The Hopf Ring for *bo* and its Connective Covers

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July 29, 2005

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Abstract

In this work we recompute the known ordinary mod 2 homology of the spectrum *bo* as a Hopf ring. In addition, our study computes the Hopf modules over the Hopf ring $H_*\underline{bo}_*$ for the connective covers $bo\langle 1\rangle, bo\langle 2\rangle$, and $bo\langle 4\rangle$. These results are calculated using the bar spectral sequence, which allows us to advance from one space to the next in the spectrum, and by using maps from our spectra to the mod 2 Eilenberg-Mac Lane spectrum $\overline{H}$ in order to facilitate the simplification of our result. These calculations yield, in turn, the Hopf ring for $bo\langle n\rangle$ for all n.

MSC: 55N15; 55P43

Keywords: Hopf ring, spectra, *bo*, homotopy

Contents

1 Introduction

The object of this paper is to compute the Hopf ring $H_*(\underline{bo}_*;\mathbb{Z}/2)$ for the connective real Bott spectrum bo. This paper is intended to follow the joint paper [3], which computed the Hopf ring $H_*(\underline{KO}_*;\mathbb{Z}/2)$ for the spectrum representing orthogonal K-theory, KO. The primary example of a Hopf ring is $F_*\underline{E}_*$, where F_* is a multiplicative homology theory and $\mathbb{E} = \{\underline{E}_*\}$ is a ring-spectrum. As the Brown representability Theorem shows that spectra are intimately connected to homology theories, there is considerable interest in the computation of Hopf rings. In addition, the computation of Hopf rings adds to our understanding the structure of fundamental classes of spaces. Frequently, the bar spectral sequence is used to compute the homology of loop spectra; given the homology of the n-th space, the bar spectral sequence converges to the homology of the $(n+1)$-st space.

One of the first Hopf rings computed (following the methods of J.P. Serre - see [9]) was that of the mod 2 Eilenberg-Mac Lane spectrum $\overline{H}$, representing mod 2 homology, $H_*(\ ;\mathbb{Z}/2)$. Since then, many other Hopf rings have been computed, such as for complex cobordism (see [4]) and for Morava K-theory (see [8]). Amongst these computations is the Hopf ring for the real orthogonal K-theory spectrum KO, which represents orthogonal K-theory (see [3]). As KO fulfills Bott periodicity, i.e. $\underline{KO}_n \cong \underline{KO}_{n+8}$, this Hopf ring can be described by solely examining the first eight spaces in the spectrum.

A natural question arises from the computations for KO: What is the Hopf ring for the connected cover of KO? (This connected cover is the non-periodic spectrum bo, which represents connective real K-theory.) We certainly expect the structure of this Hopf ring to be much more complicated than that of KO, as there are now infinitely many spaces that need to be examined, instead of just eight. But, on the other hand, there should also be patterns that appear every eight spaces. This calculation - the Hopf ring for bo - is the core of this paper.

Two maps will be utilized to accomplish our goal; the obvious map of spectra to the periodic Bott spectrum,

$$bo \to KO,$$

and the fundamental class,

$$\Theta : bo \to H \to \overline{H},$$

where H represents the integer Eilenberg-Mac Lane spectrum and $\overline{H}$ represents the mod 2 version. As stated above, the Hopf rings for both of these target spaces have been computed; we include them later in this article for completeness' sake. In particular, the computations and properties proven for the Hopf ring for the periodic case KO in [3] will form the basis for our study of non-periodic bo. Note the starting point for both bo and KO is shared: $\underline{KO}_0 = \mathbb{Z} \times BO = \underline{bo}_0$.

The sum of these maps, $\Phi : bo \to KO \times \overline{H}$, induces a *monomorphism* for all n,

$$\Phi_* : H_*\underline{bo}_n \to H_*\underline{KO}_n \otimes H_*\underline{\overline{H}}_n,$$

and we compute $H_*\underline{bo}_n$ as a sub-Hopf ring of the known Hopf ring on the right, where we have used the notation $\underline{\overline{H}}_n = K(\mathbb{Z}/2, n)$. (From this point forward, the notation H_*X will be used to mean $H_*(X; \mathbb{Z}/2)$.)

Unlike most Hopf rings that have been computed, $H_*\underline{bo}_*$ requires a large number of generators. In fact, the Hopf ring $H_*\underline{bo}_*$ is the (graded) tensor product of four families of Hopf algebras:

1. Polynomial and exterior subalgebras of $H_*\underline{KO}_*$.

2. Polynomial algebras on generators that decompose in $H_*\underline{KO}_*$, companions to the polynomial algebras in the first family.

3. Exterior algebras (subsets of $H_*\underline{\overline{H}}_*$) which arise from the second family.

4. General exterior algebras in $H_*\underline{\overline{H}}_*$ that arise from the third family by unlimited suspension.

The Hopf ring for connective complex K-theory, the spectrum bu, has been calculated in [1]. The techniques used in the Hopf ring for bu are similar to those of this paper, but as the spectrum bu has a less complicated structure than bo, the Hopf ring for bu needs many less generators than does bo.

After the calculation of $H_*\underline{bo}_*$ is complete, the calculations of the connective covers $bo\langle 4\rangle, bo\langle 2\rangle$, and $bo\langle 1\rangle$ follow the same methodology. These spectra are not ring-spectra, so their mod-2 homology groups do not form Hopf rings (in the usual sense). Thus for $bo\langle 4\rangle, bo\langle 2\rangle$ and $bo\langle 1\rangle$ we compute the Hopf module over the Hopf ring $H_*\underline{bo}_*$. As $bo\langle n+8\rangle = \Sigma^8 bo\langle n\rangle$, this actually completes the calculation for $bo\langle n\rangle$ for all n.

In [6], the cohomology of the spaces $BO\langle k\rangle$ is computed for $k = 0, 1, 2, 4 \bmod 8$. We denote the generators of $H^*(BO, \mathbb{Z}/2)$ (which is well known to be a polynomial algebra over $\mathbb{Z}/2$) as $w_i \in H^i(BO, \mathbb{Z}/2), i \geq 0, w_0 = 1$. Define

$$Q_k = \begin{cases} Sq^2 \text{ if } k \equiv 0, 1 \bmod 8 \\ Sq^3 \text{ if } k \equiv 2 \bmod 8 \\ Sq^5 \text{ if } k \equiv 4 \bmod 8 \end{cases}$$

Let $I(Q_k i_k)$ denote the ideal generated by $Q_k i_k$ and $\phi(0, k)$ denote the number of integers s such that $0 < s \leq k$ and $s \equiv 0, 1, 2,$ or $4 \bmod 8$. Additionally, define $L(i)$ as one more than the number of 1's in the dyadic expansion of $i - 1$, and θ_i as classes in $H^*(BO, \mathbb{Z}/2)$ congruent to w_i modulo decomposable elements. Then [6] proves

$$H^*(BO(k, \ldots, \infty), \mathbb{Z}/2) \cong H^*(K(\pi_k(BO), k))/I(Q_k i_k) \otimes \mathbb{Z}_2[\theta_i | L(i) > \phi(0, k)].$$

We should note that the spaces $BO(8k, \ldots, \infty) = BO\langle 8k\rangle$ are the spaces $\underline{bo}_{8k}$ examined in this paper. Also, the spaces $BO\langle 8k+1\rangle$ are the spaces $\underline{bo\langle 1\rangle}_{8k}$, the spaces $BO\langle 8k+2\rangle$ are the spaces $\underline{bo\langle 2\rangle}_{8k}$ etc. There is quite a difference in methodology, though, from this paper to [6]. Specifically, in [6], the author completes the computation for every 8^{th} space and uses induction to go from $bo\langle 0\rangle$ to $bo\langle 1\rangle$ to $bo\langle 2\rangle$ to $bo\langle 4\rangle$ to $bo\langle 8\rangle$ etc. In contrast, in this paper, we deloop to obtain all of the spaces. As these are quite different techniques, and since [6] uses cohomology whereas our object of study is homology, attempts to find an explicit isomorphism between the two calculations have not succeeded.

Over the last few years there has been much interest in examining many of the spaces that arise in or are related to the spectrum *bo*. For example, BString, MString and complex analogues appear in M. J. Hopkins's theory of Topological Modular Forms. Additionally, a paper by M. Ando, M. J. Hopkins, and N. Strickland (*Elliptic spectra, the Witten genus, and the Theorem of the cube*) uses the homology of $BU\langle 6\rangle$.

2 Hopf Rings

We include a brief collection of the basic facts about Hopf rings, from [4]. Let R be a graded associative commutative ring with unit. We let CoAlg_R be the category of graded cocommutative coassociative coalgebras with counit over R. A Hopf ring is a graded ring object in the category CoAlg_R. Hopf rings include a coproduct ψ, conjugation χ, counit ε, and two products, the $*$-product and the $\circ$-product. Additionally there are relationships that mesh these maps together.

Let $H(*) = \{H_*(n)\}_{n\in\mathbb{Z}}$ be a Hopf ring over the ring R. Let $a \in H_i(n)$, $b \in H_j(k)$, and $c \in H_q(k)$. In the following we will omit the signs on our formulae as we shall be working over $\mathbb{Z}/2$.

1. Each $H_*(n)$ is an element of CoAlg_R. Thus there is a coassociative cocommutative coproduct for all n, $\psi : H_*(n) \to H_*(n) \otimes H_*(n)$, which we write as
$$\psi(a) = \Sigma a' \otimes a''.$$
Additionally, there is a counit, $\varepsilon : H_*(n) \to R$ such that $(1_{H_*(n)} \otimes \varepsilon) \circ \psi$ is the identity.

2. Each $H_*(k)$ is an abelian group object of CoAlg_R. This gives the first product structure,
$$* : H_*(k) \otimes H_*(k) \to H_*(k)$$
which is associative and commutative. This product structure operates with the coproduct and counit: $\psi(b * c) = \psi(b) * \psi(c) = \Sigma(b' \otimes b'') * (c' \otimes c'')$, and $\varepsilon(b * c) = \varepsilon(b)\varepsilon(c)$. The abelian group object unit, or zero, is $\zeta : R \to H_*(k)$, which is in CoAlg_R. If we define $[0_k] = \zeta(1) \neq 0$, then $[0_k] * b = b$. Finally, the conjugation $\chi : H_*(k) \to H_*(k)$ has $\chi\chi =$ identity and $\zeta\varepsilon(b) = \Sigma b' * \chi(b'')$. It is the abelian group object inverse.

3. The associative map
$$\circ : H_*(n) \otimes H_*(k) \to H_*(n + k)$$

gives the second product structure. This map is in CoAlg_R: $\psi(a \circ b) = \psi(a) \circ \psi(b) = \Sigma(a' \otimes a'') \circ (b' \otimes b'')$ and $\varepsilon(a \circ b) = \varepsilon(a)\varepsilon(b)$. Multiplication by zero gives $[0_n] \circ b = \zeta\varepsilon(b)$. There is a unit map $i : R \to H_*(0)$. If we define $i(1) = [1] \in H_0(0)$, then $[1] \circ b = b$. Defining $\chi([1]) = [-1] \in H_0(0)$ yields $\chi(a) = [-1] \circ a$ and $\chi(a \circ b) = \chi(a) \circ b = a \circ \chi(b)$.

The $\circ$-product also satisfies

(a) Commutativity:

$$a \circ b = [-1]^{\circ nk} \circ b \circ a = \chi^{nk}(b \circ a),$$

where $a \in H_i(n)$ and $b \in H_j(k)$.

(b) Distributivity:

$$a \circ (b * c) = \Sigma(a' \circ b) * (a'' \circ c).$$

In our notation we use a^2 instead of a^{*2} and bc instead of $b * c$.

3 Background, Notation, and Machinery

We quickly review the tools and notation used throughout the rest of this article.

3.1 The homotopy elements $[\eta], [\beta], [\lambda]$

It is well-documented that

$$KO_* = \mathbb{Z}[\eta, \beta, \lambda^{\pm 1}]/(\eta^3, 2\eta, \eta\beta, \beta^2 - 4\lambda)$$

and

$$bo_* = \mathbb{Z}[\eta, \beta, \lambda]/(\eta^3, 2\eta, \eta\beta, \beta^2 - 4\lambda) \subset KO_*,$$

where $\deg(\eta) = 1, \deg(\beta) = 4$, and $\deg(\lambda) = 8$. (Note that λ is the Bott element.) We would like to use these elements to simplify our calculations.

Following [4], suppose $\mathcal{S}$ is a homotopy category of topological spaces (with certain properties), and $F_*(-)$ is an associative commutative multiplicative unreduced generalized homology theory with unit defined on $\mathcal{S}$. If we let $G^*(-)$ be a similar cohomology theory (also defined on $\mathcal{S}$), then $G^*(-)$ has a representing Ω-spectrum

$$\underline{G}_* = \{\underline{G}_n\}_{n \in \mathbb{Z}} \in \mathrm{gr}\mathcal{S},$$

in other words,

$$G^n(X) \simeq [X, \underline{G}_n] \text{ and } \Omega\underline{G}_{n+1} \simeq \underline{G}_n$$

(with $\mathrm{gr}\mathcal{S}$ the category of graded objects of $\mathcal{S}$). Denoting the two coefficient rings by F_* and G^*, we let $x \in G^n$ have degree $-n$ in the coefficient ring. Then $x \in G^n \simeq [\mathrm{point}, \underline{G}_n]$ and so we have a map in homology $x_* : F_* \to F_*\underline{G}_n$. We define $[x] \in F_0\underline{G}_n$ to be the image of $1 \in F_*$ under this map.

If we take $z \in G^m$ and $x, y \in G^n$, then

1. $[z] \circ [x] = [zx] = [-1]^{\circ mn} \circ [x] \circ [z]$.
2. $[x] * [y] = [x + y] = [y + x] = [y] * [x]$.
3. $\psi[z] = [z] \otimes [z]$.
4. The sub-Hopf algebra of $F_*\underline{G}_*$ generated by all $[x]$ with $x \in G^*$ is the ring-ring of G^* over F_*, i.e. $F_*[G^*]$.

Application to the case $G = KO$ and $F_*(-) = H_*(-)$ yields the ring-ring elements

$$[\eta] \in H_0\underline{KO}_{-1}, [\beta] \in H_0\underline{KO}_{-4}, \text{ and } [\lambda] \in H_0\underline{KO}_{-8}.$$

We will need these homotopy elements throughout our calculations. In particular, the Bott element $[\lambda]$ will allow us to map down eight spaces, to facilitate the simplification of the elements in our Hopf ring.

3.2 The Bar Spectral Sequence and $\boldsymbol{Tor}$

Let E be an Ω-spectrum, and let $\underline{E}'_n$ be the component of the basepoint in the n-th space of E. The bar spectral sequence is really the homology Eilenberg-Moore spectral sequence for the path-loop fibration on the space $\underline{E}'_{n+1}$,

$$\begin{array}{ccccc} \Omega\underline{E}'_{n+1} = \Omega\underline{E}_{n+1} & \longrightarrow & \text{Paths } \underline{E}'_{n+1} & \longrightarrow & \underline{E}'_{n+1}. \\ \downarrow \cong & & \downarrow \cong & & \\ \underline{E}_n & & \star & & \end{array}$$

Based on the bar resolution, it is a spectral sequence of Hopf algebras with the basic property that

$$E^2_{s,t} = \mathrm{Tor}^{H_*(\underline{E}_n)}_{s,t}(\mathbb{Z}/2, \mathbb{Z}/2) \Rightarrow H_{s+t}(\underline{E}'_{n+1}; \mathbb{Z}/2).$$

Its differentials are given by

$$d_r : E^r_{s,t} \to E^r_{s-r,t+r-1}.$$

For more information, see [2].

For the rest of this paper, we will use the notations $P(x)$ and $E(x)$ to represent the polynomial and exterior algebras, respectively, on the generator x.

When R is a bicommutative Hopf algebra over $\mathbb{Z}/2$, then $\mathrm{Tor}^R_{**}(\mathbb{Z}/2, \mathbb{Z}/2)$ is again a bicommutative Hopf algebra over $\mathbb{Z}/2$. If I is the augmentation ideal, there is a natural isomorphism $\sigma : I/I^2 \to \mathrm{Tor}^R_{1,*}(\mathbb{Z}/2, \mathbb{Z}/2)$. Moreover, if $y \in I_k$ and $y^2 = 0$ there are naturally defined divided powers $\gamma_j(\sigma(y)) \in \mathrm{Tor}^R_{j,jk}$. The following computations for $\mathrm{Tor}^R_{*,*}(\mathbb{Z}/2, \mathbb{Z}/2)$ are well known (see [5]) and will be needed throughout our calculation:

1. $\mathrm{Tor}^{P(x)}_{*,*}(\mathbb{Z}/2,\mathbb{Z}/2) = E(\sigma(x))$, the exterior algebra on $\sigma(x) \in \mathrm{Tor}_{1,*}$.
2. $\mathrm{Tor}^{E(x)}_{*,*}(\mathbb{Z}/2,\mathbb{Z}/2) = \Gamma[\sigma(x)]$, the divided power algebra on $\sigma(x)$, which is spanned by the elements $\gamma_j(\sigma(x))$.
3. $\mathrm{Tor}^{\mathbb{Z}/2[\mathbb{Z}]}_{*,*}(\mathbb{Z}/2,\mathbb{Z}/2) = E(\sigma(x))$, where $\mathbb{Z}/2[\mathbb{Z}] = \mathbb{Z}/2(x,x^{-1})$ is the group ring of $\mathbb{Z}$.
4. $\mathrm{Tor}^A \otimes \mathrm{Tor}^B \xrightarrow{\simeq} \mathrm{Tor}^{A\otimes B}$.

3.3 Frobenius and Verschiebung Maps

Given A, a bicommutative Hopf algebra over $\mathbb{Z}/p$ (p prime), the Frobenius map $F : A \to A$ is described by $F(x) = x^p$. The Verschiebung map V is defined to be the dual of F on A^*, the dual of A. In other words, if $F' : A^* \to A^*$ is defined by $x \mapsto x^p$, then $(F')^* = V : A \to A$.

From [5], these maps have the following properties (for $p = 2$):

1. With a shift of grading, V and F are Hopf algebra maps.
2. $VF = FV$:
$$V(x^2) = VF(x) = FV(x) = [V(x)]^2.$$
3. $V(x \circ y) = V(x) \circ V(y)$
4. $F(x \circ V(y)) = F(x) \circ y$
5. For the coalgebra $\Gamma(x)$,
$$V(\gamma_{2q}(x)) = \gamma_q(x)$$
$$V(\gamma_q(x)) = 0 \text{ if } q \neq 0 \mod 2.$$

4 The Hopf Rings for $\overline{H}$ and KO

As stated in the introduction, we will compute the Hopf ring for *bo* as a sub-Hopf ring of $H_*\underline{KO}_n \otimes H_*\underline{\overline{H}}_n$. For reference and for ease of reading, we record the descriptions of both known components of this Hopf ring.

4.1 The Hopf ring for $\overline{H}$

The algebra generators of $H_*\underline{\overline{H}}_*$ for $k > 0$ can be written uniquely in the form

$$\beta_{(0)}^{\circ j_0} \circ \beta_{(1)}^{\circ j_1} \circ \beta_{(2)}^{\circ j_2} \circ \dots,$$

where $|\beta_{(i)}| = 2^i$. We order the generators lexicographically, according to the sequence of indices $(j_0, j_1, j_2, \ldots)$. Whenever we introduce a new generator of the Hopf ring $H_*\underline{bo}_*$, we (generally) evaluate the map

$$\Theta : H_*\underline{bo}_k \to H_*K(\mathbb{Z}, k) \to H_*\overline{\underline{H}}_k$$

on it, ignoring decomposables and retaining only the earliest indecomposable term in the ordering above. Thus for $k > 0$

$$H_*\overline{\underline{H}}_k = E\Big(\beta_{(i_1)} \circ \ldots \circ \beta_{(i_k)} : i_k \geq \ldots \geq i_2 \geq i_1 \geq 0\Big).$$

The case $k = 0$ gives the group ring, which may also be written $E([1]-1)$. Also, $H_*\overline{\underline{H}}_0$ has a basis consisting of 1 and [1], with $[1]^2 = 1$, and $H_*K(\mathbb{Z}, 0) = \mathbb{Z}[z_0, {z_0}^{-1}]$, where $z_0 = [1]$ and ${z_0}^{-1} = [-1]$. The map Θ is a morphism of Hopf rings — it preserves $*$-products, $\circ$-products, etc. and it induces morphisms of bar spectral sequences. We note that the suspension element of $H_*\overline{\underline{H}}_*$ is $\beta_{(0)}$. Please see [4] for more details.

4.2 The Hopf ring for *KO*

This Hopf ring was computed in detail in [3], and is the springboard for our calculations in *bo*. We will need a complete description of not only the Hopf ring for KO, but also the relations that arose throughout that calculation. Since our calculations for $H_*\underline{bo}_0$ through $H_*\underline{bo}_3$ are almost identical to the corresponding cases in $H_*\underline{KO}_*$, the reader will be able to gain insight into how the Hopf ring for KO originated. The main tool used to complete this calculation was the bar spectral sequence, which allows us to ascend from $H_*\underline{KO}_*$ to $H_*\underline{KO}_{*+1}$.

The notation and terminology for $H_*\underline{KO}_*$ are as follows: push forward the usual generators of $H_*\mathbb{R}P^\infty$ along the inclusion

$$\mathbb{R}P^\infty = BO(1) \subseteq \{1\} \times BO \subseteq \mathbb{Z} \times BO = \underline{KO}_0$$

to get elements $z_k \in H_k\underline{KO}_0$. We have the element $[1] = z_0 \in H_0\underline{KO}_0$ which allows us to define $\overline{z}_k = z_k/z_0 = z_k * [-1]$. Note that we will use $\overline{z}_k$ (rather than z_k) in order to obtain generators that lie in KO' rather than in KO. (Equivalently, $z_i = \overline{z}_i * [1]$.) As a consequence of property (5) from section 3.3, for the elements $z_i \in H_*\underline{KO}_0$ we have both $V(z_{2i}) = z_i$ and $V(z_{2i+1}) = 0$. The fundamental class in $H_1\underline{KO}_1$ is denoted e, and is also the suspension class. The element $1_1 = [0_1]$ is the unit for the $*$-product in $H_0\underline{KO}_1$.

As opposed to the case for *bo*, KO is periodic, so we only record the homology for the first seven spaces. Space-by-space, the Hopf ring for KO is:

$$
\begin{array}{lll}
H_*\underline{KO}_0 = & H_*(\mathbb{Z}\times BO) & = P(\overline{z}_0^{\pm 1})\otimes P(\overline{z}_i : i>0) \\
H_*\underline{KO}_1 = & H_*(U/O) & = P(e\circ z_{2i} : i\geq 0) \\
H_*\underline{KO}_2 = & H_*(Sp/U) & = P(e^{\circ 2}\circ z_{4i} : i\geq 0) \\
H_*\underline{KO}_3 = & H_*(Sp) & = E(e^{\circ 3}\circ z_{4i} : i\geq 0) \\
H_*\underline{KO}_4 = & H_*(\mathbb{Z}\times BSp) & = P([\beta\lambda^{-1}],[-\beta\lambda^{-1}])\otimes P(\overline{z}_{4i}\circ[\beta\lambda^{-1}] : i>0) \\
H_*\underline{KO}_5 = & H_*(U/Sp) & = E(e\circ z_{4i}\circ[\beta\lambda^{-1}] : i\geq 0) \\
H_*\underline{KO}_6 = & H_*(O/U) & = E(\overline{z}_{2i}\circ[\eta^2\lambda^{-1}] : i\geq 0) \\
H_*\underline{KO}_7 = & H_*(O) & = E(\overline{z}_i\circ[\eta\lambda^{-1}] : i\geq 0)
\end{array}
$$

Any other $H_*\underline{KO}_k$ can be filled in using Bott periodicity. For example,

$$H_*\underline{KO}_8 = H_*(\mathbb{Z}\times BO) = P([\lambda^{-1}],[-\lambda^{-1}])\otimes P(\overline{z}_i\circ[\lambda^{-1}] : i>0)\cong H_*\underline{KO}_0.$$

4.2.1 Hopf ring properties

We start with three particularly helpful lemmas from [3]. To illustrate the nature of Hopf ring calculations, which will be needed throughout this paper, we also include the lemmas' proofs.

Lemma 4.1. *For all elements a with $\varepsilon a = 0$,*

$$e\circ(a*[1]) = e\circ a.$$

In particular, for any $i>0$, $e\circ\overline{z}_i = e\circ z_i$.

Proof. Using distributivity we obtain

$$e\circ(a*[1]) = (e\circ a)*(1\circ[1]) + (1\circ a)*(e\circ[1]) = (e\circ a)*[0] + 0*(e\circ[1])$$
$$= e\circ a.$$

Thus $e\circ z_i = e\circ(\overline{z}_i*[1])$ is equivalent to $e\circ\overline{z}_i$ for all $i>0$. □

Lemma 4.2. *The Verschiebung map has both of the following properties:*

1. *The element $V^j\Big(\gamma_{2^j}(\sigma(z_i))\Big)$ is detected by the element $e\circ z_i$.*
2. *If $V^j(x) = z_i$, then $x = z_{2^j\cdot i}$, mod decomposable elements.*

Proof. 1. By virtue of the fact that $V\Big(\gamma_{2q}(x)\Big) = \gamma_q(x)$, we obtain

$$V^j\Big(\gamma_{2^j}(\sigma(z_i))\Big) = V^{j-1}\Big(\gamma_{2^{j-1}}(\sigma(z_i))\Big) = \ldots = \sigma(z_i),$$

which is detected by $e\circ z_i$.

2. The facts that $V(z_{2i}) = z_i$ and $V(z_{2i+1}) = 0$ may be used to prove this equation. □

Lemma 4.3. *For any element* $[x]$, $[x] \circ \overline{z}_i = 0$ *iff* $[x] \circ z_i = 0$.

Proof. Suppose that $[x] \circ \overline{z}_i = 0$. Then the distributive property shows

$$[x] \circ z_i = [x] \circ (\overline{z}_i * [1]) = ([x] \circ \overline{z}_i) * ([x] \circ [1]) = 0.$$

Conversely, suppose $[x] \circ z_i = 0$. Then

$$[x] \circ \overline{z}_i = [x] \circ (z_i * [-1]) = ([x] \circ z_i) * ([x] \circ [-1]) = 0.$$

$\square$

There are eleven additional Hopf ring relations from $H_*\underline{KO}_*$ which will be needed for our computation (although the last three relations are not needed explicitly):

1. $F(e^{\circ n}) = (e^{\circ n})^2 = e^{\circ n} \circ z_n$, which is only nonzero if z_n is $\circ$-indecomposable, i.e. if $n = 2^i$
2. $z_j \circ z_k = \frac{(j+k)!}{j!k!} z_{j+k}$
3. $e^{\circ 4} \circ [\lambda] = [\beta] \circ \overline{z}_4$
4. $\overline{z}_{2i+1} \circ [\beta] = \overline{z}_{4i+2} \circ [\beta] = 0$
5. $e^{\circ 2} \circ [\beta] = \overline{z}_2 \circ [\eta^2]$
6. $\overline{z}_{2i+1} \circ [\eta^2] = 0$
7. $e \circ [\eta] = \overline{z}_1$
8. $\overline{z}_1 \circ \overline{z}_{2i+1} = \overline{z}_1^2 \circ \overline{z}_{2i} = F(\overline{z}_1 \circ \overline{z}_i)$ for $i > 0$ and $\overline{z}_1 \circ \overline{z}_1 = F(\overline{z}_1)$
9. $\psi(z_i) = \sum_{j+k=i} z_j \otimes z_k$
10. $\psi(e) = e \otimes 1 + 1 \otimes e = e \otimes [0] + [0] \otimes e$
11. $e \circ$ (decomposable elements) $= 0$, where we call an element x decomposable if $x = ab$, with $\varepsilon a = \varepsilon b = 0$.

Even though the lemmas and properties were proven for the connective case, they still hold true in the non-connective case for *bo*. The proofs for the Hopf ring properties are similar to those of the lemmas; some involve the Frobenius and Verschiebung, some use a map $f : KU \to KO$ that forgets the complex structure, and others use the known structure in KO or $\mathbb{R}P^\infty$, since $\mathbb{R}P^\infty = BO(1) \subseteq \{1\} \times BO \subseteq \mathbb{Z} \times BO = \underline{KO}_0$.

5 The Calculation of the Hopf Ring for $H_*\underline{bo}_*$

We are ready to proceed with the calculation at hand. Before proving the general structure theorem, we must explicitly compute the answers in two cycles of eight spaces each. In the first cycle, we map down eight spaces using multiplication by $[\lambda]$ and simplify our answers in $H_*\underline{KO}_{n-8}$, since this multiplication is a monomorphism. Throughout each step we will compute the result of the map

$$\Theta : bo \to H \to \overline{H},$$

although this map will not be utilized until the second cycle. In the second cycle, multiplication by $[\lambda]$ will no longer be a monomorphism, so we will start using the map Θ along with this multiplication to complete our solutions.

We start with the established fact that

$$H_*\underline{bo}_0 = H_*(\mathbb{Z} \times BO) = H_*\underline{KO}_0 = P(z_0^{\pm 1}) \otimes P(\overline{z}_i : i > 0).$$

The map

$$\Theta : H_*\underline{bo}_0 \to H_*K(\mathbb{Z}, 0) \to H_*\underline{\overline{H}}_0$$

is clearly given by

$$\Theta(z_0) = [1], \Theta(z_0^{-1}) = [-1] = [1], \text{ and } \Theta(z_i) = 0 \text{ for } i > 0.$$

5.1 The first cycle

In $H_*\underline{bo}_1$ through $H_*\underline{bo}_8$ we show that multiplication by $[\lambda]$,

$$\circ[\lambda] : H_*\underline{bo}_n \to H_*\underline{bo}_{n-8} = H_*\underline{KO}_{n-8} \simeq H_*\underline{KO}_n$$

is a monomorphism. As the beginning steps are identical to those in the calculation for the Hopf ring of KO in [3], some of the preliminary details will be abbreviated.

In $H_*\underline{bo}_1, H_*\underline{bo}_2, H_*\underline{bo}_4$ and $H_*\underline{bo}_8$, we find it useful to search for the lowest 0 digit in the binary expansion of i, by writing

$$i = 1 + 2 + \ldots + 2^{m-1} + 2^{m+1}q = 2^m(2q+1) - 1,$$

where $m, q \geq 0$.

5.1.1 $H_*\underline{bo}_1$

We input $H_*\underline{bo}_0 = P(z_0^{\pm 1}) \otimes P(\overline{z}_i : i > 0) = H_*\underline{KO}_0$ into the bar spectral sequence:

$$E^2_{s,t} = \mathrm{Tor}_{s,t}^{H_*\underline{bo}_0}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathrm{Tor}_{*,*}^{P(z_0^{\pm 1}) \otimes P(\overline{z}_i : i>0)}(\mathbb{Z}/2, \mathbb{Z}/2)$$

$$= E(\sigma(z_0)) \otimes E(\sigma(\overline{z}_i) : i > 0) \Rightarrow H_*\underline{bo}_1.$$

Note that we have used the Tor-properties from section 3.2. The suspension $\sigma(x)$ lies in the first filtration of the bar spectral sequence. The generators of the E^2-term are all in $E^2_{1,*}$, collapsing the bar spectral sequence.

Recall that e is the fundamental class in $H_1\underline{KO}_1$. For any $x \in H_*\underline{E}_n, e \circ x \in H_*\underline{E}_{(n+1)}$ detects the image in E^∞ of $\sigma(x)$. Thus, using Lemma 4.1, $e \circ z_i = e \circ \overline{z}_i \in H_*\underline{bo}_1$ detects $\sigma(\overline{z}_i)$. (Note that $\sigma(\overline{z}_i) = \sigma(z_i)$ for $i > 0$.)

To simplify the E^∞ term as a Hopf ring, we use the first two Hopf ring relations from section 4.2.1 (with $n = 1$ in relation (1), i.e. $e^2 = e \circ z_1$) together with the Verschiebung properties $V(z_{2k}) = z_k$ and $F(x \circ V(y)) = F(x) \circ y$ to compute the following:

$$(e \circ z_k)^2 = F(e \circ z_k) = F(e) \circ z_{2k} = (e \circ z_1) \circ z_{2k} = e \circ z_{2k+1}.$$

We apply this calculation as often as possible by writing $i = 2^m(2q+1)-1$ as above; then $e \circ z_i = F^m(e \circ z_{2q})$ so each element $e \circ z_{2i}$ is a polynomial generator. We have

$$H_*\underline{bo}_1 = P\Big(e \circ z_{2i} : i \geq 0\Big).$$

(Note that as $\underline{bo}_n = \underline{KO}_n$ for $n \leq 3$, this result is exactly as in [3].)

The map Θ preserves the suspension element e, which for H is $\beta_{(0)}$.

5.1.2 $H_*\underline{bo}_2$

This case is similar to $H_*\underline{bo}_1$; the bar spectral sequence gives

$$E^2_{s,t} = \mathrm{Tor}^{H_*\underline{bo}_1}_{s,t}(\mathbb{Z}/2, \mathbb{Z}/2) = E\Big(\sigma(e \circ z_{2i})\Big) \Rightarrow H_*\underline{bo}_2.$$

Again, the bar spectral sequence collapses at the E^2-term. The first two Hopf ring relations from section 4.2.1 (now with $n = 2$) give

$$H_*\underline{bo}_2 = P\Big(e^{\circ 2} \circ z_{4i} : i \geq 0\Big).$$

5.1.3 $H_*\underline{bo}_3$

The bar spectral sequence is

$$E^2_{s,t} = \mathrm{Tor}^{H_*\underline{bo}_2}_{s,t}(\mathbb{Z}/2, \mathbb{Z}/2) = E\Big(\sigma(e^{\circ 2} \circ z_{4i})\Big) \Rightarrow H_*\underline{bo}_3.$$

As usual, the bar spectral sequence collapses at the E^2-term.

Again, $e^{\circ 3} \circ z_{4i} \in H_*\underline{bo}_3$ detects $\sigma(e^{\circ 2} \circ z_{4i})$. The first Hopf ring relation (with $n = 3$) gives $(e^{\circ 3})^2 = 0$. Hence $F(e^{\circ 3} \circ z_{4i}) = (F(e^{\circ 3})) \circ z_{8i} = 0$, so the elements $e^{\circ 3} \circ z_{4i}$ generate an exterior algebra:

$$H_*\underline{bo}_3 = E\Big(e^{\circ 3} \circ z_{4i} : i \geq 0\Big).$$

5.1.4 $H_*\underline{bo}_4$

It is at this point that our calculation begins to diverge from that of $H_*\underline{KO}_*$. The bar spectral sequence yields

$$E^2_{s,t} = \mathrm{Tor}^{H_*\underline{bo}_3}_{s,t}(\mathbb{Z}/2, \mathbb{Z}/2) = \Gamma\Big(\sigma(e^{\circ 3} \circ z_{4i}) : i \geq 0\Big) \Rightarrow H_*\underline{bo}_4.$$

Since all elements are in even total degree, the bar spectral sequence collapses at the E^2-term. As the generators of the divided power algebra are γ_{2^j}, we may write

$$E^\infty = \Gamma\Big(\sigma(e^{\circ 3} \circ z_{4i})\Big) = E\Big(\gamma_{2^j}(\sigma(e^{\circ 3} \circ z_{4i})) : i \geq 0, j \geq 0\Big).$$

We will simplify this exterior algebra in steps.

We start our simplification with the elements $e^{\circ 4} \circ z_{4i}$, which detect the elements $\sigma(e^{\circ 3} \circ z_{4i})$ from the first filtration. Hopf ring property (1) with $n = 4$ yields

$$F(e^{\circ 4} \circ z_{4k}) = F(e^{\circ 4}) \circ z_{8k} = (e^{\circ 4} \circ z_4) \circ z_{8k} = e^{\circ 4} \circ z_{8k+4}.$$

If we once again expand the integer i into its binary form $i = 2^m(2q+1) - 1$, and repeat this process as often as possible, we obtain $e^{\circ 4} \circ z_{4i} = F^m(e^{\circ 4} \circ z_{8q})$. Therefore the elements $e^{\circ 4} \circ z_{8i}$ generate a polynomial subalgebra of $H_*\underline{bo}_4$.

Next, we turn our attention to all the elements in all filtrations. Suppose $x \in H_*\underline{bo}_4$ in filtration 2^j detects $\gamma_{2^j}\Big(\sigma(e^{\circ 3} \circ z_{8q})\Big)$. Since we have $V^j(x)$ detecting $V^j\Big(\gamma_{2^j}(\sigma(e^{\circ 3} \circ z_{8q}))\Big) = \sigma(e^{\circ 3} \circ z_{8q})$, we must have $V^j x = e^{\circ 4} \circ z_{8q}$ as both these elements are in filtration 1. Further,

$$V^j F^m x = F^m(e^{\circ 4} \circ z_{8q}) = e^{\circ 4} \circ z_{4i}$$

detects $\sigma(e^{\circ 3} \circ z_{4i})$ so $F^m x$ detects $\gamma_{2^j}\Big(\sigma(e^{\circ 3} \circ z_{4i})\Big)$ mod $\ker(V^j)$. Thus the elements x (as j and q vary) are polynomial generators of $H_*\underline{bo}_4$.

It remains for us to identify the generating elements x. To accomplish this, we consider the image of $H_*\underline{bo}_4$ under the map $\circ[\lambda]$ in the known algebra $H_*\underline{bo}_{-4} = H_*\underline{KO}_{-4} = P(\overline{z}_{4i} \circ [\beta] : i > 0) \otimes P([\beta], [\beta]^{-1})$. We use Hopf ring properties (2), (3), and (4) from section 4.2.1; then

$$V^j(x \circ [\lambda]) = e^{\circ 4} \circ z_{8q} \circ [\lambda] = [\beta] \circ \overline{z}_4 \circ z_{8q} = [\beta] \circ \overline{z}_{8q+4} + \text{decomposables}.$$

Application of Lemma 4.2 proves that

$$x \circ [\lambda] = \overline{z}_{2^j(8q+4)} \circ [\beta] + \text{decomposables}.$$

We denote the element $x + *$−decomposables by $x + D(*)$. We therefore write $x = \overline{z}_{2^j(8q+4)} \circ [\beta\lambda^{-1}] + D(*) \in H_*\underline{KO}_4$. Hopf ring relations (3) and (4) note that $\overline{z}_{2k+1} \circ [\beta] = \overline{z}_{4k+2} \circ [\beta] = 0$. Since any positive integer divisible by 4 has the form $2^j(8q+4)$

uniquely, the image of $H_*\underline{bo}_4$ is the whole polynomial ring $P(\overline{z}_{4i} \circ [\beta] : i > 0)$ and $\circ[\lambda]$ is therefore one-to-one.

We introduce the notation $\alpha(i)$; the expression $\alpha(i) = k$ means that $i = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_k}$, where $i_1 < i_2 < \ldots < i_k$. It counts the number of 1's in the binary expansion of i. Its significance is that $z_i = z_{2^{i_1}} \circ z_{2^{i_2}} \circ \ldots \circ z_{2^{i_k}}$, a k-fold $\circ$-product. We also use the notation $\alpha(i) = 0$ to mean $i = 0$.

Thus we write formally

$$\begin{aligned} H_*\underline{bo}_4 &= P\Big(\overline{z}_{4i} \circ [\beta\lambda^{-1}] : \alpha(i) \geq 1\Big) \\ &\subset H_*\underline{KO}_4 = P\Big([\beta\lambda^{-1}], [-\beta\lambda^{-1}]\Big) \otimes P\Big(\overline{z}_{4i} \circ [\beta\lambda^{-1}] : \alpha(i) \geq 1\Big). \end{aligned}$$

Note that $[\beta\lambda^{-1}]$ itself is not an element of $H_*\underline{bo}_4$.

We turn our attention to the map Θ. We clearly have

$$\Theta(\overline{z}_4 \circ [\beta\lambda^{-1}]) = \Theta(e^{\circ 4}) = {\beta_{(0)}}^{\circ 4}.$$

Since $V^i(\overline{z}_{2^{i+2}}) = \overline{z}_4$ and $V^i(\beta_{(i)}) = \beta_{(0)}$, we deduce

$$\Theta(\overline{z}_{2^{i+2}} \circ [\beta\lambda^{-1}]) = \beta_{(i)}^{\circ 4} + \ldots$$

where the unstated terms involve $\beta_{(j)}$ with $j < i$.

5.1.5 $H_*\underline{bo}_5$

The input of $H_*\underline{bo}_4$ into the bar spectral sequence gives

$$E^2_{s,t} = \mathrm{Tor}^{H_*\underline{bo}_4}_{s,t}(\mathbb{Z}/2, \mathbb{Z}/2) = E\Big(\sigma(\overline{z}_{4i} \circ [\beta\lambda^{-1}]) : \alpha(i) \geq 1\Big) \Rightarrow H_*\underline{bo}_5.$$

Once again the suspension $\sigma(x)$ lies in the first filtration, which collapses the bar spectral sequence at the E^2-term.

We use Lemmas 4.1 and 4.3 and relation (4) from section 4.2.1 to simplify $E^\infty = E\Big(\sigma(\overline{z}_{4i} \circ [\beta\lambda^{-1}]) : \alpha(i) \geq 1\Big)$. The element $e \circ \overline{z}_{4i} \circ [\beta\lambda^{-1}]$ detects $\sigma(\overline{z}_{4i} \circ [\beta\lambda^{-1}])$. We have

$$F(e \circ \overline{z}_{4i} \circ [\beta\lambda^{-1}]) = F(e \circ z_{4i} \circ [\beta\lambda^{-1}]) = F(e) \circ z_{8i} \circ [\beta\lambda^{-1}] = (e \circ z_1) \circ z_{8i} \circ [\beta] \circ [\lambda^{-1}] = 0.$$

We have thus found

$$H_*\underline{bo}_5 = E\Big(e \circ z_{4i} \circ [\beta\lambda^{-1}] : \alpha(i) \geq 1\Big).$$

Note that there is an injection from $H_*\underline{bo}_5$ to

$$H_*\underline{bo}_{-3} = H_*\underline{KO}_{-3} = E\Big(e \circ z_{4i} \circ [\beta]\Big).$$

We also have

$$\Theta(e \circ \overline{z}_{2^{i+2}} \circ [\beta\lambda^{-1}]) = \beta_{(0)} \circ \beta_{(i)}^{\circ 4} + \ldots$$

5.1.6 $H_*\underline{bo}_6$

Since $H_*\underline{bo}_5 = E\Big(e \circ z_{4i} \circ [\beta\lambda^{-1}] : \alpha(i) \geq 1\Big)$, the bar spectral sequence is

$$E^2_{s,t} = \mathrm{Tor}^{H_*\underline{bo}_5}_{s,t}(\mathbb{Z}/2, \mathbb{Z}/2) = \Gamma\Big(\sigma(e \circ z_{4i} \circ [\beta\lambda^{-1}]) : \alpha(i) \geq 1\Big) \Rightarrow H_*\underline{bo}_6.$$

Upon rewriting $\Gamma\Big(\sigma(e \circ z_{4i} \circ [\beta\lambda^{-1}]) : \alpha(i) \geq 1\Big)$ in the equivalent form $E\Big(\gamma_{2^j}(\sigma(e \circ z_{4i} \circ [\beta\lambda^{-1}])) : \alpha(i) \geq 1, j \geq 0\Big)$, we find that all elements lie in even total degree. Thus the bar spectral sequence collapses at the E^2-term, so

$$E^\infty = E\Big(\gamma_{2^j}(\sigma(e \circ z_{4i} \circ [\beta\lambda^{-1}])) : j \geq 0, \alpha(i) \geq 1\Big).$$

As with $H_*\underline{bo}_4$, the E^∞-term needs to be simplified in stages. We start with the elements in the first filtration, noting that $e^{\circ 2} \circ z_{4i} \circ [\beta\lambda^{-1}]$ detects $\sigma(e \circ z_{4i} \circ [\beta\lambda^{-1}])$. We use Hopf ring property (5) from section 4.2.1 to rewrite

$$e^{\circ 2} \circ z_{4k} \circ [\beta\lambda^{-1}] = \overline{z}_2 \circ z_{4k} \circ [\eta^2\lambda^{-1}] = \overline{z}_{4k+2} \circ [\eta^2\lambda^{-1}] + \text{decomposables}.$$

Thus the first filtration yields an exterior algebra with generators $\overline{z}_{4i+2} \circ [\eta^2\lambda^{-1}] +$ decomposables, as $H_*\underline{bo}_{-2}$ is an exterior algebra. Note that $\overline{z}_{2i+1} \circ [\eta^2\lambda^{-1}] = 0$.

Next, we examine the elements in γ_{2^j}. Suppose $x \in H_*\underline{bo}_6$ detects the exterior algebra generator $\gamma_{2^j}\Big(\sigma(e \circ z_{4i} \circ [\beta\lambda^{-1}])\Big)$. Lemma 4.2 shows that

$$V^j x = e^{\circ 2} \circ \overline{z}_{4i} \circ [\beta\lambda^{-1}] = \overline{z}_{4i+2} \circ [\eta^2\lambda^{-1}] + \text{decomposables}.$$

Thus, modulo decomposables, $x = \overline{z}_{2^j(4i+2)} \circ [\eta^2\lambda^{-1}]$. As every number $2k$ with $\alpha(k) \geq 2$ can be written uniquely in the form $2^j(4i+2)$ with $i > 0$, we have $x = \overline{z}_{2k} \circ [\eta^2\lambda^{-1}] + D(*)$. The application of Hopf ring property (6) from section 4.2.1 completes this simplification;

$$H_*\underline{bo}_6 = E\Big(\overline{z}_{2i} \circ [\eta^2\lambda^{-1}] + D(*) : \alpha(i) \geq 2\Big).$$

Clearly,

$$\Theta(\overline{z}_2 \circ \overline{z}_{2^{i+2}} \circ [\eta^2\lambda^{-1}]) = \Theta(e^{\circ 2} \circ \overline{z}_{2^{i+2}} \circ [\beta\lambda^{-1}] + D(*)) = \beta^{\circ 2}_{(0)} \circ \beta^{\circ 4}_{(i)} + \ldots$$

Since Θ commutes with V^j and $\overline{z}_2 \circ \overline{z}_{2^{i+2}} = \overline{z}_{2+2^{i+2}} + D(*)$, we deduce

$$\Theta\Big(\overline{z}_{2^{j+1}+2^{i+j+2}} \circ [\eta^2\lambda^{-1}]\Big) = \beta^{\circ 2}_{(j)} \circ \beta^{\circ 4}_{(i+j)} + \ldots$$

We rewrite this as

$$\Theta\Big(\overline{z}_{2^{i_1+1}+2^{i_2+2}} \circ [\eta^2\lambda^{-1}]\Big) = \beta^{\circ 2}_{(i_1)} \circ \beta^{\circ 4}_{(i_2)} + \ldots$$

for $i_2 \geq i_1 \geq 0$.

5.1.7 $H_*\underline{bo}_7$

Since $H_*\underline{bo}_6 = E\Big(\overline{z}_{2i} \circ [\eta^2\lambda^{-1}] + D(*) : \alpha(i) \geq 2\Big)$, the bar spectral sequence gives

$$E^2_{s,t} = \mathrm{Tor}^{H_*\underline{bo}_6}_{s,t}(\mathbb{Z}/2, \mathbb{Z}/2) = \Gamma\Big(\sigma(\overline{z}_{2i} \circ [\eta^2\lambda^{-1}]) : \alpha(i) \geq 2\Big) \Rightarrow H_*\underline{bo}_7.$$

We may again rewrite $\Gamma\Big(\sigma(\overline{z}_{2i} \circ [\eta^2\lambda^{-1}]) : \alpha(i) \geq 2\Big)$ as $E\Big(\gamma_{2^j}(\sigma(\overline{z}_{2i} \circ [\eta^2\lambda^{-1}])) : \alpha(i) \geq 2, j \geq 0\Big)$.

The suspension $\sigma(x)$ lies in the first filtration, but the elements $\gamma_{2^j}(x)$ are in the 2^j-th filtration. Thus, for the moment, we gain no information about the behavior of the bar spectral sequence.

As usual, $e \circ \overline{z}_{2i} \circ [\eta^2\lambda^{-1}]$ detects $\sigma(\overline{z}_{2i} \circ [\eta^2\lambda^{-1}])$. We now use Hopf ring relation (7) from section 4.2.1 to write this class of elements as

$$e \circ \overline{z}_{2i} \circ [\eta^2\lambda^{-1}] = \overline{z}_1 \circ \overline{z}_{2i} \circ [\eta\lambda^{-1}] = \overline{z}_{2i+1} \circ [\eta\lambda^{-1}],$$

modulo decomposables. Again we apply $\circ[\lambda]$ to map $H_*\underline{bo}_7$ into the known algebra

$$H_*\underline{bo}_{-1} = H_*\underline{KO}_{-1} = E\Big(\overline{z}_i \circ [\eta] : i > 0\Big) \otimes E\Big([\eta] - 1\Big).$$

Since the morphism of spectral sequences induced by $\lambda : \underline{bo}_7 \to \underline{bo}_{-1}$ is clearly a monomorphism on the E^2-terms, and the latter spectral sequence collapses, the spectral sequence for $\underline{bo}_7$ also collapses.

As always for E^∞-terms that begin as divided power algebras, it remains to simplify in stages. Suppose that $x \in H_*\underline{bo}_7$ detects $\gamma_{2^j}\Big(\sigma(\overline{z}_{2i} \circ [\eta^2\lambda^{-1}])\Big)$. Then

$$V^j x = e \circ \overline{z}_{2i} \circ [\eta^2\lambda^{-1}] = \overline{z}_1 \circ \overline{z}_{2i} \circ [\eta\lambda^{-1}] = \overline{z}_{2i+1} \circ [\eta\lambda^{-1}] + D(*)$$

and

$$V^j(x \circ [\lambda]) = \overline{z}_{2i+1} \circ [\eta] + D(*)$$

Hence, we must have $x \circ [\lambda] = \overline{z}_{2^j(2i+1)} \circ [\eta] +$ decomposables, where $\alpha(i) \geq 2$. Every number k with $\alpha(k) \geq 3$ can be written uniquely in the form $2^j(2i+1)$ with $\alpha(i) \geq 2$. Thus $H_*\underline{bo}_7 \to H_*\underline{bo}_{-1}$ is monic and we write

$$H_*\underline{bo}_7 = E\Big(\overline{z}_i \circ [\eta\lambda^{-1}] + D(*) : \alpha(i) \geq 3\Big).$$

Clearly,

$$\Theta\Big(\overline{z}_1 \circ \overline{z}_{2^{i_1+1}+2^{i_2+2}} \circ [\eta\lambda^{-1}]\Big) = \Theta\Big(e \circ \overline{z}_{2^{i_1+1}+2^{i_2+2}} \circ [\eta^2\lambda^{-1}]\Big) = \beta_{(0)} \circ \beta^{\circ 2}_{(i_1)} \circ \beta^{\circ 4}_{(i_2)} + \ldots$$

for $i_2 \geq i_1$. Since Θ commutes with V^j, we deduce

$$\Theta\Big(\overline{z}_{2^j+2^{j+i_1+1}+2^{j+i_2+2}} \circ [\eta\lambda^{-1}]\Big) = \beta_{(j)} \circ \beta^{\circ 2}_{(j+i_1)} \circ \beta^{\circ 4}_{(j+i_2)} + \ldots$$

For any i with $\alpha(i) = 3$, we therefore have, after reindexing,

$$\Theta(\overline{z}_i \circ [\eta\lambda^{-1}]) = \beta_{(i_1)} \circ \beta^{\circ 2}_{(i_2)} \circ \beta^{\circ 4}_{(i_3)} + \ldots$$

where $i = 2^{i_1} + 2^{i_2+1} + 2^{i_3+2}$ is the binary expansion of i with $i_3 \geq i_2 \geq i_1$.

5.1.8 $H_*\underline{bo}_8$

We complete the first cycle of eight, reaching $\underline{bo}_8$. As $H_*\underline{bo}_7 = E\Big(\overline{z}_i \circ [\eta\lambda^{-1}] + D(*) : \alpha(i) \geq 3\Big)$, the bar spectral sequence yields

$$E^2_{s,t} = \mathrm{Tor}^{H_*\underline{bo}_7}_{s,t}(\mathbb{Z}/2, \mathbb{Z}/2) = \Gamma\Big(\sigma(\overline{z}_i \circ [\eta\lambda^{-1}]) : \alpha(i) \geq 3\Big) \Rightarrow H_*\underline{bo}_8.$$

We note that $\Gamma\Big(\sigma(\overline{z}_i \circ [\eta\lambda^{-1}]) : \alpha(i) \geq 3\Big)$ is equivalent to $E\Big(\gamma_{2^j}(\sigma(\overline{z}_i \circ [\eta\lambda^{-1}])) : \alpha(i) \geq 3, j \geq 0\Big)$. To see that the bar spectral sequence collapses, apply $\circ[\lambda]$ to map to the known bar spectral sequence for H_*BO, which does collapse.

The element $\sigma(\overline{z}_i \circ [\eta\lambda^{-1}])$ is once again detected by $e \circ \overline{z}_i \circ [\eta\lambda^{-1}]$. As usual, we will start our simplification with elements in the first filtration. Using Hopf ring property (7) from section 4.2.1 and applying $\circ[\lambda]$ in order to work in $\underline{bo}_0$, we can rewrite $e \circ \overline{z}_i \circ [\eta\lambda^{-1}] \circ [\lambda]$ as $\overline{z}_1 \circ \overline{z}_i$. Hopf ring property (8) from section 4.2.1 gives

$$\overline{z}_1 \circ \overline{z}_{2k+1} = \overline{z}_1^2 \circ \overline{z}_{2k} = F(\overline{z}_1) \circ \overline{z}_{2k} = F(\overline{z}_1 \circ \overline{z}_k)$$

when $k > 0$. In the case that $k = 0$, we have

$$\overline{z}_1 \circ \overline{z}_1 = F(\overline{z}_1).$$

We apply this relation as often as possible, by expanding each integer in binary form and looking for the lowest 0 digit: write $i = 2^m(2q+1) - 1$, where $m, q \geq 0$ and $\alpha(i) \geq 3$. Then for $q > 0$

$$\overline{z}_1 \circ \overline{z}_i = F^m(\overline{z}_1 \circ \overline{z}_{2q}) = F^m(\overline{z}_{2q+1}) + F^m D(*) \tag{1}$$

For $q = 0$ we have simply $\overline{z}_1 \circ \overline{z}_i = F^m(\overline{z}_1)$. Note that $\alpha(i) = \alpha(q) + m$ and $\alpha(2q+1) = \alpha(q) + 1$.

In the spectral sequence for $\underline{bo}_8$, suppose x detects $\gamma_{2^j}(\sigma(\overline{z}_i \circ [\eta\lambda^{-1}]))$. Then $V^j(x)$ detects $\sigma(\overline{z}_i \circ [\eta\lambda^{-1}])$ and $V^j(x \circ [\lambda])$ detects $\sigma(\overline{z}_i \circ [\eta])$. Since this lies in filtration one, we have

$$V^j(x \circ [\lambda]) = e \circ \overline{z}_i \circ [\eta] = F^m(\overline{z}_{2q+1}) + F^m D(*).$$

Then $x \circ [\lambda] = F^m(\overline{z}_r) + F^m D(*)$, where $r = 2^j(2q+1)$. Since $\alpha(r) + m = \alpha(i) + 1$, we do not get all pairs (m, r), only those for which $\alpha(r) + m \geq 4$. So $\overline{z}_r$ lifts to $\underline{bo}_8$ only if $\alpha(r) \geq 4$; if $\alpha(r) = 4 - k$, where $k > 0$, $F^k(\overline{z}_r) + F^k D(*)$ lifts to $\underline{bo}_8$. Thus

$$\begin{aligned} H_*\underline{bo}_8 &= P\Big(\overline{z}_i \circ [\lambda^{-1}] + D(*) : \alpha(i) \geq 4\Big) \\ &\quad \otimes P\Big(F^j(\overline{z}_i \circ [\lambda^{-1}]) + F^j D(*) : \alpha(i) + j = 4, i, j \geq 1\Big). \end{aligned}$$

We need the value of Θ on $F^m(\overline{z}_k \circ [\lambda^{-1}]) + F^m D(*)$ whenever $\alpha(k) + m = 4$ (including the case $m = 0$). As above, we write $k = 2^j(2q+1)$, so that $\alpha(k) = \alpha(q)+1$. Then mod decomposables

$$V^j\Big(F^m(\overline{z}_k \circ [\lambda^{-1}])\Big) = F^m(\overline{z}_{2q+1} \circ [\lambda^{-1}]) = \overline{z}_1 \circ \overline{z}_i \circ [\lambda^{-1}] = e \circ \overline{z}_i \circ [\eta\lambda^{-1}],$$

where

$$i = 2^m(2q+1) - 1 = (2^m - 1) + 2^{m+1}q,$$

so that $\alpha(i) = m + \alpha(q) = m + \alpha(k) - 1 = 3$. Thus we let

$$i = 2^{i_1} + 2^{i_2+1} + 2^{i_3+2}$$

be the binary expansion of i, where $i_3 \geq i_2 \geq i_1$. We know

$$\Theta(e \circ \overline{z}_i \circ [\eta\lambda^{-1}]) = \beta_{(0)} \circ \beta_{(i_1)} \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 4} + \dots$$

Since Θ commutes with V, we must have

$$\Theta\Big(F^m(\overline{z}_k \circ [\lambda^{-1}])\Big) = \beta_{(j)} \circ \beta_{(j+i_1)} \circ \beta_{(j+i_2)}^{\circ 2} \circ \beta_{(j+i_3)}^{\circ 4} + \dots$$

We reindex as

$$\Theta\Big(F^j(\overline{z}_i \circ [\lambda^{-1}])\Big) = \beta_{(i_1)} \circ \beta_{(i_2)} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4} + \dots$$

whenever $\alpha(i) + j = 4$. Here, we write $i = 2^{i_1}(2q+1)$, where 2^{i_1} is the largest power of 2 that divides i, and use the binary expansion

$$\alpha(2^j i - 2^{i_1}) = \alpha\Big((2^{j+i_1} - 2^{i_1}) + 2^{j+i_1+1}q\Big) = j + \alpha(q) = j + \alpha(i) - 1 = 3.$$

Thus

$$2^j i - 2^{i_1} = 2^{i_2} + 2^{i_3+1} + 2^{i_4+2},$$

where $i_4 \geq i_3 \geq i_2 \geq i_1$. We note that $i_2 > i_1$ if $j = 0$, but that $i_2 = i_1$ if $j > 0$.

5.2 The second cycle

From $H_*\underline{bo}_9$ and on, $\circ[\lambda]$ will no longer be a monomorphism. Instead, we treat $H_*\underline{bo}_n$ as a sub-Hopf algebra of $H_*\underline{bo}_{n-8} \otimes H_*\underline{\overline{H}}_n \subset H_*\underline{KO}_n \otimes H_*\underline{\overline{H}}_n$ by means of $\circ[\lambda]$ and Θ. To check that the bar spectral sequence for $H_*\underline{bo}_n$ collapses, it is only necessary to verify that the map $\underline{bo}_{n-1} \to \underline{KO}_{n-1} \times \underline{\overline{H}}_{n-1}$ induces a monomorphism on the E^2-terms, thus embedding the spectral sequence in one that is known to collapse. This will be clear in practice. We make frequent appeal to the structure of $H_*\underline{bo}_{n-8}$. Each factor of $H_*\underline{bo}_{n-1}$ will lead to one or more factors of $H_*\underline{bo}_n$.

5.2.1 $H_*\underline{bo}_9$

Since $H_*\underline{bo}_8$ is known, the bar spectral sequence is

$$\begin{aligned} E^2_{s,t} &= \mathrm{Tor}_{s,t}^{H_*\underline{bo}_8}(\mathbb{Z}/2, \mathbb{Z}/2) \\ &= E\Big(\sigma(\overline{z}_i \circ [\lambda^{-1}]) : \alpha(i) \geq 4\Big) \\ &\quad \otimes E\Big(\sigma(F^j(\overline{z}_i \circ [\lambda^{-1}])) : \alpha(i) + j = 4, i, j \geq 1\Big) \\ &\Rightarrow H_*\underline{bo}_9. \end{aligned}$$

This is the last step in which we will explicitly state the bar spectral sequence.

As usual, $e\circ z_i\circ[\lambda^{-1}]+D(*) = e\circ\overline{z}_i\circ[\lambda^{-1}]+D(*)$ detects $\sigma(\overline{z}_i\circ[\lambda^{-1}])$. We use the relation $e\circ z_{2k+1} = F(e\circ z_k)$ as often as possible, as in $H_*\underline{bo}_1$. Write $i = 2^m(2q+1)-1$ where $\alpha(i) = m+\alpha(q) \geq 4$; then

$$e\circ z_i\circ[\lambda^{-1}] + D(*) = F^m(e\circ z_{2q}\circ[\lambda^{-1}]) + F^m D(*)$$

detects $\sigma(\overline{z}_i\circ[\lambda^{-1}])$. Thus if $\alpha(q)\geq 4$, the element $e\circ z_{2q}\circ[\lambda^{-1}]+D(*)$ is a polynomial generator of $H_*\underline{bo}_9$. But if $\alpha(q) < 4$ the polynomial generator is $F^m(e\circ z_{2q}\circ[\lambda^{-1}]) + F^m D(*)$. instead, with $m = 4-\alpha(q)$. This takes care of the first exterior algebra in $E^2 = E^\infty$.

The method of applying $\circ[\lambda]$ *fails* for the second exterior algebra. Although $F^j(\overline{z}_i\circ[\lambda^{-1}]) + F^j D(*)$ is a generator of $H_*\underline{bo}_8$, it becomes decomposable in $H_*\underline{bo}_0$ and $\circ[\lambda]$ annihilates $\sigma\Big(F^j(\overline{z}_i\circ[\lambda^{-1}])\Big)$ at the E^2-level. Instead, we apply the morphism Θ of bar spectral sequences to the bar spectral sequence for $H_*\overline{\underline{H}}_9$. Suppose $x\in H_*\underline{bo}_9$ is the element that detects $\sigma\Big(F^j(\overline{z}_i\circ[\lambda^{-1}])\Big)$; then $x\circ[\lambda]=0$, and by $H_*\underline{bo}_8$, $\Theta(x)$ detects $\sigma(\beta_{(i_1)}\circ\beta_{(i_2)}\circ\beta_{(i_3)}^{\circ 2}\circ\beta_{(i_4)}^{\circ 4}+\ldots)$ and therefore is

$$\Theta(x) = \beta_{(0)}\circ\beta_{(i_1)}\circ\beta_{(i_2)}\circ\beta_{(i_3)}^{\circ 2}\circ\beta_{(i_4)}^{\circ 4}+\ldots$$

Also, since $j>0$, we have $i_2 = i_1$ here. Hence $\circ[\lambda]$ and Θ define a monomorphism

$$H_*\underline{bo}_9 \to H_*\underline{KO}_9\otimes H_*\overline{\underline{H}}_9 = H_*(\underline{KO}_9\times\overline{\underline{H}}_9)$$

which makes it clear that $x^2 = 0$, as $H_*\overline{\underline{H}}_9$ is an exterior algebra. We therefore treat $H_*\underline{bo}_9$ as a subalgebra of $H_*\underline{KO}_9\otimes H_*\overline{\underline{H}}_9$, and label those generators that map trivially to $H_*\underline{KO}_9$ by their images under Θ instead. Thus, we write

$$\begin{aligned} H_*\underline{bo}_9 &= P\Big(e\circ z_{2i}\circ[\lambda^{-1}] : \alpha(i)\geq 4\Big)\\ &\otimes P\Big(F^j(e\circ z_{2i}\circ[\lambda^{-1}]) + F^j D(*) : \alpha(i)+j = 4, j\geq 1\Big)\\ &\otimes E\Big(\beta_{(i_1)}\circ\beta_{(i_2)}^{\circ 2}\circ\beta_{(i_3)}^{\circ 2}\circ\beta_{(i_4)}^{\circ 4}+\ldots : i_4\geq i_3\geq i_2\geq i_1 = 0\Big). \end{aligned}$$

The generators of the third factor are already defined by their images under Θ. We need the images of the generators of the form $F^j(e\circ z_{2i}\circ[\lambda^{-1}]) + F^j D(*)$ with $\alpha(i)+j = 4$ (including the case $j=0$). We again use $F^j(e\circ z_{2i}) = e\circ z_k$, where $k = 2^j(2i+1)-1$, and note that $\alpha(k) = j+\alpha(i) = 4$. By $H_*\underline{bo}_8$,

$$\begin{aligned} \Theta\Big(F^j(e\circ z_{2i}\circ[\lambda^{-1}]) + F^j D(*)\Big) &= \Theta(e\circ z_k\circ[\lambda^{-1}] + D(*))\\ &= \beta_{(i_1)}\circ\beta_{(i_2)}\circ\beta_{(i_3)}\circ\beta_{(i_4)}^{\circ 2}\circ\beta_{(i_5)}^{\circ 4}+\ldots \end{aligned}$$

with indices defined by the binary expansion $k = 2^{i_2}+2^{i_3}+2^{i_4+1}+2^{i_5+2}$, with $i_5\geq i_4\geq i_3 > i_2\geq i_1 = 0$. Since $i_3 > i_2$, this is different from all the generators in the third factor of $H_*\underline{bo}_9$.

5.2.2 $H_*\underline{bo}_{10}$

The first factor in E^2 is

$$E\Big(\sigma(e \circ z_{2i} \circ [\lambda^{-1}]) : \alpha(i) \geq 4\Big),$$

where $e^{\circ 2} \circ z_{2i} \circ [\lambda^{-1}]$ detects $\sigma(e \circ z_{2i} \circ [\lambda^{-1}])$. As in $H_*\underline{bo}_2$ we write $i = 2^m(2q+1) - 1$ where $\alpha(i) = m + \alpha(q) \geq 4$; then

$$e^{\circ 2} \circ z_{2i} \circ [\lambda^{-1}] = F^m(e^{\circ 2} \circ z_{4q} \circ [\lambda^{-1}])$$

detects $\sigma(e \circ z_{2i} \circ [\lambda^{-1}])$. As in $H_*\underline{bo}_9$ (with indices doubled), we deduce polynomial generators $e^{\circ 2} \circ z_{4q} \circ [\lambda^{-1}]$ (with $\alpha(q) \geq 4$) and $F^m(e^{\circ 2} \circ z_{4q} \circ [\lambda^{-1}])$ (with $\alpha(q) + m = 4, m \geq 1$) in $H_*\underline{bo}_{10}$.

The second factor in E^2 is

$$E\Big(\sigma(F^j(e \circ z_{2i} \circ [\lambda^{-1}])) : \alpha(i) + j = 4, j \geq 1\Big),$$

which is annihilated by $\circ[\lambda]$, so we therefore apply Θ instead. By $H_*\underline{bo}_9$,

$$\begin{aligned} \Theta\Big(\sigma(e \circ z_k \circ [\lambda^{-1}]) + D(*)\Big) &= \Theta\Big(\sigma(F^j(e \circ z_{2i} \circ [\lambda^{-1}]))\Big) \\ &= \sigma(\beta_{(0)} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\circ 2} \circ \beta_{(i_5)}^{\circ 4} + \ldots) \end{aligned}$$

where we use the binary expansion

$$k = 2^j(2i+1) - 1 = 2^{i_2} + 2^{i_3} + 2^{i_4+1} + 2^{i_5+2}$$

with $i_5 \geq i_4 \geq i_3 > i_2$. Since $j > 0$, k is odd and $i_2 = 0$. This element is therefore detected by $\beta_{(0)}^{\circ 3} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\circ 2} \circ \beta_{(i_5)}^{\circ 4} + \ldots$

We rewrite the third factor

$$\Gamma\Big(\sigma(\beta_{(i_1)} \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4}) : i_4 \geq i_3 \geq i_2 \geq i_1 = 0\Big)$$

in E^2 as

$$E\Big(\gamma_{2^j}(\sigma(\beta_{(0)} \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4})) : i_4 \geq i_3 \geq i_2 \geq 0, j \geq 0\Big).$$

The generator shown is thus detected by

$$\beta_{(j)} \circ \beta_{(j)} \circ \beta_{(j+i_2)}^{\circ 2} \circ \beta_{(j+i_3)}^{\circ 2} \circ \beta_{(j+i_4)}^{\circ 4} \ + \ \ldots$$

Generally, a factor

$$E(\beta_{(j_1)} \circ \ldots \circ \beta_{(j_m)})$$

in $H_*\underline{bo}_n$ gives a factor

$$\Gamma\Big(\sigma(\beta_{(j_1)} \circ \ldots \circ \beta_{(j_m)})\Big)$$

in E^2. This yields the factor

$$E\Big(\beta_{(j)} \circ \beta_{(j+j_1)} \circ \ldots \circ \beta_{(j+j_m)} : j \geq 0\Big)$$

in $H_*\underline{bo}_{n+1}$, which we reindex. In future, we use this without comment.

Therefore,

$$\begin{aligned} H_*\underline{bo}_{10} &= P\Big(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-1}] : \alpha(i) \geq 4\Big) \\ &\otimes P\Big(F^j(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-1}]) + F^j D(*) : \alpha(i) + j = 4, j \geq 1\Big) \\ &\otimes E\Big(\beta_{(i_1)}^{\circ 3} \circ \beta_{(i_2)} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4} : i_4 \geq i_3 \geq i_2 > i_1 = 0\Big) \\ &\otimes E\Big(\beta_{(i_1)}^{\circ 2} \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4} : i_4 \geq i_3 \geq i_2 \geq i_1 \geq 0\Big), \end{aligned}$$

and we again have a monomorphism

$$H_*\underline{bo}_{10} \to H_*\underline{KO}_2 \otimes H_*\underline{\overline{H}}_{10} = H_*(\underline{KO}_{10} \times \underline{\overline{H}}_{10}).$$

We need to compute $\Theta\Big(F^j(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-1}])\Big)$ whenever $\alpha(i) + j = 4$ and $i, j \geq 0$. Since

$$F^j(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-1}]) = e^{\circ 2} \circ z_{2k} \circ [\lambda^{-1}] = e \circ (e \circ z_{2k} \circ [\lambda^{-1}]),$$

where $k = 2^j(2i+1) - 1$ (thus giving $\alpha(k) = j + \alpha(i) = 4$), $H_*\underline{bo}_9$ gives

$$\Theta\Big(F^j(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-1}])\Big) = \beta_{(i_1)}^{\circ 2} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\circ 2} \circ \beta_{(i_5)}^{\circ 4} + \ldots,$$

where we use the binary expansion $2k = 2^{i_2} + 2^{i_3} + 2^{i_4+1} + 2^{i_5+2}$ with indices $i_5 \geq i_4 \geq i_3 > i_2 > i_1 = 0$. Again, we observe that the conditions on the indices ensure that this element differs from all the previously mentioned generators of $H_*\underline{bo}_{10}$. (In the future, we suppress any mention of this point.)

5.2.3 $H_*\underline{bo}_{11}$

The factor $P\Big(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-1}] : \alpha(i) \geq 4\Big)$ in $H_*\underline{bo}_{10}$ leads to

$$E\Big(e^{\circ 3} \circ z_{4i} \circ [\lambda^{-1}] : \alpha(i) \geq 4\Big),$$

just as in $H_*\underline{bo}_3$.

The factor

$$P\Big(F^j(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-1}]) + F^j D(*) : \alpha(i) + j = 4, j \geq 1\Big)$$

gives rise to the factor

$$E\Big(\sigma(F^j(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-1}])) : \alpha(i) + j = 4, j \geq 1\Big)$$

in E^2. However, $F^j(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-1}])$ decomposes in $H_*\underline{bo}_2$ and we must apply Θ instead. By $H_*\underline{bo}_{10}$, $\sigma\Big(F^j(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-1}])\Big)$ is detected by

$$\beta_{(0)} \circ (\beta_{(0)}^{\circ 2} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\circ 2} \circ \beta_{(i_5)}^{\circ 4}),$$

where $i_5 \geq i_4 \geq i_3 > i_2$. Since $j > 0$, we have $i_2 = 1$.

The two exterior factors are handled as usual.

Thus

$$\begin{aligned} H_*\underline{bo}_{11} \;=\; & E\Big(e^{\circ 3} \circ z_{4i} \circ [\lambda^{-1}] : \alpha(i) \geq 4\Big) \\ & \otimes E\Big(\beta_{(i_1)}^{\circ 3} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\circ 2} \circ \beta_{(i_5)}^{\circ 4} + \ldots : \\ & \qquad i_5 \geq i_4 \geq i_3 > i_2 = 1, i_1 = 0\Big) \\ & \otimes E\Big(\beta_{(i_1)}^{\circ 4} \circ \beta_{(i_2)} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4} + \ldots : i_4 \geq i_3 \geq i_2 > i_1 \geq 0\Big) \\ & \otimes E\Big(\beta_{(i_1)} \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 2} \circ \beta_{(i_5)}^{\circ 4} + \ldots : \\ & \qquad i_5 \geq i_4 \geq i_3 \geq i_2 \geq i_1 \geq 0\Big). \end{aligned}$$

We need to evaluate $\Theta(e^{\circ 3} \circ z_{4i} \circ [\lambda^{-1}])$ when $\alpha(i) = 4$. By factoring $e^{\circ 3} \circ z_{4i} \circ [\lambda^{-1}] = e^{\circ 3} \circ (z_{4i} \circ [\lambda^{-1}])$, we see from $H_*\underline{bo}_8$ that

$$\Theta(e^{\circ 3} \circ z_{4i} \circ [\lambda^{-1}]) = \beta_{(0)}^{\circ 3} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\circ 2} \circ \beta_{(i_5)}^{\circ 4} + \ldots$$

where we use the binary expansion $4i = 2^{i_2} + 2^{i_3} + 2^{i_4+1} + 2^{i_5+2}$, with $i_5 \geq i_4 \geq i_3 > i_2 \geq 2$.

The notations $A(s)$ and $C(n, k)$

Now we define some notation for the basic families of elements. We assume implicitly that $i_m \geq i_n \geq 0$ whenever $m > n$. We define

$$A(s) = \text{the set of all } \beta_{(i_1)} \circ \beta_{(i_2)} \circ \ldots \circ \beta_{(i_s)}, \text{ for } s \geq 1,$$
$$A(0) = [1]$$

and inductively

$$C(n, k) = \text{the set of all } \beta_{(i_k)} \circ \beta_{(i_{k+1})} \circ \beta_{(i_{k+2})}^{\circ 2} \circ \beta_{(i_{k+3})}^{\circ 4} \circ C(n-1, k+4),$$

where $i_{k+1} > i_k \geq i_{k-1} + 3$, starting from $C(0, k) = [1]$. Thus $C(n, k)$ depends on i_{k-1} as well.

Thus we rewrite

$$\begin{aligned} H_*\underline{bo}_{11} \;=\; & E\Big(e^{\circ 3} \circ z_{4i} \circ [\lambda^{-1}] : \alpha(i) \geq 4\Big) \\ & \otimes E\Big(\beta_{(i_1)}^{\circ 3} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\circ 2} \circ \beta_{(i_5)}^{\circ 4} + \ldots : i_3 > i_2 = 1, i_1 = 0\Big) \\ & \otimes E\Big(\beta_{(i_1)}^{\circ 4} \circ \beta_{(i_2)} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4} + \ldots : i_2 > i_1\Big) \\ & \otimes E\Big(A(1) \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 2} \circ \beta_{(i_5)}^{\circ 4} + \ldots\Big). \end{aligned}$$

5.2.4 $H_*\underline{bo}_{12}$

Only the first factor of $H_*\underline{bo}_{11}$ requires any discussion. It gives rise to the factor

$$\Gamma\Big(\sigma(e^{\circ 3}\circ z_{4i}\circ[\lambda^{-1}]) : \alpha(i)\geq 4\Big)$$

in the E^2-term, which we rewrite as

$$E\Big(\gamma_{2^j}(\sigma(e^{\circ 3}\circ z_{4i}\circ[\lambda^{-1}])) : \alpha(i)\geq 4, j\geq 0\Big).$$

By $H_*\underline{bo}_4$, $\gamma_{2^j}\Big(\sigma(e^{\circ 3}\circ z_{4i}\circ[\lambda^{-1}])\Big)$ is detected by the element written formally as $F^m\Big(\overline{z}_{2^j(8q+4)}\circ[\beta\lambda^{-2}]\Big)+F^m D(*)\in H_*\underline{bo}_{12}$, where as before we write $i=2^m(2q+1)-1$, so that $\alpha(i)=m+\alpha(q)\geq 4$.

Therefore

$$\begin{aligned} H_*\underline{bo}_{12} &= P\Big(\overline{z}_{4i}\circ[\beta\lambda^{-2}] : \alpha(i)\geq 5\Big)\\ &\otimes P\Big(F^j(\overline{z}_{4i}\circ[\beta\lambda^{-2}])+F^jD(*) : \alpha(i)+j=5, i,j\geq 1\Big)\\ &\otimes E\Big(\beta^{\circ 4}_{(i_1)}\circ\beta_{(i_2)}\circ\beta_{(i_3)}\circ\beta^{\circ 2}_{(i_4)}\circ\beta^{\circ 4}_{(i_5)}+\ldots : i_3>i_2=i_1+1\Big)\\ &\otimes E\Big(A(1)\circ\beta^{\circ 4}_{(i_2)}\circ\beta_{(i_3)}\circ\beta^{\circ 2}_{(i_4)}\circ\beta^{\circ 4}_{(i_5)}+\ldots : i_3>i_2\Big)\\ &\otimes E\Big(A(2)\circ\beta^{\circ 2}_{(i_3)}\circ\beta^{\circ 2}_{(i_4)}\circ\beta^{\circ 2}_{(i_5)}\circ\beta^{\circ 4}_{(i_6)}+\ldots\Big). \end{aligned}$$

From $H_*\underline{bo}_{11}$,

$$\Theta\Big(e^{\circ 4}\circ z_{4i}\circ[\lambda^{-1}]\Big)=\beta^{\circ 4}_{(0)}\circ\beta_{(i_2)}\circ\beta_{(i_3)}\circ\beta^{\circ 2}_{(i_4)}\circ\beta^{\circ 4}_{(i_5)}$$

whenever $\alpha(i)=4$, where we use the binary expansion

$$4i=2^{i_2}+2^{i_3}+2^{i_4+1}+2^{i_5+2},$$

so that $i_3>i_2\geq 2$. By $H_*\underline{bo}_4$, we can rewrite

$$e^{\circ 4}\circ z_{4i}\circ[\lambda^{-1}]=F^m(\overline{z}_{8q+4}\circ[\beta\lambda^{-2}])+F^mD(*)$$

where $i=2^m(2q+1)-1$, so that $\alpha(8q+4)+m=\alpha(q)+1+m=\alpha(i)+1=5$. Since Θ commutes with V^j, we deduce that

$$\Theta\Big(F^m(\overline{z}_{2^j(8q+4)}\circ[\beta\lambda^{-2}])+F^mD(*)\Big)=\beta^{\circ 4}_{(j)}\circ\beta_{(j+i_2)}\circ\beta_{(j+i_3)}\circ\beta^{\circ 2}_{(j+i_4)}\circ\beta^{\circ 4}_{(j+i_5)}+\ldots$$

We reindex this as

$$\Theta\Big(F^j(\overline{z}_{4i}\circ[\beta\lambda^{-2}])+F^jD(*)\Big)=\beta^{\circ 4}_{(i_1)}\circ\beta_{(i_2)}\circ\beta_{(i_3)}\circ\beta^{\circ 2}_{(i_4)}\circ\beta^{\circ 4}_{(i_5)}+\ldots,$$

where 2^{i_1} is the largest power of 2 that divides i, which we have set equal to $2^{i_1}(2q+1)$, as above, and we use the binary expansion

$$2^j\cdot 4i-2^{i_1+2}=2^{i_2}+2^{i_3}+2^{i_4+1}+2^{i_5+2},$$

where $i_3>i_2\geq i_1+2$. We note that $i_2=i_1+2$ if $j>0$, and that $i_2\geq i_1+3$ if $j=0$.

5.2.5 $H_*\underline{bo}_{13}$

By $H_*\underline{bo}_5$, the factor

$$P\Big(\overline{z}_{4i} \circ [\beta\lambda^{-2}] + D(*) : \alpha(i) \geq 5\Big)$$

in $H_*\underline{bo}_{12}$ yields the factor

$$E\Big(e \circ \overline{z}_{4i} \circ [\beta\lambda^{-2}] : \alpha(i) \geq 5\Big)$$

in $H_*\underline{bo}_{13}$.

The factor

$$P\Big(F^j(\overline{z}_{4i} \circ [\beta\lambda^{-2}]) + F^j D(*) : \alpha(i) + j = 5, i, j \geq 1\Big)$$

in $H_*\underline{bo}_{12}$ yields the factor

$$E\Big(\sigma(F^j(\overline{z}_{4i} \circ [\beta\lambda^{-2}])) : \alpha(i) + j = 5, i, j \geq 1\Big)$$

in the E^2-term. As in $H_*\underline{bo}_{11}$, the generators map trivially to $H_*\underline{bo}_5$ and we must apply Θ instead. By $H_*\underline{bo}_{12}$, the generator shown is detected by

$$\beta_{(0)} \circ \beta_{(i_1)}^{\circ 4} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\circ 2} \circ \beta_{(i_5)}^{\circ 4},$$

where $i_3 > i_2 = i_1 + 2$ (since $j > 0$).

Thus,

$$\begin{aligned} H_*\underline{bo}_{13} \;=\; & E\Big(e \circ z_{4i} \circ [\beta\lambda^{-2}] : \alpha(i) \geq 5\Big) \\ & \otimes E\Big(\beta_{(i_1)} \circ \beta_{(i_2)}^{\circ 4} \circ \beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta_{(i_5)}^{\circ 2} \circ \beta_{(i_6)}^{\circ 4} + \ldots : \\ & \qquad i_4 > i_3 = i_2 + 2, i_1 = 0\Big) \\ & \otimes E\Big(A(1) \circ \beta_{(i_2)}^{\circ 4} \circ \beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta_{(i_5)}^{\circ 2} \circ \beta_{(i_6)}^{\circ 4} + \ldots : i_4 > i_3 = i_2 + 1\Big) \\ & \otimes E\Big(A(2) \circ \beta_{(i_3)}^{\circ 4} \circ \beta_{(i_4)} \circ \beta_{(i_5)}^{\circ 2} \circ \beta_{(i_6)}^{\circ 4} + \ldots : i_4 > i_3\Big) \\ & \otimes E\Big(A(3) \circ \beta_{(i_4)}^{\circ 2} \circ \beta_{(i_5)}^{\circ 2} \circ \beta_{(i_6)}^{\circ 2} \circ \beta_{(i_7)}^{\circ 4} + \ldots\Big). \end{aligned}$$

We need to know $\Theta(e \circ \overline{z}_{4i} \circ [\beta\lambda^{-2}])$ whenever $\alpha(i) = 5$. By decomposing within $H_*\underline{bo}_*$, we see from $H_*\underline{bo}_5$ and $H_*\underline{bo}_8$ that

$$\Theta(e \circ z_{4i} \circ [\beta\lambda^{-2}]) = \Big(\beta_{(0)} \circ \beta_{(i_2)}^{\circ 4}\Big) \circ \Big(\beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta_{(i_5)}^{\circ 2} \circ \beta_{(i_6)}^{\circ 4}\Big) + \ldots$$

where we use the binary expansion

$$4i = 2^{i_2+2} + 2^{i_3} + 2^{i_4} + 2^{i_5+1} + 2^{i_6+2},$$

so that $i_4 > i_3 \geq i_2 + 3$.

5.2.6 $H_*\underline{bo}_{14}$

Now we can use much of $H_*\underline{bo}_6$ to find $H_*\underline{bo}_{14}$.

Thus

$$
\begin{aligned}
H_*\underline{bo}_{14} \quad = \quad & E\Big(\overline{z}_{2i} \circ [\eta^2\lambda^{-2}] : \alpha(i) \geq 6\Big) \\
& \otimes E\Big(\beta_{(i_1)}^{\circ 2} \circ \beta_{(i_2)}^{\circ 4} \circ \beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta_{(i_5)}^{\circ 2} \circ \beta_{(i_6)}^{\circ 4} + \ldots : i_4 > i_3 = i_2 + 2\Big) \\
& \otimes E\Big(A(2) \circ \beta_{(i_3)}^{\circ 4} \circ \beta_{(i_4)} \circ \beta_{(i_5)} \circ \beta_{(i_6)}^{\circ 2} \circ \beta_{(i_7)}^{\circ 4} + \ldots : i_5 > i_4 = i_3 + 1\Big) \\
& \otimes E\Big(A(3) \circ \beta_{(i_4)}^{\circ 4} \circ \beta_{(i_5)} \circ \beta_{(i_6)}^{\circ 2} \circ \beta_{(i_7)}^{\circ 4} + \ldots : i_5 > i_4\Big) \\
& \otimes E\Big(A(4) \circ \beta_{(i_5)}^{\circ 2} \circ \beta_{(i_6)}^{\circ 2} \circ \beta_{(i_7)}^{\circ 2} \circ \beta_{(i_8)}^{\circ 4} + \ldots\Big).
\end{aligned}
$$

We need to know $\Theta(\overline{z}_{2i} \circ [\eta^2\lambda^{-2}] + D(*))$, whenever $\alpha(i) = 6$. Again we can factor within $H_*\underline{bo}_*$ and read off from $H_*\underline{bo}_6$ and $H_*\underline{bo}_8$ that

$$
\Theta(\overline{z}_{2i} \circ [\eta^2\lambda^{-2}] + D(*)) = \beta_{(i_1)}^{\circ 2} \circ \beta_{(i_2)}^{\circ 4} \circ \beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta_{(i_5)}^{\circ 2} \circ \beta_{(i_6)}^{\circ 4} + \ldots,
$$

where we use the binary expansion

$$
2i = 2^{i_1+1} + 2^{i_2+2} + 2^{i_3} + 2^{i_4} + 2^{i_5+1} + 2^{i_6+2},
$$

so that $i_4 > i_3 \geq i_2 + 3$.

5.2.7 $H_*\underline{bo}_{15}$

We use everything we learned from $H_*\underline{bo}_7$ to find $H_*\underline{bo}_{15}$. Thus

$$
\begin{aligned}
H_*\underline{bo}_{15} \quad = \quad & E\Big(\overline{z}_{i} \circ [\eta\lambda^{-2}] : \alpha(i) \geq 7\Big) \\
& \otimes E\Big(A(1) \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 4} \circ \beta_{(i_4)} \circ \beta_{(i_5)} \circ \beta_{(i_6)}^{\circ 2} \circ \beta_{(i_7)}^{\circ 4} + \ldots : \\
& \qquad i_5 > i_4 = i_3 + 2\Big) \\
& \otimes E\Big(A(3) \circ \beta_{(i_4)}^{\circ 4} \circ \beta_{(i_5)} \circ \beta_{(i_6)} \circ \beta_{(i_7)}^{\circ 2} \circ \beta_{(i_8)}^{\circ 4} + \ldots : i_6 > i_5 = i_4 + 1\Big) \\
& \otimes E\Big(A(4) \circ \beta_{(i_5)}^{\circ 4} \circ \beta_{(i_6)} \circ \beta_{(i_7)}^{\circ 2} \circ \beta_{(i_8)}^{\circ 4} + \ldots : i_6 > i_5\Big) \\
& \otimes E\Big(A(5) \circ \beta_{(i_6)}^{\circ 2} \circ \beta_{(i_7)}^{\circ 2} \circ \beta_{(i_8)}^{\circ 2} \circ \beta_{(i_9)}^{\circ 4} + \ldots\Big).
\end{aligned}
$$

For $\alpha(i) = 7$ (by using $H_*\underline{bo}_7$ and $H_*\underline{bo}_8$), we have

$$
\Theta(\overline{z}_{i} \circ [\eta\lambda^{-2}]) = \beta_{(i_1)} \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 4} \circ \beta_{(i_4)} \circ \beta_{(i_5)} \circ \beta_{(i_6)}^{\circ 2} \circ \beta_{(i_7)}^{\circ 4} + \ldots,
$$

where we use the binary expansion

$$
i = 2^{i_1} + 2^{i_2+1} + 2^{i_3+2} + 2^{i_4} + 2^{i_5} + 2^{i_6+1} + 2^{i_7+2},
$$

so that $i_5 > i_4 \geq i_3 + 3$.

5.3 The structure theorem

The general pattern should now be apparent.

Theorem 5.3.1. *The Hopf ring $H_*\underline{bo}_*$ is a sub-Hopf ring of $H_*(\underline{KO}_* \times \overline{\underline{H}}_*)$ and is the (graded) tensor product of the following four families of Hopf algebras:*

1. *Polynomial and exterior subalgebras of $H_*\underline{KO}_*$:*

$$
\begin{aligned}
&P\Big(\overline{z}_i \circ [\lambda^{-n}] + D(*) : i > 0, \alpha(i) \geq 4n\Big), \text{ on } \underline{bo}_{8n}\\
&P\Big([\lambda^{-n}], [\lambda^{-n}]^{-1}\Big), \text{ on } \underline{bo}_{8n}, \text{ for } n \leq 0\\
&P\Big(e \circ z_{2i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n\Big), \text{ on } \underline{bo}_{8n+1}\\
&P\Big(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n\Big), \text{ on } \underline{bo}_{8n+2}\\
&E\Big(e^{\circ 3} \circ z_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n\Big), \text{ on } \underline{bo}_{8n+3}\\
&P\Big(\overline{z}_{4i} \circ [\beta\lambda^{-(n+1)}] + D(*) : i > 0, \alpha(i) \geq 4n+1\Big), \text{ on } \underline{bo}_{8n+4}\\
&P\Big([\beta\lambda^{-n}], [\beta\lambda^{-n}]^{-1}\Big), \text{ on } \underline{bo}_{8n-4}, \text{ for } n \leq 0\\
&E\Big(e \circ z_{4i} \circ [\beta\lambda^{-(n+1)}] : \alpha(i) \geq 4n+1\Big), \text{ on } \underline{bo}_{8n+5}\\
&E\Big(\overline{z}_{2i} \circ [\eta^2\lambda^{-(n+1)}] + D(*) : i > 0, \alpha(i) \geq 4n+2\Big), \text{ on } \underline{bo}_{8n+6}\\
&E\Big([\eta^2\lambda^{-n}] - 1\Big), \text{ on } \underline{bo}_{8n-2}, \text{ for } n \leq 0\\
&E\Big(\overline{z}_i \circ [\eta\lambda^{-(n+1)}] + D(*) : i > 0, \alpha(i) \geq 4n+3\Big), \text{ on } \underline{bo}_{8n+7}\\
&E\Big([\eta\lambda^{-n}] - 1\Big), \text{ on } \underline{bo}_{8n-1}, \text{ for } n \leq 0.
\end{aligned}
$$

2. *Polynomial algebras on generators that decompose in $H_*\underline{KO}_*$, companions to the polynomial algebras in the first family:*

$$
\begin{aligned}
&P\Big(F^j(\overline{z}_i \circ [\lambda^{-n}]) + F^jD(*) : \alpha(i) + j = 4n, i, j \geq 1\Big), \text{ on } \underline{bo}_{8n}\\
&P\Big(F^j(e \circ z_{2i} \circ [\lambda^{-n}]) + F^jD(*) : \alpha(i) + j = 4n, j \geq 1\Big), \text{ on } \underline{bo}_{8n+1}\\
&P\Big(F^j(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-n}]) + F^jD(*) : \alpha(i) + j = 4n, j \geq 1\Big), \text{ on } \underline{bo}_{8n+2}\\
&P\Big(F^j(\overline{z}_{4i} \circ [\beta\lambda^{-(n+1)}]) + F^jD(*) : \alpha(i) + j = 4n+1, i, j \geq 1\Big), \text{ on } \underline{bo}_{8n+4}.
\end{aligned}
$$

3. *Exterior algebras involving $\beta_{(0)}$ that arise from the second family:*

$$
\begin{aligned}
&E\Big(\beta_{(0)} \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4} \circ C(n,5) + \ldots\Big), \text{ on } \underline{bo}_{8n+9}\\
&E\Big(\beta_{(0)}^{\circ 3} \circ \beta_{(i_2)} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4} \circ C(n,5) + \ldots : i_2 > 0\Big), \text{ on } \underline{bo}_{8n+10}\\
&E\Big(\beta_{(0)}^{\circ 3} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\circ 2} \circ \beta_{(i_5)}^{\circ 4} \circ C(n,6) + \ldots : i_3 > i_2 = 1\Big), \text{ on } \underline{bo}_{8n+11}\\
&E\Big(\beta_{(0)} \circ \beta_{(i_2)}^{\circ 4} \circ \beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta_{(i_5)}^{\circ 2} \circ \beta_{(i_6)}^{\circ 4} \circ C(n,7) + \ldots : i_4 > i_3 = i_2 + 2\Big),\\
&\quad \text{on } \underline{bo}_{8n+13}.
\end{aligned}
$$

4. General exterior algebras that arise from the third family by unlimited suspension:

$$E\Big(A(s)\circ\beta^{\circ 2}_{(i_{s+1})}\circ\beta^{\circ 2}_{(i_{s+2})}\circ\beta^{\circ 2}_{(i_{s+3})}\circ\beta^{\circ 4}_{(i_{s+4})}\circ C(n,s+5)+\dots\Big)$$
$$E\Big(A(s)\circ\beta^{\circ 4}_{(i_{s+1})}\circ\beta_{(i_{s+2})}\circ\beta^{\circ 2}_{(i_{s+3})}\circ\beta^{\circ 4}_{(i_{s+4})}\circ C(n,s+5)+\dots : i_{s+2}>i_{s+1}\Big)$$
$$E\Big(A(s)\circ\beta^{\circ 4}_{(i_{s+1})}\circ\beta_{(i_{s+2})}\circ\beta_{(i_{s+3})}\circ\beta^{\circ 2}_{(i_{s+4})}\circ\beta^{\circ 4}_{(i_{s+5})}\circ C(n,s+6)+\dots : i_{s+3}>i_{s+2}=i_{s+1}+1\Big)$$
$$E\Big(A(s)\circ\beta^{\circ 2}_{(i_{s+1})}\circ\beta^{\circ 4}_{(i_{s+2})}\circ\beta_{(i_{s+3})}\circ\beta_{(i_{s+4})}\circ\beta^{\circ 2}_{(i_{s+5})}\circ\beta^{\circ 4}_{(i_{s+6})}\circ C(n,s+7)+\dots : i_{s+4}>i_{s+3}=i_{s+2}+2\Big).$$

Proof. We have the result for $n\le 1$. For $n\le 0$, only the first family of Hopf algebras exists, and for $k\le 0$, $H_*\underline{bo}_k = H_*\underline{KO}_k$. We note that there is no conflict between the many classes of generators we have exhibited. By induction, we assume that $H_*\underline{bo}_k$ is a subalgebra of $H_*\underline{KO}_k\otimes H_*\underline{\overline{H}}_k$, as stated. The bar spectral sequence for $H_*\underline{bo}_{k+1}$ collapses, because the bar spectral sequence for $H_*(\underline{KO}_{k+1}\times\underline{\overline{H}}_{k+1})$ is known to collapse; to see this, we only need to verify that the inclusion induces a monomorphism on the E^2-terms.

Each listed Hopf algebra in $H_*\underline{bo}_k$ gives rise to one or two Hopf algebras in $H_*\underline{bo}_{k+1}$. In $H_*\underline{bo}_k$ there is exactly one algebra from the first family. It is a subalgebra of $H_*\underline{KO}_k$, and we see from the steps already given how it gives rise to a first family algebra in $H_*\underline{bo}_{k+1}$, with the appropriate restriction on $\alpha(i)$. In addition, in half the cases it spawns a polynomial algebra in the second family, with the appropriate condition on $\alpha(i)+j$.

We treat each algebra of the third or fourth families in $H_*\underline{bo}_k$ as a subalgebra of $H_*\underline{\overline{H}}_k$, and we have already noted how such an algebra gives rise to an algebra of the fourth family in $H_*\underline{bo}_{k+1}$.

The only case that requires any discussion is how an algebra from the second family in $H_*\underline{bo}_k$ gives rise to an algebra from the third family in $H_*\underline{bo}_{k+1}$. Because the generators decompose in $H_*\underline{KO}_k$, we must map them into $H_*\underline{\overline{H}}_k$ instead. Thus we need to compute $\Theta\Big(F^j(\overline{z}_i\circ[\lambda^{-n}])\Big)$, etc.

We need $\Theta\Big(F^m(\overline{z}_k\circ[\lambda^{-n}])+F^mD(*)\Big)$ whenever $\alpha(k)+m=4n$ and $m\ge 1$. We write $k=2^j(2q+1)$, so that $\alpha(k)=\alpha(q)+1$. By $H_*\underline{bo}_8$, we have

$$V^j\Theta\Big(F^m(\overline{z}_k\circ[\lambda^{-n}])\Big)=\Theta(e\circ\overline{z}_i\circ[\eta\lambda^{-n}]),$$

where

$$i=2^m(2q+1)-1=(2^m-1)+2^{m+1}q,$$

so that i is odd and $\alpha(i)=\alpha(q)+m=4n-1$. We can evaluate this by factoring $e\circ\overline{z}_i\circ[\eta\lambda^{-n}]$ in $H_*\underline{bo}_*$. Let

$$i=(2^{i_1}+2^{i_2+1}+2^{i_3+2})+(2^{i_4}+2^{i_5}+2^{i_6+1}+2^{i_7+2})+\dots$$

$$+(2^{i_{4n-4}}+2^{i_{4n-3}}+2^{i_{4n-2}+1}+2^{i_{4n-1}+2})$$

be the binary expansion of i, so that $i_1=0$, $i_5>i_4\geq i_3+3$, $i_9>i_8\geq i_7+3,\ldots$ Then

$$e\circ\overline{z}_i\circ[\eta\lambda^{-n}]=e\circ\Big(\overline{z}_{s_1}\circ[\eta\lambda^{-1}]\Big)\circ\Big(\overline{z}_{s_2}\circ[\lambda^{-1}]\Big)\circ\ldots\circ\Big(\overline{z}_{s_n}\circ[\lambda^{-1}]\Big),$$

where $s_1=2^{i_1}+2^{i_2+1}+2^{i_3+2}$, $s_2=2^{i_4}+2^{i_5}+2^{i_6+1}+2^{i_7+2}$, etc. By $H_*\underline{bo}_7$ and $H_*\underline{bo}_8$,

$$\begin{aligned}\Theta(e\circ\overline{z}_i\circ[\eta\lambda^{-n}]) &= \beta_{(0)}\circ\Big(\beta_{(0)}\circ\beta_{(i_2)}^{\circ 2}\circ\beta_{(i_3)}^{\circ 4}\Big)\circ\Big(\beta_{(i_4)}\circ\beta_{(i_5)}\circ\beta_{(i_6)}^{\circ 2}\circ\beta_{(i_7)}^{\circ 4}\Big)\circ\ldots\\ &= \beta_{(0)}^{\circ 2}\circ\beta_{(i_2)}^{\circ 2}\circ\beta_{(i_3)}^{\circ 4}\circ c(n-1,4),\end{aligned}$$

where $c(n-1,4)$ denotes a typical generator of $C(n-1,4)$. Then

$$\Theta\Big(F^m(\overline{z}_k\circ[\lambda^{-n}])\Big)=\beta_{(j)}^{\circ 2}\circ\beta_{(j+i_2)}^{\circ 2}\circ\beta_{(j+i_3)}^{\circ 4}\circ c'(n-1,4)+\ldots,$$

where $c'(n-1,4)$ denotes $c(n-1,4)$ with all indices raised by j.

In the bar spectral sequence for $H_*\underline{bo}_{8n+1}$, the polynomial algebra

$$P\Big(F^m(\overline{z}_k\circ[\lambda^{-n}])+F^mD(*):\alpha(k)+m=4n\Big)$$

gives the factor

$$E\Big(\sigma(\beta_{(i_1)}^{\circ 2}\circ\beta_{(i_2)}^{\circ 2}\circ\beta_{(i_3)}^{\circ 4}\circ C(n-1,4))\Big)$$

in the E^2-term, which corresponds to

$$E\Big(\beta_{(0)}\circ\beta_{(i_2)}^{\circ 2}\circ\beta_{(i_3)}^{\circ 2}\circ\beta_{(i_4)}^{\circ 4}\circ C(n-1,5)\Big)$$

in $H_*\underline{bo}_{8n+1}$.

Next, we need $\Theta\Big(F^j(e\circ z_{2i}\circ[\lambda^{-n}])+F^jD(*)\Big)$ for $\alpha(i)+j=4n, j\geq 1$. By $H_*\underline{bo}_1$, $F^j(e\circ z_{2i})=e\circ z_k$, where $k=2^j(2i+1)-1=(2^j-1)+2^{j+1}i$, so that $\alpha(k)=j+\alpha(i)=4n$. Again we factor $e\circ z_k\circ[\lambda^{-n}]$ in $H_*\underline{bo}_*$ and find

$$\Theta\Big(F^j(e\circ z_{2i}\circ[\lambda^{-n}])\Big)=\beta_{(0)}\circ\Big(\beta_{(i_1)}\circ\beta_{(i_2)}\circ\beta_{(i_3)}^{\circ 2}\circ\beta_{(i_4)}^{\circ 4}\Big)\circ\Big(\beta_{(i_5)}\circ\ldots\Big)\circ\ldots,$$

where we use the binary expansion

$$k=(2^{i_1}+2^{i_2}+2^{i_3+1}+2^{i_4+2})+(2^{i_5}+2^{i_6}+2^{i_7+1}+2^{i_8+2})+\ldots$$

Since $j\geq 1, k$ is odd and $i_1=0$. We obtain the factor

$$E\Big(\beta_{(0)}^{\circ 3}\circ\beta_{(i_2)}\circ\beta_{(i_3)}^{\circ 2}\circ\beta_{(i_4)}^{\circ 4}\circ C(n-1,5)+\ldots:i_2>0\Big)$$

in $H_*\underline{bo}_{8n+2}$.

The case $\Theta\Big(F^j(e^{\circ 2}\circ z_{4i}\circ[\lambda^{-n}])+F^jD(*)\Big)$ is almost identical to the previous case, with indices doubled. We use $F^j(e^{\circ 2}\circ z_{4i}) = e^{\circ 2}\circ z_{2k}$, where k is odd as above. This time, to evaluate $\Theta(e^{\circ 2}\circ z_{2k}\circ[\lambda^{-n}])$, we need the binary expansion

$$2k = (2^{i_1} + 2^{i_2} + 2^{i_3+1} + 2^{i_4+2}) + (2^{i_5} + \dots) + \dots,$$

so that $i_1 = 1$ and $i_2 > 1$. We obtain the factor

$$E\Big(\beta_{(0)}^{\circ 3}\circ\beta_{(i_1)}\circ\beta_{(i_2)}\circ\beta_{(i_3)}^{\circ 2}\circ\beta_{(i_4)}^{\circ 4}\circ C(n-1,5)+\dots : i_2 > i_1 = 1\Big)$$

in $H_*\underline{bo}_{8n+3}$.

Finally, we need $\Theta\Big(F^j(\overline{z}_{4i}\circ[\beta\lambda^{-(n+1)}])+F^jD(*)\Big)$ whenever $\alpha(i)+j = 4n+1$ and $i,j\geq 1$. We reindex and note that by $H_*\underline{bo}_{12}$

$$\begin{aligned} V^j\Theta\Big(F^m(\overline{z}_{2^j(8q+4)}\circ[\beta\lambda^{-(n+1)}])+F^mD(*)\Big) &= \Theta\Big(F^m(\overline{z}_{8q+4}\circ[\beta\lambda^{-(n+1)}])+F^mD(*)\Big) \\ &= \Theta\Big(e^{\circ 4}\circ z_{4i}\circ[\lambda^{-n}]+D(*)\Big), \end{aligned}$$

where $i = 2^m(2q+1)-1$, so that $\alpha(i) = m+\alpha(q) = 4n$. We factor $e^{\circ 4}\circ z_{4i}\circ[\lambda^{-n}]$ in $H_*\underline{bo}_*$ and use $H_*\underline{bo}_8$ to obtain

$$\beta_{(0)}^{\circ 4}\circ\Big(\beta_{(i_1)}\circ\beta_{(i_2)}\circ\beta_{(i_3)}^{\circ 2}\circ\beta_{(i_4)}^{\circ 4}\Big)\circ c(n-1,5)+\dots,$$

where we use the binary expansion

$$4i = (2^{i_1} + 2^{i_2} + 2^{i_3+1} + 2^{i_4+2}) + (2^{i_5} + \dots) + \dots$$

Since $m\geq 1$, i is odd and $i_2 > i_1 = 2$. Then

$$\begin{aligned} &\Theta\Big(F^m(\overline{z}_{2^j(8q+4)}\circ[\beta\lambda^{-(n+1)}])+F^mD(*)\Big) \\ &= \beta_{(j)}^{\circ 4}\circ\Big(\beta_{(j+i_1)}\circ\beta_{(j+i_2)}\circ\beta_{(j+i_3)}^{\circ 2}\circ\beta_{(j+i_4)}^{\circ 4}\Big)\circ c'(n-1,5)+\dots \end{aligned}$$

and $H_*\underline{bo}_{8n+5}$ contains the factor (after reindexing)

$$E\Big(\beta_{(0)}\circ\beta_{(i_2)}^{\circ 4}\circ\beta_{(i_3)}\circ\beta_{(i_4)}\circ\beta_{(i_5)}^{\circ 2}\circ\beta_{(i_6)}^{\circ 4}\circ C(n-1,7) : i_4 > i_3 = i_2+2\Big).$$

Thus we have proven our hypothesis. ■

6 The Computation of the Hopf Modules $H_*\underline{bo\langle 4\rangle}_*$, $H_*\underline{bo\langle 2\rangle}_*$ and $H_*\underline{bo\langle 1\rangle}_*$ over the Hopf Ring $H_*\underline{bo}_*$

6.1 The computation of $H_*\underline{bo\langle 4\rangle}_*$

To calculate $H_*\underline{bo\langle 4\rangle}_*$, we keep in mind that we have the following exact triangle of spectra:

$$\Phi : \Sigma^3 H \to \Sigma^8 bo \to bo\langle 4\rangle \to \Sigma^4 H.$$

We map $\Sigma^4 H \to \Sigma^4 \overline{H}$ to simplify calculations.

Thus we have the maps

$$\zeta : H_*\underline{bo}_{n+8} \to H_*\underline{bo\langle 4\rangle}_n$$

and

$$\Theta : H_*\underline{bo\langle 4\rangle}_n \to H_*\underline{\overline{H}}_{n+4}.$$

We note that all of the generators of $H_*\underline{bo\langle 4\rangle}_n$ either map non-trivially to $H_*\underline{\overline{H}}_{n+4}$ or map from $H_*\underline{bo}_{n+8}$, but no element does both.

Our conclusion will be that the maps

$$\underline{bo\langle 4\rangle}_n \to \underline{bo}_n \to \underline{KO}_n$$

and

$$\Theta : \underline{bo\langle 4\rangle}_n \to \underline{\overline{H}}_{n+4}$$

embed $H_*\underline{bo\langle 4\rangle}_n$ as a sub-Hopf algebra of $H_*(\underline{KO}_n \times \underline{\overline{H}}_{n+4})$, which we describe.

We use

$$\Phi : \underline{bo}_0 \xrightarrow{\simeq} \underline{bo\langle 4\rangle}_{-8} = \underline{bo}_{-8}.$$

We also have the fibration

$$\Phi : \underline{bo}_4 \longrightarrow \underline{bo\langle 4\rangle}_{-4} \longrightarrow K(\mathbb{Z}, 0).$$

We keep in mind that $\underline{bo\langle 4\rangle}_{-4} = \underline{bo}_4 \times \mathbb{Z} = \underline{KO}_{-4}$. This is our starting place. Also, $\underline{bo\langle 4\rangle}_k = \underline{bo}_k$ for $k \le -4$.

The general method of computation involves the same type of argument that we used in the proof of $H_*\underline{bo}_*$. Thus, we use the bar spectral sequence, properties of Hopf rings, and the properties of the Verschiebung and Frobenius maps. We must keep track of the map Θ to determine the structure of the exterior algebras in $H_*\underline{\overline{H}}_*$, as we did in $H_*\underline{bo}_*$. We also use the map ζ to determine the structure of $H_*\underline{bo\langle 4\rangle}_*$ in $H_*\underline{bo}_*$. As in $H_*\underline{bo}_*$, we also assume implicitly that $i_m \ge i_n \ge 0$ whenever $m > n$. We illustrate the methodology for $H_*\underline{bo\langle 4\rangle}_*$, and note that in subsequent sections all proofs will be left to the reader.

6.1.1 $H_*\underline{bo\langle 4\rangle}_{-4}$

We recall that

$$H_*\underline{\overline{H}}_0 = \mathbb{Z}[z_0, {z_0}^{-1}],$$

where $z_0 = [1]$ and ${z_0}^{-1} = [-1]$,

$$H_*\underline{bo}_4 = P\Big(\overline{z}_{4i} \circ [\beta\lambda^{-1}] : \alpha(i) \ge 1\Big),$$

and

$$H_*\underline{KO}_{-4} = P\Big(\overline{z}_{4i} \circ [\beta] : i > 0\Big) \otimes P\Big([\beta], [\beta]^{-1}\Big) = H_*\underline{bo\langle 4\rangle}_{-4}.$$

We clearly have

$$\zeta : \overline{z}_{4i} \circ [\beta\lambda^{-1}] \longmapsto \overline{z}_{4i} \circ [\beta]$$

for $i > 0$, and we also have the maps

$$\Theta([\beta]) = [1] \text{ and } \Theta([\beta]^{-1}) = [-1] = [1].$$

6.1.2 $H_*\underline{bo\langle 4\rangle}_{-3}$ through $H_*\underline{bo\langle 4\rangle}_{-1}$

The facts that

$$H_*\underline{\overline{H}}_k = E(\beta_{(i_1)} \circ \ldots \circ \beta_{(i_k)}) \text{ for } k \geq 1$$

and

$$\begin{aligned} H_*\underline{bo}_5 &= E(e \circ z_{4i} \circ [\beta\lambda^{-1}] : \alpha(i) \geq 1) \\ H_*\underline{bo}_6 &= E(\overline{z}_{2i} \circ [\eta^2\lambda^{-1}] + D(*) : \alpha(i) \geq 2) \\ H_*\underline{bo}_7 &= E(\overline{z}_i \circ [\eta\lambda^{-1}] + D(*) : \alpha(i) \geq 3) \end{aligned}$$

yield the following:

$$\begin{aligned} H_*\underline{bo\langle 4\rangle}_{-3} &= E(e \circ z_{4i} \circ [\beta] : i \geq 0) \\ H_*\underline{bo\langle 4\rangle}_{-2} &= E(\overline{z}_{2i} \circ [\eta^2] + D(*) : \alpha(i) \geq 1) \\ H_*\underline{bo\langle 4\rangle}_{-1} &= E(\overline{z}_i \circ [\eta] + D(*) : \alpha(i) \geq 2) \end{aligned}$$

Again the maps ζ and Θ are clear:

$$\begin{aligned} &\zeta : e \circ z_{4i} \circ [\beta\lambda^{-1}] \longmapsto e \circ z_{4i} \circ [\beta] \text{ on } H_*\underline{bo}_{-3} \text{ for } \alpha(i) \geq 1 \\ &\zeta : \overline{z}_{2i} \circ [\eta^2\lambda^{-1}] \longmapsto \overline{z}_{2i} \circ [\eta^2] \text{ on } H_*\underline{bo}_{-2} \text{ for } \alpha(i) \geq 2 \\ &\zeta : \overline{z}_i \circ [\eta\lambda^{-1}] \longmapsto \overline{z}_i \circ [\eta] \text{ on } H_*\underline{bo}_{-1} \text{ for } \alpha(i) \geq 3 \end{aligned}$$

$$\begin{aligned} &\Theta : e \circ [\beta] \longmapsto \beta_{(0)} \text{ on } H_*\underline{bo\langle 4\rangle}_{-3} \\ &\Theta : \overline{z}_{2^{i_1+1}} \circ [\eta^2] \longmapsto \beta_{(i_1)}^{\circ 2} \text{ on } H_*\underline{bo\langle 4\rangle}_{-2} \\ &\Theta : \overline{z}_{2^{i_1}+2^{i_2+1}} \circ [\eta] \longmapsto \beta_{(i_1)} \circ \beta_{(i_2)}^{\circ 2} \text{ on } H_*\underline{bo\langle 4\rangle}_{-1} \end{aligned}$$

6.1.3 $H_*\underline{bo\langle 4\rangle}_0$

Here $H_*\underline{\overline{H}}_4 = E(\beta_{(i_1)} \circ \ldots \circ \beta_{(i_4)})$,

$$\begin{aligned} H_*\underline{bo}_8 \;&=\; P\Big(\overline{z}_i \circ [\lambda^{-1}] + D(*) : \alpha(i) \geq 4\Big) \\ &\quad \otimes P\Big(F^j(\overline{z}_i \circ [\lambda^{-1}]) + F^jD(*) : \alpha(i) + j = 4, i, j \geq 1\Big), \end{aligned}$$

and

$$H_*\underline{bo\langle 4\rangle}_0 = P\Big(\overline{z}_i : \alpha(i) \geq 3\Big) \otimes P\Big(F^j(\overline{z}_i) + F^jD(*) : \alpha(i) + j = 3, i, j \geq 1\Big).$$

The elements that are not hit by ζ are the elements in

$$E(\overline{z}_i : \alpha(i) = 3) \otimes E\Big(F^j(\overline{z}_i) + F^jD(*) : \alpha(i) + j = 3, i, j \geq 1\Big).$$

These are also the suspensions of the elements mapped out by Θ from the previous section.

We have

$$\Theta : \overline{z}_{2^{i_1}} \circ \overline{z}_{2^{i_2}+2^{i_3}+1} \longmapsto \beta_{(i_1)} \circ \beta_{(i_2)} \circ \beta_{(i_3)}^{\circ 2}.$$

(Note that, as in $H_*\underline{bo}_{8k}$, we can simplify the elements $\overline{z}_{2^{i_1}} \circ \overline{z}_{2^{i_2}+2^{i_3}+1}$.)

6.1.4 $H_*\underline{bo\langle 4\rangle}_1$

As in $H_*\underline{bo}_{8k+1}$, when we take the suspension of any element of the form x^2, it maps to zero in $H_*\underline{bo\langle 4\rangle}_*$, thus at this step we must map it to $H_*\underline{\overline{H}}_{*+4}$. These were the elements that in the calculation for $H_*\underline{bo\langle 4\rangle}_0$ mapped by

$$\Theta : \overline{z}_{2^{i_1}} \circ \overline{z}_{2^{i_2}+2^{i_3}+1} \longmapsto \beta_{(i_1)} \circ \beta_{(i_2)} \circ \beta_{(i_3)}^{\circ 2} \text{ for } i_3 \geq i_2 = i_1.$$

We will not be including the $H_*\underline{\overline{H}}_*$ parts of $H_*\underline{bo}_*$.

Thus $H_*\underline{\overline{H}}_5 = E(\beta_{(i_1)} \circ \ldots \circ \beta_{(i_5)})$,

$$H_*\underline{bo}_9 = P\Big(e \circ z_{2i} \circ [\lambda^{-1}] : \alpha(i) \geq 4\Big) \otimes P\Big(F^j(e \circ z_{2i} \circ [\lambda^{-1}]) + F^j D(*) : \alpha(i) + j = 4, j \geq 1\Big),$$

and

$$\begin{aligned} H_*\underline{bo\langle 4\rangle}_1 &= P\Big(e \circ z_{2i} : \alpha(i) \geq 3\Big) \otimes P\Big(F^j(e \circ z_{2i}) + F^j D(*) : \alpha(i) + j = 3, j \geq 1\Big) \\ &\quad \otimes E(\beta_{(i_1)} \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 2} : i_1 = 0). \end{aligned}$$

The elements that are not hit by ζ are the elements in

$$E(e \circ z_{2i} : \alpha(i) = 3) \otimes E\Big(F^j(e \circ z_{2i}) + F^j D(*) : \alpha(i) + j = 3, j \geq 1\Big).$$

These are also the suspensions of the elements mapped out by Θ from the calculation of $H_*\underline{bo\langle 4\rangle}_0$. Thus

$$\Theta : e \circ z_{2^{i_2}+2^{i_3}+2^{i_4}+1} \longmapsto \beta_{(i_1)} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\circ 2} \text{ for } i_3 > i_2 \geq i_1 = 0.$$

(We note that the elements $(e \circ z_{1+2^{i_3}+2^{i_4}+1})$ may be simplified as in $H_*\underline{bo}_9$.)

6.1.5 $H_*\underline{bo\langle 4\rangle}_2$

As in $H_*\underline{bo}_{8k+2}$, we must map the elements that were of the form x^2 in the calculation for $H_*\underline{bo\langle 4\rangle}_1$ to $H_*\underline{\overline{H}}_*$. These were the elements that (in the previous section) mapped by

$$\Theta : e \circ z_{2^{i_2}+2^{i_3}+2^{i_4}+1} \longmapsto \beta_{(i_1)} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)}^{\circ 2} \text{ for } i_3 > i_2 = i_1 = 0.$$

We therefore have $H_*\underline{\overline{H}}_6 = E(\beta_{(i_1)} \circ \ldots \circ \beta_{(i_6)})$,

$$\begin{aligned} H_*\underline{bo}_{10} &= P\Big(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-1}] : \alpha(i) \geq 4\Big) \\ &\quad \otimes P\Big(F^j(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-1}]) + F^j D(*) : \alpha(i) + j = 4, j \geq 1\Big), \end{aligned}$$

and

$$H_*\underline{bo\langle 4\rangle}_2 = P\Big(e^{\circ 2}\circ z_{4i} : \alpha(i)\geq 3\Big)\otimes P\Big(F^j(e^{\circ 2}\circ z_{4i}) + F^jD(*) : \alpha(i)+j=3, j\geq 1\Big)$$
$$\otimes E(\beta^{\circ 2}_{(i_1)}\circ\beta^{\circ 2}_{(i_2)}\circ\beta^{\circ 2}_{(i_3)})\otimes E(\beta^{\circ 3}_{(i_1)}\circ\beta_{(i_2)}\circ\beta^{\circ 2}_{(i_3)} : i_2>i_1=0).$$

The elements that are not hit by ζ are in

$$E(e^{\circ 2}\circ z_{4i} : \alpha(i)=3)\otimes E\Big(F^j(e^{\circ 2}\circ z_{4i}) + F^jD(*) : \alpha(i)+j=3, j\geq 1\Big).$$

These are also the suspensions of the elements mapped out by Θ from the calculations for $H_*\underline{bo\langle 4\rangle}_1$. Thus

$$\Theta : e^{\circ 2}\circ z_{2^{i_2}+2^{i_3}+2^{i_4+1}} \longmapsto \beta^{\circ 2}_{(i_1)}\circ\beta_{(i_2)}\circ\beta_{(i_3)}\circ\beta^{\circ 2}_{(i_4)} \text{ for } i_3>i_2>i_1=0.$$

6.1.6 $H_*\underline{bo\langle 4\rangle}_3$

We define

$$A(s) = \text{the set of all } \beta_{(i_1)}\circ\beta_{(i_2)}\circ\ldots\circ\beta_{(i_s)}, \text{ for } s\geq 1,$$
$$A(0)=[1],$$

as before, but now we inductively define

$$C(n,k) = \text{the set of all } \beta^{\circ 4}_{(i_k)}\circ\beta_{(i_{k+1})}\circ\beta_{(i_{k+2})}\circ\beta^{\circ 2}_{(i_{k+3})}\circ C(n-1,k+4),$$

where $i_{k+2}>i_{k+1}\geq i_k+3$, starting from $C(0,k)=[1]$.

Again we must map the elements that were of the form $F(x)$ from $H_*\underline{bo\langle 4\rangle}_2$ to $H_*\underline{\overline{H}}_*$. These were the elements that mapped by

$$\Theta : e^{\circ 2}\circ z_{2^{i_2}+2^{i_3}+2^{i_4+1}} \longmapsto \beta^{\circ 2}_{(i_1)}\circ\beta_{(i_2)}\circ\beta_{(i_3)}\circ\beta^{\circ 2}_{(i_4)} \text{ for } i_3>i_2=1, i_1=0.$$

Thus $H_*\underline{\overline{H}}_7 = E(\beta_{(i_1)}\circ\ldots\circ\beta_{(i_7)})$,

$$H_*\underline{bo}_{11} = E\Big(e^{\circ 3}\circ z_{4i}\circ[\lambda^{-1}] : \alpha(i)\geq 4\Big),$$

and

$$H_*\underline{bo\langle 4\rangle}_3 = E\Big(e^{\circ 3}\circ z_{4i} : \alpha(i)\geq 3\Big)$$
$$\otimes E(A(1)\circ\beta^{\circ 2}_{(i_2)}\circ\beta^{\circ 2}_{(i_3)}\circ\beta^{\circ 2}_{(i_4)})$$
$$\otimes E(\beta^{\circ 4}_{(i_1)}\circ\beta_{(i_2)}\circ\beta^{\circ 2}_{(i_3)} : i_2>i_1)$$
$$\otimes E(\beta^{\circ 3}_{(i_1)}\circ\beta_{(i_2)}\circ\beta_{(i_3)}\circ\beta^{\circ 2}_{(i_4)} : i_3>i_2=1, i_1=0).$$

The elements that are not hit by ζ are the elements in

$$E(e^{\circ 3}\circ z_{4i} : \alpha(i)=3).$$

Thus

$$\Theta : e^{\circ 3}\circ z_{2^{i_2}+2^{i_3}+2^{i_4+1}} \longmapsto \beta^{\circ 3}_{(i_1)}\circ\beta_{(i_2)}\circ\beta_{(i_3)}\circ\beta^{\circ 2}_{(i_4)} \text{ for } i_3>i_2\geq 2, i_1=0.$$

6.1.7 $H_*\underline{bo\langle 4\rangle}_k$ for $k > 3$

The steps from here on are all similar. We know $H_*\overline{\underline{H}}_{k+4}$ and $H_*\underline{bo}_{k+8}$ so can find $H_*\underline{bo\langle 4\rangle}_k$. We have elements of the form $F(x)$, and we keep track of the maps Θ and ζ as we move through the process.

For example, in $H_*\underline{bo\langle 4\rangle}_4$ we have

$$\Theta : \overline{z}_{2^{i_1+2}} \circ \overline{z}_{2^{i_2}+2^{i_3}+2^{i_4+1}} \circ [\beta\lambda^{-1}] \longmapsto \beta^{\circ 4}_{(i_1)} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta^{\circ 2}_{(i_4)} \text{ for } i_3 > i_2 \geq i_1 + 2.$$

and in $H_*\underline{bo\langle 4\rangle}_5$ we have

$$\Theta : e \circ z_{2^{i_2+2}+2^{i_3}+2^{i_4}+2^{i_5+1}} \circ [\beta\lambda^{-1}] \longmapsto \beta_{(i_1)} \circ C(1,2) \text{ for } i_1 = 0.$$

All remaining proofs are left to the reader.

Thus $H_*\underline{bo\langle 4\rangle}_*$ is the tensor product of the following four families of Hopf algebras:

1. Polynomial and exterior subalgebras of $H_*\underline{bo}_*$:

$$\begin{aligned}
&P\Big(\overline{z}_i \circ [\lambda^{-n}] + D(*) : i > 0, \alpha(i) \geq 4n+3\Big), \text{on } \underline{bo\langle 4\rangle}_{8n}\\
&P\Big([\lambda^{-n}], [\lambda^{-n}]^{-1}\Big), \text{on } \underline{bo\langle 4\rangle}_{8n}, \text{ for } n < 0\\
&P\Big(e \circ z_{2i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n+3\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+1}\\
&P\Big(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n+3\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+2}\\
&E\Big(e^{\circ 3} \circ z_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n+3\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+3}\\
&P\Big(\overline{z}_{4i} \circ [\beta\lambda^{-(n+1)}] + D(*) : i > 0, \alpha(i) \geq 4n+4\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+4}\\
&P\Big([\beta\lambda^{-n}], [\beta\lambda^{-n}]^{-1}\Big), \text{on } \underline{bo\langle 4\rangle}_{8n-4}, \text{ for } n \leq 0\\
&E\Big(e \circ z_{4i} \circ [\beta\lambda^{-(n+1)}] : \alpha(i) \geq 4n+4\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+5}\\
&E\Big(\overline{z}_{2i} \circ [\eta^2\lambda^{-(n+1)}] + D(*) : i > 0, \alpha(i) \geq 4n+5\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+6}\\
&E\Big([\eta^2\lambda^{-n}] - 1\Big), \text{on } \underline{bo\langle 4\rangle}_{8n-2}, \text{ for } n < 0\\
&E\Big(\overline{z}_i \circ [\eta\lambda^{-(n+1)}] + D(*) : i > 0, \alpha(i) \geq 4n+6\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+7}\\
&E\Big([\eta\lambda^{-n}] - 1\Big), \text{on } \underline{bo\langle 4\rangle}_{8n-1}, \text{ for } n < 0.
\end{aligned}$$

2. Polynomial algebras on generators that decompose in $H_*\underline{bo}_*$, companions to the polynomial algebras in the first family:

$$\begin{aligned}
&P\Big(F^j(\overline{z}_i \circ [\lambda^{-n}]) + F^jD(*) : \alpha(i) + j = 4n+3, i, j \geq 1\Big), \text{on } \underline{bo\langle 4\rangle}_{8n}\\
&P\Big(F^j(e \circ z_{2i} \circ [\lambda^{-n}]) + F^jD(*) : \alpha(i) + j = 4n+3, j \geq 1\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+1}\\
&P\Big(F^j(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-n}]) + F^jD(*) : \alpha(i) + j = 4n+3, j \geq 1\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+2}\\
&P\Big(F^j(\overline{z}_{4i} \circ [\beta\lambda^{-(n+1)}]) + F^jD(*) : \alpha(i) + j = 4n+4, i, j \geq 1\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+4}.
\end{aligned}$$

3. Exterior algebras involving $\beta_{(0)}$ that arise from the second family:

$$E\Big(\beta_{(0)} \circ \beta^{\circ 2}_{(i_2)} \circ \beta^{\circ 2}_{(i_3)} \circ C(n,4) + \dots\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+1}$$
$$E\Big(\beta^{\circ 3}_{(0)} \circ \beta_{(i_2)} \circ \beta^{\circ 2}_{(i_3)} \circ C(n,4) + \dots : i_2 > 0\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+2}$$
$$E\Big(\beta^{\circ 3}_{(0)} \circ \beta_{(i_2)} \circ \beta_{(i_3)} \circ \beta^{\circ 2}_{(i_4)} \circ C(n,5) + \dots : i_3 > i_2 = 1\Big), \text{on } \underline{bo\langle 4\rangle}_{8n+3}$$
$$E\Big(\beta_{(0)} \circ \beta^{\circ 4}_{(i_2)} \circ \beta_{(i_3)} \circ \beta_{(i_4)} \circ \beta^{\circ 2}_{(i_5)} \circ C(n,6) + \dots : i_4 > i_3 = i_2 + 2\Big), \text{ on } \underline{bo\langle 4\rangle}_{8n+5}.$$

4. General exterior algebras that arise from the third family by unlimited suspension:

$$E\Big(A(s) \circ \beta^{\circ 2}_{(i_{s+1})} \circ \beta^{\circ 2}_{(i_{s+2})} \circ \beta^{\circ 2}_{(i_{s+3})} \circ C(n,s+4) + \dots\Big)$$
$$E\Big(A(s) \circ \beta^{\circ 4}_{(i_{s+1})} \circ \beta_{(i_{s+2})} \circ \beta^{\circ 2}_{(i_{s+3})} \circ C(n,s+4) + \dots : i_{s+2} > i_{s+1}\Big)$$
$$E\Big(A(s) \circ \beta^{\circ 4}_{(i_{s+1})} \circ \beta_{(i_{s+2})} \circ \beta_{(i_{s+3})} \circ \beta^{\circ 2}_{(i_{s+4})} \circ C(n,s+5) + \dots : i_{s+3} > i_{s+2} = i_{s+1} + 1\Big)$$
$$E\Big(A(s) \circ \beta^{\circ 2}_{(i_{s+1})} \circ \beta^{\circ 4}_{(i_{s+2})} \circ \beta_{(i_{s+3})} \circ \beta_{(i_{s+4})} \circ \beta^{\circ 2}_{(i_{s+5})} \circ C(n,s+6) + \dots : i_{s+4} > i_{s+3} = i_{s+2} + 2\Big).$$

6.2 The computation of $H_*\underline{bo\langle 2\rangle}_*$

To calculate $H_*\underline{bo\langle 2\rangle}_*$ we use the following exact triangle of spectra:

$$\Phi : \Sigma\overline{H} \to bo\langle 4\rangle \to bo\langle 2\rangle \to \Sigma^2\overline{H}.$$

Once again we define

$$\Theta : H_*\underline{bo\langle 2\rangle}_n \to H_*\underline{\overline{H}}_{n+2}$$

and

$$\zeta : H_*\underline{bo\langle 4\rangle}_n \to H_*\underline{bo\langle 2\rangle}_n.$$

The starting point is

$$\Phi : \underline{bo\langle 4\rangle}_{-2} \to \underline{bo\langle 2\rangle}_{-2} \to \underline{\overline{H}}_0,$$

where we note that

$$H_*\underline{bo\langle 2\rangle}_{-2} = H_*\underline{KO}_{-2} = E\Big(\overline{z}_{2i} \circ [\eta^2] : i > 0\Big) \otimes E\Big([\eta^2] - 1\Big).$$

Also, $\underline{bo\langle 2\rangle}_k = \underline{bo}_k$ for $k \le -2$. Thus we start with

$$\Theta\Big([\eta^2] - 1\Big) = [1]$$

and

$$\zeta : \overline{z}_{2i} \circ [\eta^2] \mapsto \overline{z}_{2i} \circ [\eta^2],$$

for $i > 0$.

Again we define

$$A(s) = \text{ the set of all } \beta_{(i_1)} \circ \beta_{(i_2)} \circ \ldots \circ \beta_{(i_s)}, \text{ for } s \geq 1,$$

$$A(0) = [1],$$

but now we inductively define

$$C(n,k) = \text{the set of all } \beta^{\circ 2}_{(i_k)} \circ \beta^{\circ 4}_{(i_{k+1})} \circ \beta_{(i_{k+2})} \circ \beta_{(i_{k+3})} \circ C(n-1, k+4),$$

where $i_{k+3} > i_{k+2} \geq i_{k+1} + 3$, starting from $C(0,k) = [1]$.

Our conclusion is that the maps

$$\underline{bo\langle 2\rangle}_n \to \underline{bo}_n \to \underline{KO}_n$$

and

$$\Theta : \underline{bo\langle 2\rangle}_n \to \underline{\overline{H}}_{n+2}$$

embed $H_*\underline{bo\langle 2\rangle}_n$ as a sub-Hopf algebra of $H_*(\underline{KO}_n \times \underline{\overline{H}}_{n+2})$, which we describe. Thus $H_*\underline{bo\langle 2\rangle}_*$ is the tensor product of the following four families of Hopf algebras:

1. Polynomial and exterior subalgebras of $H_*\underline{bo}_*$:

$$P\Big(\overline{z}_i \circ [\lambda^{-n}] + D(*) : i > 0, \alpha(i) \geq 4n+2\Big), \text{on } \underline{bo\langle 2\rangle}_{8n}$$
$$P\Big([\lambda^{-n}], [\lambda^{-n}]^{-1}\Big), \text{on } \underline{bo\langle 2\rangle}_{8n}, \text{ for } n < 0$$
$$P\Big(e \circ z_{2i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n+2\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+1}$$
$$P\Big(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n+2\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+2}$$
$$E\Big(e^{\circ 3} \circ z_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n+2\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+3}$$
$$P\Big(\overline{z}_{4i} \circ [\beta\lambda^{-(n+1)}] + D(*) : i > 0, \alpha(i) \geq 4n+3\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+4}$$
$$P\Big([\beta\lambda^{-n}], [\beta\lambda^{-n}]^{-1}\Big), \text{on } \underline{bo\langle 2\rangle}_{8n-4}, \text{ for } n \leq 0$$
$$E\Big(e \circ z_{4i} \circ [\beta\lambda^{-(n+1)}] : \alpha(i) \geq 4n+3\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+5}$$
$$E\Big(\overline{z}_{2i} \circ [\eta^2\lambda^{-(n+1)}] + D(*) : i > 0, \alpha(i) \geq 4n+4\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+6}$$
$$E\Big([\eta^2\lambda^{-n}] - 1\Big), \text{on } \underline{bo\langle 2\rangle}_{8n-2}, \text{ for } n \leq 0$$
$$E\Big(\overline{z}_i \circ [\eta\lambda^{-(n+1)}] + D(*) : i > 0, \alpha(i) \geq 4n+5\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+7}$$
$$E\Big([\eta\lambda^{-n}] - 1\Big), \text{on } \underline{bo\langle 2\rangle}_{8n-1}, \text{ for } n < 0.$$

2. Polynomial algebras on generators that decompose in $H_*\underline{bo}_*$, companions to the polynomial algebras in the first family:

$$
\begin{aligned}
&P\Big(F^j(\overline{z}_i\circ[\lambda^{-n}])+F^jD(*):\alpha(i)+j=4n+2, i,j\geq 1\Big), \text{on } \underline{bo\langle 2\rangle}_{8n}\\
&P\Big(F^j(e\circ z_{2i}\circ[\lambda^{-n}])+F^jD(*):\alpha(i)+j=4n+2, j\geq 1\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+1}\\
&P\Big(F^j(e^{\circ 2}\circ z_{4i}\circ[\lambda^{-n}])+F^jD(*):\alpha(i)+j=4n+2, j\geq 1\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+2}\\
&P\Big(F^j(\overline{z}_{4i}\circ[\beta\lambda^{-(n+1)}])+F^jD(*):\alpha(i)+j=4n+3, i,j\geq 1\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+4}.
\end{aligned}
$$

3. Exterior algebras involving $\beta_{(0)}$ that arise from the second family:

$$
\begin{aligned}
&E\Big(\beta_{(0)}\circ\beta^{\circ 2}_{(i_2)}\circ C(n,3)+\dots\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+1}\\
&E\Big(\beta^{\circ 3}_{(0)}\circ\beta_{(i_2)}\circ C(n,3)+\dots : i_2>0\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+2}\\
&E\Big(\beta^{\circ 3}_{(0)}\circ\beta_{(i_2)}\circ\beta_{(i_3)}\circ C(n,4)+\dots : i_3>i_2=1\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+3}\\
&E\Big(\beta_{(0)}\circ\beta^{\circ 4}_{(i_2)}\circ\beta_{(i_3)}\circ\beta_{(i_4)}\circ C(n,5)+\dots : i_4>i_3=i_2+2\Big), \text{on } \underline{bo\langle 2\rangle}_{8n+5}.
\end{aligned}
$$

4. General exterior algebras that arise from the third family by unlimited suspension:

$$
\begin{aligned}
&E\Big(A(s)\circ\beta^{\circ 2}_{(i_{s+1})}\circ\beta^{\circ 2}_{(i_{s+2})}\circ C(n,s+3)+\dots\Big)\\
&E\Big(A(s)\circ\beta^{\circ 4}_{(i_{s+1})}\circ\beta_{(i_{s+2})}\circ C(n,s+3+\dots): i_{s+2}>i_{s+1}\Big)\\
&E\Big(A(s)\circ\beta^{\circ 4}_{(i_{s+1})}\circ\beta_{(i_{s+2})}\circ\beta_{(i_{s+3})}\circ C(n,s+4)+\dots : i_{s+3}>i_{s+2}=i_{s+1}+1\Big)\\
&E\Big(A(s)\circ\beta^{\circ 2}_{(i_{s+1})}\circ\beta^{\circ 4}_{(i_{s+2})}\circ\beta_{(i_{s+3})}\circ\beta_{(i_{s+4})}\circ C(n,s+5)+\dots :\\
&\qquad i_{s+4}>i_{s+3}=i_{s+2}+2\Big).
\end{aligned}
$$

6.3 The computation of $H_*\underline{bo\langle 1\rangle}_*$

To calculate $H_*\underline{bo\langle 1\rangle}_*$ we keep in mind that we have the following exact triangle of spectra:

$$\Phi:\overline{H}\to bo\langle 2\rangle\to bo\langle 1\rangle\to\Sigma^1\overline{H}.$$

We again define

$$\Theta: H_*\underline{bo\langle 1\rangle}_n\to H_*\underline{\overline{H}}_{n+1}$$

and

$$\zeta: H_*\underline{bo\langle 2\rangle}_n\to H_*\underline{bo\langle 1\rangle}_n.$$

The starting point is

$$\Phi:\underline{bo\langle 2\rangle}_{-1}\to\underline{bo\langle 1\rangle}_{-1}\to\underline{\overline{H}}_0,$$

where we note that

$$H_*\underline{bo\langle 1\rangle}_{-1} = H_*\underline{KO}_{-1} = E\Big(\overline{z}_i \circ [\eta] : i > 0\Big) \otimes E\Big([\eta] - 1\Big).$$

Also, $\underline{bo\langle 1\rangle}_k = \underline{bo}_k$ for $k \leq -1$.

We define

$$A(s) = \text{the set of all } \beta_{(i_1)} \circ \beta_{(i_2)} \circ \ldots \circ \beta_{(i_s)}, \text{ for } s \geq 1,$$

$$A(0) = [1]$$

as before, but now we inductively define

$$C(n,k) = \text{the set of all } \beta_{(i_k)} \circ \beta^{\circ 2}_{(i_{k+1})} \circ \beta^{\circ 4}_{(i_{k+2})} \circ \beta_{(i_{k+3})} \circ C(n-1, k+4),$$

where $i_{k+3} \geq i_{k+2} + 3$ and $i_k > i_{k-1}$, starting from $C(0,k) = [1]$.

Our conclusion is that the maps

$$\underline{bo\langle 1\rangle}_n \rightarrow \underline{bo}_n \rightarrow \underline{KO}_n$$

and

$$\Theta : \underline{bo\langle 1\rangle}_n \rightarrow \overline{\underline{H}}_{n+1}$$

embed $H_*\underline{bo\langle 1\rangle}_n$ as a sub-Hopf algebra of $H_*(\underline{KO}_n \times \overline{\underline{H}}_{n+1})$, which we describe. Thus $H_*\underline{bo\langle 1\rangle}_*$ is the tensor product of the following four families of Hopf algebras:

1. Polynomial and exterior subalgebras of $H_*\underline{bo}_*$:

$$\begin{aligned}
&P\Big(\overline{z}_i \circ [\lambda^{-n}] + D(*) : i > 0, \alpha(i) \geq 4n+1\Big), \text{on } \underline{bo\langle 1\rangle}_{8n} \\
&P\Big([\lambda^{-n}], [\lambda^{-n}]^{-1}\Big), \text{on } \underline{bo\langle 1\rangle}_{8n}, \text{ for } n < 0 \\
&P\Big(e \circ z_{2i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n+1\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+1} \\
&P\Big(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n+1\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+2} \\
&E\Big(e^{\circ 3} \circ z_{4i} \circ [\lambda^{-n}] : \alpha(i) \geq 4n+1\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+3} \\
&P\Big(\overline{z}_{4i} \circ [\beta\lambda^{-(n+1)}] + D(*) : i > 0, \alpha(i) \geq 4n+2\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+4} \\
&P\Big([\beta\lambda^{-n}], [\beta\lambda^{-n}]^{-1}\Big), \text{on } \underline{bo\langle 1\rangle}_{8n-4}, \text{ for } n \leq 0 \\
&E\Big(e \circ z_{4i} \circ [\beta\lambda^{-(n+1)}] : \alpha(i) \geq 4n+2\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+5} \\
&E\Big(\overline{z}_{2i} \circ [\eta^2\lambda^{-(n+1)}] + D(*) : i > 0, \alpha(i) \geq 4n+3\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+6} \\
&E\Big([\eta^2\lambda^{-n}] - 1\Big), \text{on } \underline{bo\langle 1\rangle}_{8n-2}, \text{ for } n \leq 0 \\
&E\Big(\overline{z}_i \circ [\eta\lambda^{-(n+1)}] + D(*) : i > 0, \alpha(i) \geq 4n+4\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+7} \\
&E\Big([\eta\lambda^{-n}] - 1\Big), \text{on } \underline{bo\langle 1\rangle}_{8n-1}, \text{ for } n \leq 0.
\end{aligned}$$

2. Polynomial algebras on generators that decompose in $H_*\underline{bo}_*$, companions to the polynomial algebras in the first family:

$$P\Big(F^j(\overline{z}_i \circ [\lambda^{-n}]) + F^jD(*) : \alpha(i) + j = 4n+1, i, j \geq 1\Big), \text{on } \underline{bo\langle 1\rangle}_{8n}$$
$$P\Big(F^j(e \circ z_{2i} \circ [\lambda^{-n}]) + F^jD(*) : \alpha(i) + j = 4n+1, j \geq 1\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+1}$$
$$P\Big(F^j(e^{\circ 2} \circ z_{4i} \circ [\lambda^{-n}]) + F^jD(*) : \alpha(i) + j = 4n+1, j \geq 1\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+2}$$
$$P\Big(F^j(\overline{z}_{4i} \circ [\beta\lambda^{-(n+1)}]) + F^jD(*) : \alpha(i) + j = 4n+2, i, j \geq 1\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+4}.$$

3. Exterior algebras involving $\beta_{(0)}$ that arise from the second family:

$$E\Big(\beta_{(0)}^{\circ 3} \circ C(n,2) + \dots\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+2}$$
$$E\Big(\beta_{(0)}^{\circ 3} \circ \beta_{(i_2)} \circ C(n,3) + \dots : i_2 = 1\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+3}$$
$$E\Big(\beta_{(0)} \circ \beta_{(i_2)}^{\circ 4} \circ \beta_{(i_3)} \circ C(n,4) + \dots : i_3 = i_2 + 2\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+5}$$
$$E\Big(\beta_{(0)} \circ \beta_{(i_2)}^{\circ 2} \circ \beta_{(i_3)}^{\circ 2} \circ \beta_{(i_4)}^{\circ 4} \circ \beta_{(i_5)} \circ C(n,6) + \dots : i_5 \geq i_4 + 3\Big), \text{on } \underline{bo\langle 1\rangle}_{8n+9}.$$

4. General exterior algebras that arise from the third family by unlimited suspension:

$$E\Big(A(s) \circ \beta_{(i_{s+1})}^{\circ 4} \circ C(n, s+2) + \dots\Big)$$
$$E\Big(A(s) \circ \beta_{(i_{s+1})}^{\circ 4} \circ \beta_{(i_{s+2})} \circ C(n, s+3) + \dots : i_{s+2} = i_{s+1} + 1\Big)$$
$$E\Big(A(s) \circ \beta_{(i_{s+1})}^{\circ 2} \circ \beta_{(i_{s+2})}^{\circ 4} \circ \beta_{(i_{s+3})} \circ C(n, s+4) + \dots : i_{s+3} = i_{s+2} + 2\Big)$$
$$E\Big(A(s) \circ \beta_{(i_{s+1})}^{\circ 2} \circ \beta_{(i_{s+2})}^{\circ 2} \circ \beta_{(i_{s+3})}^{\circ 2} \circ \beta_{(i_{s+4})}^{\circ 4} \circ \beta_{(i_{s+5})} \circ C(n, s+6) + \dots : i_{s+5} \geq i_{s+4} + 3\Big).$$

6.4 The computation of $H_*\underline{bo}_*$

We note that we can use this same process to come full circle, calculating $H_*\underline{bo}_*$ from $H_*\underline{bo\langle 1\rangle}_*$. Here we start with the exact triangle of spectra

$$\Phi : \Sigma^{-1}H \to bo\langle 1\rangle \to bo\langle 0\rangle \to H,$$

and proceed from this point as we did in the three previous calculations. ■

As noted in the introduction, as $bo\langle n+8\rangle = \Sigma^8 bo\langle n\rangle$, this completes the calculation for $bo\langle n\rangle$ for all n.

References

[1] S. Hara, *The Hopf rings for connective Morava K-theory and connective complex K-theory*, J. Math. Kyoto Univ. **31-1** (1991), 43-70.

[2] J. McCleary, *A User's Guide to Spectral Sequences*, Mathematics Lecture Series **12**, Publish or Perish Press, Wilmington, Del, (1985).

[3] D. S. Cowen Morton and N. P. Strickland *The Hopf rings for KO and KU*, J. Pure Appl. Algebra **166** (2002), 247-265.

[4] D. C. Ravenel and W. S. Wilson, *The Hopf Ring for Complex Cobordism*, J. Pure Appl. Algebra **9** (1977), 241-280.

[5] D. C. Ravenel and W. S. Wilson, *The Morava K-theories of Eilenberg-Mac Lane Spaces and the Conner-Floyd Conjecture*, Amer. J. Math. **102**, (1980), 691-748.

[6] R. E. Stong, *Determination of* $H^*(BO(k,\ldots,\infty),\mathbb{Z}_2)$ *and* $H^*(BU(k,\ldots,\infty),\mathbb{Z}_2)$, Trans. Am. Math. Soc. **107**, (1963), 526-544.

[7] N. Strickland, *Bott Periodicity and Hopf Rings*, PhD thesis, Manchester University (1992).

[8] W. S. Wilson, *The Hopf ring for Morava K-theory*, Publ. Res. Inst. Math. Sci., **20** (1984), 1025-1036.

[9] W. S. Wilson, *Hopf Rings in Algebraic Topology*, Expo. Math. **18** (2000), 369-388.