Directions: You have 50 minutes in which to complete this exam. Please make sure that you read through this entire exam before attempting any problems. You must show all work, or risk losing credit. Be sure to answer all questions asked.

To receive full credit on problems, they must not only be mathematically correct, but they must also be solved using the correct notation and terminology.

Note that I have included the Invertible Matrix Theorem at the end of this exam.

Good luck!
1. (23 points) Let $T : \mathbb{P}_2 \rightarrow \mathbb{P}_3$ be defined by $T(a + bx + cx^2) = a + b(x^3 + 1) + c$.

a. Find the kernel for $T$ and a basis for the kernel of $T$. (Show all work.) You do not need to prove that you have a basis, but you do need to show all work as to how you arrived to your basis. Then find dim of ker $T$.

b. Find the range of $T$ and a basis for the range of $T$. (Show all work.) You do not need to prove that you have a basis, but you do need to show all work as to how you arrived to your basis. Then find dim of range $T$. 
2. (13 points) Let $B = \{1, 1-t, -2+4t^2\}$ be a basis for $\mathbb{P}_2$. Find the coordinate vector of $p(t) = 7 - 8t + 3t^2$ relative to $B$. (Show all work.)

3. (11 points) Prove: If $\lambda$ is an eigenvalue of $A$ and $x$ is a corresponding eigenvector, then $1/\lambda$ is an eigenvalue of $A^{-1}$ and $x$ is a corresponding eigenvector. Note: this problem assumes that $A^{-1}$ exists. **Hint:** Start this proof with the definition of $\lambda$ is an eigenvalue of $A$ and $x$ is a corresponding eigenvector.
4. (27 points) Consider the following matrices:

\[
A = \begin{bmatrix}
1 & -2 & 0 & 4 & 1 \\
3 & 1 & 1 & 0 & 3 \\
-1 & -5 & -1 & 8 & 0 \\
3 & 8 & 2 & -12 & -2
\end{bmatrix},
B = \begin{bmatrix}
1 & 0 & \frac{2}{7} & \frac{4}{7} & 1 \\
0 & 1 & \frac{1}{7} & \frac{-12}{7} & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix},
C = 5 \times 9 \text{ matrix (unspecified)}
\]

It turns out that \textbf{B is the reduced row echelon form of A} (do not check this – it is true.)

a. Find rank \(A\). Explain (1 sentence.)

b. Find \(\dim \text{Nul } A\). Explain (1 sentence.)

c. Find a basis for \(\text{Col } A\). Explain (1 sentence.)

d. Find a basis for \(\text{Row } A\). Explain (1 sentence.)

e. Find a basis for \(\text{Nul } A\). Show all work.
5. (26 points) For the matrix $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$ find:

a. The characteristic equation

b. The eigenvalue(s)

c. The basis for the eigenspace corresponding to the eigenvalue(s) from part b (you do not have to prove this is actually a basis).
6. (20 points) Consider the bases $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$, $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix} \right\}$.

a. Find $P_{\mathcal{C} \leftarrow \mathcal{B}}$. Show all work.

b. Let $\mathbf{w} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$. It turns out that (don't check this) $[\mathbf{w}]_B = \begin{bmatrix} 1 \\ \frac{1}{3} \\ \frac{4}{3} \end{bmatrix}$. Use the theorem involving the matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ and the vector $[\mathbf{w}]_B$ to find the vector $[\mathbf{w}]_C$. Show all work (i.e. give the formula in the theorem and then show how the theorem is used). Then confirm that your solution is indeed equal to $[\mathbf{w}]_C$.
Invertible Matrix Theorem (Continued) Let $A$ be a square $n \times n$ matrix. Then the following statements are all equivalent. That is, for a given $A$, the statements are either all true or all false.

1. $A$ is an invertible matrix.
2. $A$ is row equivalent to the $n \times n$ identity matrix $I_n$.
3. $A$ has $n$ pivot positions.
4. The equation $Ax = 0$ has only the trivial solution.
5. The columns of $A$ form a linearly independent set.
6. The linear transformation $x \mapsto Ax$ is one-to-one.
7. The equation $Ax = b$ has at least one solution for each $b$ in $\mathbb{R}^n$. (Note: This could also have been written as “The equation $Ax = b$ has a unique solution for each $b$ in $\mathbb{R}^n$."
8. The columns of $A$ span $\mathbb{R}^n$.
9. The linear transformation $x \mapsto Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
10. There is an $n \times n$ matrix $C$ such that $CA = I$.
11. There is an $n \times n$ matrix $D$ such that $AD = I$.
12. $A^T$ is an invertible matrix.
13. The columns of $A$ form a basis of $\mathbb{R}^n$.
14. $\text{Col} A = \mathbb{R}^n$.
15. $\dim \text{Col} A = n$.
16. $\text{rank} A = n$.
17. $\text{Nul} A = \{0\}$.
18. $\dim \text{Nul} A = 0$.
19. The number 0 is not an eigenvalue of $A$.
20. The determinant of $A$ is not zero.