are able to attain the number of their games: namely if the first wins one of them which he lacks, and one of the others, two which they lack; for they would have played only three games, instead that, when there were only two players, the make-believe condition and the true agreed for the advantages of the players in all; and it is this which sets the extreme difference between the make-believe condition and the real.

But if the players, finding themselves in the state of the hypothesis, that is if the first lacks *one* game and the second *two* and the third *two*, wish now mutually and agree to that condition that one will play *three* complete games, and that those who will have attained the number lacking to them will take the entire sum, if it is found one alone who had attained it, or, if it is found that two had attained it, that they will divide it equally, *in this case*, the division must be made as I just gave it, that the first has 16, the second $5\frac{1}{2}$, the third $5\frac{1}{2}$ of 27 pistoles, and that this bears its demonstration by itself by supposing this condition thus.

But if they play simply with the condition, not that one necessarily plays three games, but only until one of among them has attained his games, and that then the game ceases without giving means to another to arrive there, then 17 pistoles belong to the first, 5 to the second, 5 to the third, of 27.

And this is found by my general method which determines also that under the preceding condition, 16 of them is necessary to the first, $5\frac{1}{2}$ to the second, and $5\frac{1}{2}$ to the third without serving myself of combinations, for it goes especially alone and without obstacle.

6. Here are, Sir, my thoughts on this subject about which I have no other advantage over you than the one of having much more mediation; but it is a small thing in your regard, since your first views are more penetrating than the length of my efforts.

I do not permit to offer to you my reasons for awaiting judgment from you. I believe you to have made understood thence that the method of combinations is good between two players by accident, as it is also sometimes between three players, as when one lacks *one* game, the other *one* and the other *two*, because in this case the number of games in which the game will be achieved do not suffice in order to make two win; but it is not general and is good generally only in the case solely if one is compelled to play a certain number of games exactly.

So that, as you did not have my method when you have proposed to me the division of many players, but only that of combinations, I believe that you are of different sentiments on this subject.

I beg you to send me in what way you proceed in the research of this division. I will receive your response with respect and with joy, even when your sentiment will be contrary to mine. I am, etc.

PASCAL

Extract of Letter LXXIII from Fermat to Pascal

Saturday, 29 August 1654
Written prior to receiving the preceding letter.
Pascal does not reply to this letter.

SIR,

1. Our blows always continue and I am as glad as you in the admiration that our thoughts are arranged so exactly that it seems that they have taken one same route and make one same path. Your last Treatise on the *Arithmetic triangle* and on *its application* are an authentic proof of it: and if my calculation does not deceive me, your eleventh consequence ran through the post from Paris to Toulouse, while my proposition on the figurate numbers, which indeed is the same, went from Toulouse to Paris.

I have not refrained from failing while I will encounter in this sort, and I am persuaded that the true way in order to be prevented from failing is the one to agree with you. But, I said further, the thing would keep a compliment, and we have banished this enemy from the soft and easy conversations.

It would be now my turn to produce for you some one of my numerical inventions, but the end of parliament increases my occupations, and I dare hope of your goodness that you will accord to me a just and somewhat necessary respite.

2. However, I will respond to your question about the three players, who play to two games. When the first has one of them and the others have none, your first solution is correct, and the division of the money must be made in 17, 5 and 5: from which the reason is manifest and is taken always from the same principle, the combinations showing first that the first has for him 17 equal chances, when each of the ⟨two⟩ others have only 5.

⋮

I am, Sir, your, etc.

FERMAT

Extract of Letter LXXIV from Fermat to Pascal

Friday, 25 September 1654

This is a response to Pascal's letter of 24 August.

SIR,

1. Do not fear that our agreement is refuted, you have confirmed it yourself even in trying to destroy it, and it seems to me that by responding to M. de Roberval for yourself, you have also responded for me.

I take the example of three players, the first of whom lacks one game, and to each of the two other two, which is the case that you have opposed me.

I find only 17 combinations for the first and 5 for each the other two: for, when you say that the combination acc is good for the first and for the third, it seems that you yourself no longer remember that all that which is done after one of the players has won, serves no longer for anything. Now, this combination having made the first win as soon as the first game, of what import that the third wins the next two of them, when he would win thirty, all that would be superfluous?

That which comes from this that, as you have quite well remarked, this fiction of extending the game to a particular number of games serves only to facilitate the rule and (according to my sentiment) to render all the chances equal, or else, more intelligibly, to reduce all the fractions to one same denominator.

And so that you no longer doubt, if instead of *three* games, you extend, in the case proposed, the make-believe to four, there would be not only 27 combinations, but 81, and it would be necessary to see how many combinations would make the first win one game earlier than two to each of the others, and how many make win two games to each of the two others before one to the first. You will find that the combinations for the gain of the first will be 51 and that of each of the other two 15, that which returns to the same reason.

That if you take five games or such other number as it will please you, you will find always three numbers in proportion of 17, 5, 5.

And thus I have right to say that the combination acc is only for the first and not for the third, and that cca is only for the third and not for the first, and hence my rule of the combinations is the same for three players as for two, and generally for all numbers.

2. You have already been able to see from my preceding that I have not hesitated toward the true solution of the question of the three players of which I have sent you the three decisive numbers 17, 5, 5. But since M. de Roberval would be perhaps more easy to see a solution without pretending anything, and since it is sometimes able to produce some shortening in many cases, here is the proposed example of it:

The first is able to win, either in a single game, or in two or in three.

If he wins in a single game, it is necessary that with one die which has three faces, he encounters the favorable with the first throw. A single die produces three chances: this player has therefore for him $\frac{1}{3}$ of the chance, when one plays only a single game.

If one plays two games, he is able to win in two ways, either when the second player wins the first and to him the second, or when the third wins the first and to him the second. Now, two dice produce 9 chances: this player has for him $\frac{2}{9}$ of the chances, when one plays two games.

If one plays three of them, he is able to win only in two ways, either when the second wins the first, the third the second and to him the third, or when the third wins the first, the second the second and to him the third; for, if the second or the third player won the first two, he would win the game, and not the first player. Now, three dice have 27 chances: therefore this first player has $\frac{2}{27}$ of the chances when one plays three games.

The sum of the chances which make this first player win is consequently $\frac{1}{3}$, $\frac{2}{9}$ and $\frac{3}{27}$, this which makes in all $\frac{17}{27}$.

And the rule is good and general in all cases, so that, without recourse to the make-believe, the true combinations in each number of games bear the solution and show that which I have said at the beginning, that the extension to a certain number of games is nothing other than the reduction of diverse fractions to one same denomination. Here is in a few words the whole mystery, which will bring us back without doubt to good understanding, since we seek only the reason and the truth.

⋮

FERMAT

This 25 September

I wish the health of M. de Carcavi as mine and am all to him.

I write to you from the country, and it is that which will delay by accident my responses during these vacations.

Extract of Letter LXXV from Pascal to Fermat

Tuesday, 27 October 1654

Response to the previous letter.

SIR,

Your last letter has satisfied me perfectly. I admire your method for the division, so much more as I understand it quite well; it is entirely yours, and it has nothing in common with mine; and arrives to the same but simply. Here is our understanding restored.

⋮

PASCAL

Paris, 27 October 1654

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[2] Michel Le Guern, editor. *Œuvres Complètes de Pascal*. Bibliothèque de la Pléide, Gallimard edition, 1998–1999. Two volumes. Volume contains the correspondence between Pascal and Fermat.

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[4] B. Pascal. *Œuvres Complètes de Blaise Pascal*, volume III. Paris Librairie Hachette, 1880.