Algebraic Structure in a Family of Nim-like Arrays

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Abstract

We study aspects of the algebraic structure shared by a certain family of recursively generated arrays related to the operation of Nim-addition. We first observe that each individual array represents a countably infinite, commutative loop (in the sense of quasigroups). We then prove that each loop in the family is monogenic (generated by a single element in a non-associative fashion), and use this to prove that the only loop homomorphisms between loops in the family are either trivial or an identity map.

Keywords: quasigroups and loops, monogenic, Nim, Sprague-Grundy, sequential compound

MSC: 20N05, 91A46

1 Introduction

It is well known that the operation of Nim-addition arising in the study of combinatorial games may be represented as a recursively generated array [**?**]. The purpose of this paper is to give a detailed algebraic description of the members of a family $\mathcal{A}_* = \{\mathcal{A}_s\}_{s\in\mathbb{N}\cup\{0\}}$ of related recursively generated arrays; the article [**?**] deals with the graphical and combinatorics-on-words points of view regarding these arrays. The array $\mathcal{A}_0$ is the Nim-addition table itself, and the array $\mathcal{A}_1$ arises from misère play [**?**]. The array $\mathcal{A}_2$ was first mentioned in [**?**], where Stromquist and Ullman commented that it "reveals

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many curiosities but few simple patterns." The results and observations in this paper were developed by the authors to algebraically explain some of those many curiosities.

Until recently, there appears to have been no other discussion in the literature of $\mathcal{A}_*$ or the "sequential compound" operation introduced in [**?**] which gave rise to these arrays, other than a brief mention in a list of problems compiled by Richard Guy [**?**, Problem 41]. Recently, however, Rice described each of the arrays $\mathcal{A}_s$ as endowing $\mathbb{N} \cup \{0\}$ with the algebraic structure of a quasigroup [**?**].

In contrast to the situation for $\mathcal{A}_*$, there has been a fair amount of discussion regarding an array arising in the study of Wythoff's game [**?**, **?**, **?**, **?**, **?**, **?**]. In the recent paper [**?**], Rice defines a family of arrays generalizing Wythoff's game in essentially the same way as $\mathcal{A}_*$ generalizes Nim.

We now describe the structure of this paper. In Section **??** we construct the arrays $\mathcal{A}_s$. The section closes with an algebraic perspective which shows that the arrays $\mathcal{A}_s$ may in fact be viewed as each providing the structure of a loop, which is a quasigroup with identity [**?**, **?**].

In Section **??** we collect some basic results on recurring patterns in the arrays. These are important mainly for their uses in later sections.

In Section **??** we prove the Monogenicity Theorem (Theorem **??**), which asserts that the loop $\mathcal{A}_s$ is generated by a single element if and only if the seed s satisfies $s \geq 2$. Moreover, for seed $s = 2$ every element $n > s$ is a generator, and for seed $s > 2$ every element $n \neq s$ is a generator. Drawing on the classical work of Evans in [**?**] we then prove an additional result showing that none of the loops $\mathcal{A}_s$ are finitely-represented. This implies that the results in Evans' sequel [**?**] regarding loop homomorphisms do not apply in our context.

In Section **??** we prove the Loop Homomorphism Theorem (Theorem **??**), which gives a complete description of all homomorphisms between the arrays $\mathcal{A}_s$ for most values of s. In particular, the only loop homomorphism $f : \mathcal{A}_s \to \mathcal{A}_t$ for $s \neq t$ and $s \geq 2$ or $t \geq 2$ is the trivial map $\mathcal{A}_s \to \{t\}$. For $s = t \geq 2$, a loop homomorphism f is either the trivial map $\mathcal{A}_s \to \{s\}$ or the identity map.

A graphical approach to the arrays $\mathcal{A}_s$, as well as proofs of various periodicity properties enjoyed by these arrays, may be found in our paper [**?**]. Further algebraic properties, beyond those that appear in this article, will be discussed elsewhere. We conclude in Section **??** with a description of one of these properties.

We note that our algebraic analysis of the arrays $\mathcal{A}_s$ differs from that in [**?**]. Our approach naturally gives rise to an identity element, and no result analogous to the Monogenicity Theorem appears in [**?**]. Although an analogue of the Loop Homomorphism Theorem appears in [**?**], our proof differs in that it relies essentially on Section **??**.

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\ 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\ 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\ 5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\ 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

Figure 1: $\mathcal{A}_0(7,7)$

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2 Mex and the Arrays $\mathcal{A}_s$

We begin by constructing a family of infinite arrays using the mex operation:

Definition 2.1 *For a set X of non-negative integers we define* **mex X** *to be the smallest non-negative integer not contained in X. Here,* **mex** *stands for* **m***inimal* **ex***cluded value.*

Definition 2.2 *For any 2-dimensional array $\mathcal{M}$ indexed by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, let $\boldsymbol{a_{i,j}}$ denote the entry in row i, column j, where $i, j \geq 0$. The* **principal (i,j) subarray $\mathcal{M}(i,j)$** *is the subarray of $\mathcal{M}$ consisting of entries $a_{p,q}$ with indices $(p,q) \in \{0,\ldots,i\} \times \{0,\ldots,j\}$. For $j \geq 0$ define* **Left(i,j)** *to be the set of all entries in row i to the left of the entry $a_{i,j}$, and for $i \geq 0$ define* **Up(i,j)** *to be the set of entries in column j above $a_{i,j}$. (Note that Left$(i,0)$=Up$(0,j)$=$\emptyset$.)*

Definition 2.3 *The infinite array $\boldsymbol{\mathcal{A}_s}$, for $s \in \mathbb{N}_0$, is constructed recursively: The* **seed** *$a_{0,0}$ is set to s and for $(i,j) \neq (0,0)$,*

$$a_{i,j} := mex\Big(Left(i,j) \cup Up(i,j)\Big).$$

See, for example, Figures **??** and **??**.

The array $\mathcal{A}_0$ is well known as the Nim addition table, and has been extensively studied in the setting of combinatorial game theory. In particular, the i,j-entry of $\mathcal{A}_0$ is the Nim-value

$$\begin{bmatrix}
2 & 0 & 1 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
0 & 1 & 2 & 4 & 3 & 6 & 5 & 8 & 7 & 10 & 9 & 12 & 11 & 14 & 13 & 16 \\
1 & 2 & 0 & 5 & 6 & 3 & 4 & 9 & 10 & 7 & 8 & 13 & 14 & 11 & 12 & 17 \\
3 & 4 & 5 & 0 & 1 & 2 & 7 & 6 & 9 & 8 & 11 & 10 & 13 & 12 & 15 & 14 \\
4 & 3 & 6 & 1 & 0 & 7 & 2 & 5 & 11 & 12 & 13 & 8 & 9 & 10 & 16 & 18 \\
5 & 6 & 3 & 2 & 7 & 0 & 1 & 4 & 12 & 11 & 14 & 9 & 8 & 15 & 10 & 13 \\
6 & 5 & 4 & 7 & 2 & 1 & 0 & 3 & 13 & 14 & 12 & 15 & 10 & 8 & 9 & 11 \\
7 & 8 & 9 & 6 & 5 & 4 & 3 & 0 & 1 & 2 & 15 & 14 & 16 & 17 & 11 & 10 \\
8 & 7 & 10 & 9 & 11 & 12 & 13 & 1 & 0 & 3 & 2 & 4 & 5 & 6 & 17 & 19 \\
9 & 10 & 7 & 8 & 12 & 11 & 14 & 2 & 3 & 0 & 1 & 5 & 4 & 16 & 6 & 20 \\
10 & 9 & 8 & 11 & 13 & 14 & 12 & 15 & 2 & 1 & 0 & 3 & 6 & 4 & 5 & 7 \\
11 & 12 & 13 & 10 & 8 & 9 & 15 & 14 & 4 & 5 & 3 & 0 & 1 & 2 & 7 & 6 \\
12 & 11 & 14 & 13 & 9 & 8 & 10 & 16 & 5 & 4 & 6 & 1 & 0 & 3 & 2 & 21 \\
13 & 14 & 11 & 12 & 10 & 15 & 8 & 17 & 6 & 16 & 4 & 2 & 3 & 0 & 1 & 5 \\
14 & 13 & 12 & 15 & 16 & 10 & 9 & 11 & 17 & 6 & 5 & 7 & 2 & 1 & 0 & 3 \\
15 & 16 & 17 & 14 & 18 & 13 & 11 & 10 & 19 & 20 & 7 & 6 & 21 & 5 & 3 & 0
\end{bmatrix}$$

Figure 2: $\mathcal{A}_2(15, 15)$

of the direct sum $G_1 \oplus G_2$ of a game G_1 with Nim-value i and a game G_2 with Nim-value j; see [**?**] for more details. Consideration of what is known as "misère play" gives rise to the array $\mathcal{A}_1$. The reader can easily verify that this change of seed from 0 to 1 has a minimal effect; other than the top left 2×2 block, the pattern of this array is exactly the same as that of $\mathcal{A}_0$.

In [**?**], Stromquist and Ullman define an operation on games called "sequential compound". Essentially, the sequential compound $G \rightarrow H$ of games G and H is the game in which play proceeds in G until it is exhausted, at which point play switches to H. They point out that misère play can be described as sequential compound with the single-stone Nim game $*1$. Considering the sequential compound $G \rightarrow *s$ with the s-stone, single-pile Nim game $*s$ gives rise to $\mathcal{A}_s$. Here, the i, j-entry of $\mathcal{A}_s$ is the Nim-value of the game $(G_1 \oplus G_2) \rightarrow *s$ where G_1, G_2 have Nim-values i and j, respectively.

Several properties of $\mathcal{A}_s$ follow as immediate consequences of the recursive construction:

Proposition 2.1 *For each s, the array $\mathcal{A}_s$ is symmetric, and each nonnegative integer appears exactly once in each row (and, by symmetry, each column).*

While this holds for $\mathcal{A}_0$ and $\mathcal{A}_2$ equally, it is evident from Figure **??** above that $\mathcal{A}_2$ is not at all regular, in direct contrast to $\mathcal{A}_0$. Although the entries in $\mathcal{A}_0$ can be calculated directly (*i.e.,* non-recursively) using bit-wise XOR [**?**], we have not found any non-recursive way to calculate entries of $\mathcal{A}_s$ for any $s \geq 2$ (and suspect that such an algorithm does not exist). A different recursive algorithm for constructing $\mathcal{A}_s$ (an analogue of "Algorithm WSG" in

[**?**]) is described in [**?**].

We now describe our algebraic perspective on $\mathcal{A}_s$. For a fixed s, view $\mathcal{A}_s$ as providing the "multiplication table" for an operation $*\colon \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$, where row 0 and column 0 correspond to multiplication by the seed s. Thus, s is the $*$-identity, and $i * j := a_{i',j'}$ where i' and j' are such that $a_{i',0} = i$ and $a_{0,j'} = j$. In practice, to perform the $*$ operation and find $i * j$, one simply looks at the intersection of the row with initial entry i and the column with initial entry j. For example, for $j > s$ we have $a_{0,j} = s * j$, and for $j > s \geq 1$ we have $a_{1,j} = 0 * j$. For all s, if $i, j > s$ then $a_{i,j} = i * j$. Thus, in $\mathcal{A}_2$ we have $1 * 4 = 6$ and $3 * 6 = 7$ (see Figure **??**).

Definition 2.4 ([?, ?]) *A* **quasigroup** $(Q, *)$ *is a set* Q *with binary operation* $* : Q \times Q \to Q$ *such that for every* $i, j \in Q$ *there exist unique* $p, q \in Q$ *such that* $i * p = j$ *and* $q * i = j$. *A* **loop** $(L, *)$ *is a quasigroup with identity element* $e \in L$ *such that for every* $i \in L$, $e * i = i = i * e$.

Theorem 2.1 *For each* s*, the array* $\mathcal{A}_s$ *defines a countably infinite, commutative loop structure on* $\mathbb{N}_0$*.*

Proof. By Proposition **??**, $*$ is a commutative operation. Moreover, that proposition also shows that for each $j \in \mathbb{N}_0$, the left- and right-multiplication maps $L_j, R_j\colon \mathbb{N}_0 \to \mathbb{N}_0$ given by $L_j(i) = j * i$ and $R_j(i) = i * j$ for all $i \in \mathbb{N}_0$ are bijections, *i.e.,* $*$ has a cancellation property. Thus, the algebraic structure $(\mathbb{N}_0, *)$ is a quasigroup. Since $(\mathbb{N}_0, *)$ has a $*$-identity (namely, s) it is moreover a loop. □

It is important to note that although $(\mathbb{N}_0, *)$ is a group when $s = 0$ (in that case, $*$ is bitwise-XOR), for $s \geq 1$ the operation $*$ is not even associative. For example, in seed $s = 1$ we have $(2 * 2) * 4 = 0 * 4 = 5$, but $2 * (2 * 4) = 2 * 6 = 4$.

In the following sections we will refer to $\mathcal{A}_s$ both as an array and as the corresponding loop.

3 Pattern Properties for $\mathcal{A}_s$

We collect in this section various results describing the pattern of entries in $\mathcal{A}_s$; these will be used heavily in other sections. Properties **??**-**??** describe multiplication by s, 0, 1, 2, and 3 (for most seeds). Properties **??**-**??** give the locations of the elements 0, 1 and 3 in the array (for various seeds). Properties **??** and **??** present two relations involving iterated products of a single element. Although we observed and proved these properties independently, proofs of some of the observations in this section have been recorded in [**?**]. All proofs are by strong induction. For Properties **??**-**??** note that the placement of an entry equal to i depends only on the placement of entries less than or equal to i.

The first property shows that s is the $*$-identity in $\mathcal{A}_s$ for all s.

Property 3.1 *For all* $n \in \mathbb{N}_0$,

$$s * n = n.$$

The next property gives the result of $*$-multiplication by 0 except when $s = 0$; the latter case is covered by Property **??**.

Property 3.2 *If* $s > 0$ *and* $n \leq s$ *then*

$$0 * n = \begin{cases} 0 & \text{if } n = s \\ n+1 & \text{otherwise} \end{cases}$$

If $s > 0$ *and* $n > s$ *then*

$$0 * n = \begin{cases} n-1 & \text{if } n - s \equiv 0 \bmod 2 \\ n+1 & \text{if } n - s \equiv 1 \bmod 2 \end{cases}$$

The next property gives the result of $*$-multiplication by 1 (except when $s = 1$, which is covered by Property **??**).

Property 3.3 *If* $s = 0$ *then*

$$1 * n = \begin{cases} n+1 & \text{if } n \equiv 0 \bmod 2 \\ n-1 & \text{if } n \equiv 1 \bmod 2 \end{cases}$$

If $s \geq 2$ *and* $n < s$ *then*

$$1 * n = \begin{cases} n+2 & \text{if } n \equiv 0, 2 \bmod 3 \\ n-1 & \text{if } n \equiv 1 \bmod 3 \end{cases}$$

If $s \geq 2$ *and* $n > s$ *then*

$$1 * n = \begin{cases} n-1 & \text{if } s \equiv 0 \bmod 3 \text{ and } n = s+1 \\ n-2 & \text{if } s \equiv 1 \bmod 3 \text{ and } n = s+1 \\ & \\ n+1 & \text{if } s \equiv 0, 1 \bmod 3 \text{ and } n > s+1 \text{ and } n - s \equiv 0 \bmod 2 \\ n-1 & \text{if } s \equiv 0, 1 \bmod 3 \text{ and } n > s+1 \text{ and } n - s \equiv 1 \bmod 2 \\ & \\ n-2 & \text{if } s \equiv 2 \bmod 3 \text{ and } n - s \equiv 0, 3 \bmod 4 \\ n+2 & \text{if } s \equiv 2 \bmod 3 \text{ and } n - s \equiv 1, 2 \bmod 4 \end{cases}$$

Note that the given cases when $s \equiv 2 \bmod 3$ include the possibility that $n = s + 1$.

The next property gives all but the first few results of $*$-multiplication by 2 for seeds $s \geq 5$.

Property 3.4 *If* $4 \leq n < s$ *then*

$$2 * n = \begin{cases} n+3 & \textit{if } n \equiv 0, 1, 5 \bmod 9 \\ n-1 & \textit{if } n \equiv 2, 3, 6, 8 \bmod 9 \\ n+2 & \textit{if } n \equiv 4, 7 \bmod 9 \end{cases}$$

If $n > s \geq 5$ *then*

$$2 * n = \begin{cases} n-2 & \textit{if } s \equiv 0, 4 \bmod 9 \;\; \textit{and} \;\; n-s \equiv 0, 3 \bmod 4 \\ n+2 & \textit{if } s \equiv 0, 4 \bmod 9 \;\; \textit{and} \;\; n-s \equiv 1, 2 \bmod 4 \\[1em] n+2 & \textit{if } s \equiv 1, 6 \bmod 9 \;\; \textit{and} \;\; n > s+1 \;\; \textit{and} \;\; n-s \equiv 0 \bmod 2 \\ n-2 & \textit{if } s \equiv 1, 6 \bmod 9 \;\; \textit{and} \;\; n > s+1 \;\; \textit{and} \;\; n-s \equiv 1 \bmod 2 \\[1em] n+2 & \textit{if } s \equiv 2 \bmod 9 \;\; \textit{and} \;\; n-s \equiv 0, 3 \bmod 4 \\ n-2 & \textit{if } s \equiv 2 \bmod 9 \;\; \textit{and} \;\; n-s \equiv 1, 2 \bmod 4 \\[1em] n-2 & \textit{if } s \equiv 3, 7 \bmod 9 \;\; \textit{and} \;\; n > s+1 \;\; \textit{and} \;\; n-s \equiv 0, 1 \bmod 4 \\ n+2 & \textit{if } s \equiv 3, 7 \bmod 9 \;\; \textit{and} \;\; n > s+1 \;\; \textit{and} \;\; n-s \equiv 2, 3 \bmod 4 \\[1em] n+1 & \textit{if } s \equiv 5, 8 \bmod 9 \;\; \textit{and} \;\; n > s+1 \;\; \textit{and} \;\; n-s \equiv 0 \bmod 2 \\ n-1 & \textit{if } s \equiv 5, 8 \bmod 9 \;\; \textit{and} \;\; n > s+1 \;\; \textit{and} \;\; n-s \equiv 1 \bmod 2 \\[1em] n-1 & \textit{if } s \equiv 1, 5, 7 \bmod 9 \;\; \textit{and} \;\; n = s+1 \\ n-2 & \textit{if } s \equiv 3, 6, 8 \bmod 9 \;\; \textit{and} \;\; n = s+1 \end{cases}$$

The next property gives all but the first few results of $*$-multiplication by 3 for seed $s = 2$.

Property 3.5 *For* $s = 2$ *and* $n \geq 6$ *we have*

$$3 * n = \begin{cases} n+1 & \textit{if } n \equiv 0 \bmod 2 \\ n-1 & \textit{if } n \equiv 1 \bmod 2 \end{cases}$$

The following property describes the placement of entries equal to 0 for all seeds s.

Property 3.6 *For all seeds* s *and all* n*,*

$$n * n = \begin{cases} 1 & \textit{if } n = 0 \;\; \textit{and} \;\; s > 0 \\ s & \textit{if } n = s \\ 0 & \textit{otherwise} \end{cases}$$

The next property describe the placements of the entries equal to 1 for seeds $s = 0, 1, 2$.

Property 3.7 *For $s = 0, 1$ and $m, n \geq 2$ we have $m * n = 1$ if and only if $\{m, n\} = \{2k,\ 2k + 1\}$ for some $k \geq 1$. For $s = 2$ and $m, n \geq 3$ we have $m * n = 1$ if and only if $\{m, n\} = \{2k + 1,\ 2k + 2\}$ for some $k \geq 1$.*

The next property describes the placement of the entries equal to 3 for the case seed $s = 2$. We will use this in the proof of the Monogenicity Theorem (Theorem **??**).

Property 3.8 *For $s = 2$ and $m, n \geq 6$ we have $m * n = 3$ if and only if $\{m, n\} = \{2k,\ 2k + 1\}$ for some $k \geq 3$.*

The final properties in this section will also be useful in the proof of Theorem **??**.

Property 3.9 *For $s = 2$ and $n \neq 0, s$ we have $n * (n * (n * n)) = 1$.*

Property 3.10 *In $\mathcal{A}_s$ where $s \geq 2$, we have $\underbrace{0 * (... * (0 * 0)))}_{n} = n - 1$ for all $1 \leq n \leq s + 1$.*

4 Monogenicity

Let $\langle\!| x, \diamond |\!\rangle$ denote the free unital groupoid [?] with operation $\diamond$ on a single generator x. Thus, $\langle\!| x, \diamond |\!\rangle$ contains a $\diamond$-identity $e_\diamond$ and all possible parenthesizations of $\diamond$-products of x, each of which is a distinct element. A loop $\mathcal{L}$ with identity element $e_\mathcal{L}$ is said to be **monogenic** if there is an element $n \in \mathcal{L}$ such that the operation-respecting map (groupoid homomorphism) $\phi_n \colon \langle\!| x, \diamond |\!\rangle \to \mathcal{L}$ determined by $\phi_n(e_\diamond) = e_\mathcal{L}$ and $\phi_n(x) := n$ is surjective. In this case, n is said to be a **generator** of $\mathcal{L}$. Note that our notion of monogenicity is slightly stronger than the one used in [?], which would correspond to using a free *loop* on a single generator in place of $\langle\!| x, \diamond |\!\rangle$.

Given an element $\ell \in \mathcal{L}$, we refer to any element $q \in \langle\!| x, \diamond |\!\rangle$ such that $\phi_n(q) = \ell$ as a **shape** of ℓ. Since the map ϕ_n need not be injective, an element of $\mathcal{L}$ can have more than one shape. For simplicity, we write x^k to denote $(L_x)^k(e_\diamond)$ (recall that L_x denotes the left-multiplication map; see Theorem **??**), and n^k to denote $\phi_n(x^k)$. For example, if $\gamma \in \langle\!| x, \diamond |\!\rangle$ is the shape $(x \diamond x) \diamond (x \diamond x)$ then $\phi_n(\gamma) = (n * n) * (n * n) = (n^2)^2$. Observe that in this notation Property **??** refers to n^2, Property **??** refers to n^4, and Property **??** refers to 0^n.

Theorem 4.1 (Monogenicity Theorem) *The loop $\mathcal{A}_s$ is monogenic if and only if the seed s satisfies $s \geq 2$. For seed $s = 2$, every element $n > s$ is a generator, and for $s > 2$, every element $n \neq s$ is a generator.*

Proof. This proof refers repeatedly to the properties of Section **??**; in particularly we will use Property **??** without further comment. There are, in addition, a number of "special case" computations which the reader can easily carry out by hand.

Case 1: $s = 0$.
We have $n * n = 0$ for all $n \in \mathbb{N}$. Since 0 is the $*$-identity, each element n generates only $\{0, n\}$.

Case 2: $s = 1$.
For $n > 1$, we have $n * n = 0$, so n such that $n > 1$ generates either $\{n, 0, n-1, 1\}$ (if n is odd) or $\{n, 0, n+1, 1\}$ (if n is even). It is easy to check that for $n = 0$ just $\{0, 1\}$ is generated. Of course, $n = 1$ does not generate, since it is the $*$-identity.

Case 3: $s = 2$.
First note that none of the elements $0, 1, 2$ generate; in fact, $\{0, 1, 2\}$ is a group under $*$. We proceed by showing that the element 3 generates, and then that for each $n > 3$, the element n generates 3.

Since $3^2 = 0, 3^3 = 4, 3^4 = 1, 3^5 = 5$, and $3^6 = 2$, we see that 3 generates all elements $2k$ and $2k + 1$ for $k \leq 2$. Proceeding by induction on k, if 3 generates $2k$ and $2k + 1$, then since $3^2 * (2k+1) = 0 * (2k+1) = 2k+2$ and $3 * (2k+2) = 2k+3$, we see that 3 generates $2(k+1)$ and $2(k+1)+1$. This shows that 3 is a generator when $s = 2$.

We now consider $n > 3$ when $s = 2$. We have $4^3 = 3$, $5 * (5^2 * 5^2) = 3$, and $6^2 * 6^5 = 3$, so $n = 4, 5, 6$ all generate. Letting $\phi_n \colon \langle\!\langle x, \diamond \rangle\!\rangle \to \mathcal{A}_2$ denote "evaluation at n," we now note that for each $n \geq 7$ either $\phi_n(x^3 \diamond x^5) = 3$ or $\phi_n(x \diamond [x^3 \diamond x^4]) = 3$. To show this, we argue mod 4 using various properties from Section **??**:

1. If $n = 4k+1$ for some $k \geq 2$, then $n*[n^3*n^4] = n*[(n*0)*1] = (4k+1)*[(4k+2)*1] = (4k+1)*(4k) = 3$.
2. If $n = 4k+2$ for some $k \geq 2$, then $n^3 * n^5 = (n*0)*(n*1) = (4k+1)*(4k) = 3$.
3. If $n = 4k+3$ for some $k \geq 1$, then $n^3 * n^5 = (n*0)*(n*1) = (4k+4)*(4k+5) = 3$.
4. If $n = 4k+4$ for some $k \geq 1$, then $n*[n^3*n^4] = n*[(n*0)*1] = (4k+4)*[(4k+3)*1] = (4k+4)*(4k+5) = 3$

It follows that when $s = 2$ every $n > 2$ generates the element 3, hence the entire loop $\mathcal{A}_2$.

Case 4: $s > 2$.
In these cases we will show that 0 is a generator. From this it will follow that every n with $n \neq s$ is a generator of $\mathcal{A}_s$, since for $n \neq 0, s$ we have $n^2 = 0$. Of course, s is the identity element, so it does not generate.

By Property **??** we have $0^k = k - 1$ for all seeds $s > 2$ and for $k = 1, 2, 3, \ldots s + 1$. Thus 0 generates all of the elements $0, \ldots, s$.

For seeds $s \equiv 0, 1 \bmod 3$ we have $0^2 * 0^s = 1 * (s-1) = s+1$; for seeds $s \equiv 2 \bmod 9$ we have $0^3 * 0^{s-1} = 2*(s-2) = s+1$; and for seeds $s \equiv 5, 8 \bmod 9$ we have $0^3 * 0^s = 2*(s-1) = s+1$.

Thus 0 generates $s+1$.

For all seeds s, $0 * (s+1) = s+2$. Thus 0 generates all of the elements $0, \ldots, s+2$.

Sub-Case 4a: $s > 2$ and $s \equiv 0, 1 \bmod 3$.
We have $0^2 * (s+j) = 1 * (s+j) = s+j+1$ for j even with $j \geq 2$, and $0 * (s+j) = s+j+1$ for j odd with $j \geq 1$. Thus, a simple proof by induction shows that 0 generates $\mathcal{A}_s$ for seeds $s \equiv 0, 1 \bmod 3$.

Sub-Case 4b: $s > 2$ and $s \equiv 2 \bmod 3$.
We have $0^2 * (s+1) = 1 * (s+1) = s+3$ and $0^2 * (s+2) = 1 * (s+2) = s+4$. Thus, we can generate through $s+4$.

For seeds $s \equiv 5, 8 \bmod 9$ we have $0^3 * (s+4) = 2 * (s+4) = s+5$ and for seeds $s \equiv 2 \bmod 9$ we have $0^3 * (s+3) = 2 * (s+3) = s+5$. Thus 0 generates $0, \ldots, s+5$, for $s \equiv 2 \bmod 3$.

Proceeding by induction on a variable j we assume that 0 generates $s+1$, $s+4j+2$, $s+4j+3$, $s+4j+4$, and $s+4j+5$ for $0 \leq j \leq J$; we have proven this for $j=0$ above. Since for all $k \geq 1$ we have (noting that $1 = 0^2$ and $2 = 0^3$)

$$\begin{aligned}
0 * (s+4k+1) &= s+4k+2 \\
1 * (s+4k+1) &= s+4k+3 \\
0 * (s+4k+3) &= s+4k+4 \\
2 * (s+4k+4) &= s+4k+5 \quad \text{for } s \equiv 5, 8 \bmod 9 \\
2 * (s+4k+3) &= s+4k+5 \quad \text{for } s \equiv 2 \bmod 9,
\end{aligned}$$

the induction is completed by taking $k = J+1$ in these identities. $\square$

A loop is said to be **finitely-related** if it can be described in terms of generators and relations using only finitely many relations.

Proposition 4.1 *The loop $\mathcal{A}_s$ is not finitely-related for any seed s.*

Proof. We first show that for each seed s, each element n in $\mathcal{A}_s$ satisfies a relation $\phi_n(w) = s$ for some $w \in \langle\!\langle\, x, \diamond \,\rangle\!\rangle$; from this it follows that $\mathcal{A}_s$ does not contain a free subloop. In $\mathcal{A}_0$ we may take $w = x^2$. In $\mathcal{A}_1$ we may take $w = (x^2)^2$. In $\mathcal{A}_s$ for $s \geq 2$ we take $w = (x^2)^{s+1}$ for $n \neq 0$, and $w = x^{s+1}$ for $n = 0$. That each of these choices of w has the required property follows from Propositions **??**, **??**, and **??**.

Now $\mathcal{A}_s$ is infinite and, by Theorem **??**, finitely generated. Since an infinite finitely-generated loop which is finitely-related contains a free subloop [**?**, §3.3], it must be that $\mathcal{A}_s$ is not finitely related. $\square$

5 The Loop Homomorphism Theorem

Theorem 5.1 (Loop Homomorphism Theorem) *The only loop homomorphism* $f : \mathcal{A}_s \to \mathcal{A}_t$ *for* $s \neq t$ *and either* $s \geq 2$ *or* $t \geq 2$ *(or both) is the trivial map* $\mathcal{A}_s \to \{t\}$*. For* $s = t \geq 2$ *a homomorphism* f *is either the trivial map* $\mathcal{A}_s \to \{s\}$ *or the identity map.*

Proof. Let $s, t \in \mathbb{N}_0$ and let$f : \mathcal{A}_s \to \mathcal{A}_t$ be a loop homomorphism. Recall that s is the identity element of $\mathcal{A}_s$ for all s. We have $f(s) * f(s) = f(s * s) = f(s) = f(s) * t$ so $f(s) = t$ for all f, by the cancellation property (see Theorem **??**). We let $\phi_m\colon \langle\!\langle\, x; \diamond \,\rangle\!\rangle \to \mathcal{A}_s$ and $\psi_m\colon \langle\!\langle\, x; \diamond \,\rangle\!\rangle \to \mathcal{A}_t$ denote evaluation homomorphisms as in Section **??**.

As before, we use various properties from Section **??** without explicit mention. The proof is broken into cases, according to the values of s and t.

Case 1: $s, t \geq 2$ and $f(a) = 0$ or $f(a) = t$ for some $a \neq 0, s$.
Suppose first $f(a) = t$ for some $a \neq 0, s$. Then

$$f(0) = f(a * a) = f(a) * f(a) = t * t = t$$

so $f(0) = t$ also. But then for all $i \neq 0, s$ we have

$$t = f(0) = f(i * i) = f(i) * f(i),$$

which can only occur if $f(i) = t$. Thus f is the trivial map.

Suppose now that $f(a) = 0$ for some $a \neq 0, s$. Then

$$f(0) = f(a * a) = f(a) * f(a) = 0 * 0 = 1.$$

Let b, b' be mutual inverses in $\mathcal{A}_s$, where $b, b' \neq 0, a, s$. Then $1 = f(0) = f(b*b) = f(b)*f(b)$ means that $f(b) = 0$, and likewise $f(b') = 0$. But $t = f(s) = f(b * b') = f(b) * f(b') = 0 * 0 = 1$, which is a contradiction to $t \geq 2$.

Case 2: $s, t \geq 2$ and $f(a) \neq 0, t$ for any $a \neq 0, s$.
Suppose $a \neq 0, s$. We then have

$$f(0) = f(a * a) = f(a) * f(a) = 0,$$

and Property **??** now gives $n = 0^{n+1} = f(0^{n+1}) = f(n)$ for all $1 \leq n \leq s$. Thus f is the identity map for all $0 \leq n \leq s$, and in particular, $f(s) = s$. Since we know that $f(s) = t$, this gives us that Case 2 may only occur if $s = t$, *i.e.,* different seeds give non-isomorphic loops.

Note that for $s = t > 2$, the fact that 0 is a generator (by Theorem **??**), together with $f(0) = 0$ as shown above, forces f to be the identity map $\mathcal{A}_s \to \mathcal{A}_s$.

Now we consider $s = t = 2$. Let $f\colon \mathcal{A}_2 \to \mathcal{A}_2$ be a loop endomorphism. By the hypothesis of Case 2 we have $f(3) \neq 0, 2$. It is also true that $f(3) \neq 1$, since otherwise we have

$$1 = f(3) = f(0 * 4) = f(0) * f(4) = 0 * f(4),$$

which forces $f(4) = 0$, contradicting the hypothesis of Case 2. Thus $f(3) \geq 3$. We first show that $f(3) \leq 6$, then verify that $f(3) \neq 4, 5, 6$.

Let $\omega \in \langle\!\langle x, \diamond \rangle\!\rangle$ denote the shape $\omega = \left[x^2 \diamond \left(\left(x^2\right)^2 \diamond x\right)\right]$ and let $\alpha, \beta \in \langle\!\langle x, \diamond \rangle\!\rangle$ denote the shapes

$$\alpha = (x^2)^2 \diamond \omega \qquad \text{and} \qquad \beta = \big((x^2)^2 \diamond x\big) \diamond (x \diamond \omega)\,.$$

Using various Pattern Properties we can calculate that $\phi_3(\alpha) = 4 = \phi_3(\beta)$. This gives

$$\phi_{f(3)}(\alpha) \;=\; f \circ \phi_3(\alpha) \;=\; f \circ \phi_3(\beta) \;=\; \phi_{f(3)}(\beta).$$

We will now show that for $s = t = 2$ and $f(3) = n > 6$ we in fact have $\phi_n(\alpha) \neq \phi_n(\beta)$, and thus $f(3) = n \leq 6$. (The fact that this works only for $n > 6$ is due to the hypothesis of Property **??**.)

First, we calculate $\phi_n(\alpha)$ for $n > 6$:

$$\begin{aligned}
\phi_n(\alpha) = 1 * [0 * (1 * n)] \quad &= \quad \begin{cases} 1 * [0 * (n-2)] & \text{if } n - 2 \equiv 0, 3 \bmod 4 \\ 1 * [0 * (n+2)] & \text{if } n - 2 \equiv 1, 2 \bmod 4 \end{cases} \\[1em]
&= \quad \begin{cases} 1 * [n-3] & \text{if } n - 2 \equiv 0 \bmod 4 \\ 1 * [n-1] & \text{if } n - 2 \equiv 3 \bmod 4 \\ 1 * [n+3] & \text{if } n - 2 \equiv 1 \bmod 4 \\ 1 * [n+1] & \text{if } n - 2 \equiv 2 \bmod 4 \end{cases} \\[1em]
&= \quad \begin{cases} n-1 & \text{if } n - 2 \equiv 0 \bmod 2 \\ n+1 & \text{if } n - 2 \equiv 1 \bmod 2 \end{cases}
\end{aligned}$$

Now we calculate $\phi_n(\beta)$ for $n > 6$:

$$\begin{aligned}
&\phi_n(\beta) = \\
&(1 * n) * (n * [0 * (1 * n)]) \quad = \quad \begin{cases} (n-2) * (n * [0 * (n-2)]) & \text{if } n - 2 \equiv 0, 3 \bmod 4 \\ (n+2) * (n * [0 * (n+2)]) & \text{if } n - 2 \equiv 1, 2 \bmod 4 \end{cases} \\[1em]
&\qquad\qquad\qquad\qquad\qquad\quad\; = \quad \begin{cases} (n-2) * (n * [n-3]) & \text{if } n - 2 \equiv 0 \bmod 4 \\ (n-2) * (n * [n-1]) & \text{if } n - 2 \equiv 3 \bmod 4 \\ (n+2) * (n * [n+3]) & \text{if } n - 2 \equiv 1 \bmod 4 \\ (n+2) * (n * [n+1]) & \text{if } n - 2 \equiv 2 \bmod 4 \end{cases}
\end{aligned}$$

By Properties **??** and **??** we have $\phi_n(\beta) = (n+2) * 3 = n+3$ for $n \equiv 0 \bmod 4$ and $\phi_n(\beta) = (n-2) * 3 = n-3$ for $n \equiv 1 \bmod 4$. Thus $\phi_n(\alpha) \neq \phi_n(\beta)$ for $n \equiv 0, 1 \bmod 4$, where $n > 6$.

To show that $\phi_n(\alpha) \neq \phi_n(\beta)$ for other values of n, note that we must show that

$$(n-2) * (n * [n-3]) \neq n-1 \text{ when } n \equiv 2 \bmod 4,$$

and that

$$(n+2) * (n * [n+3]) \neq n+1 \text{ when } n \equiv 3 \bmod 4.$$

But by Property **??**, for any n with $n \equiv 2 \bmod 4$, $n > 6$ the unique solution y to $(n-2)*y = n-1$ is 3, and for any n with $n \equiv 3 \bmod 4$, $n > 6$ the unique solution y to $(n+2)*y = n+1$ is also 3. By Property **??**, it is not the case that $n * [n-3] = 3$, nor that $n * [n+3] = 3$, and thus $\phi_n(\alpha) \neq \phi_n(\beta)$ for $n \equiv 2, 3 \bmod 4$, where $n > 6$.

From this we see that $f(3) = n$ must satisfy $f(3) \leq 6$.

Now let $\omega \in \langle\!\langle x, \diamond \rangle\!\rangle$ denote the shape $\omega = x^2 \diamond \left(x \diamond \left[\left(x^2\right)^2 \diamond \left(x^2 \diamond x\right)\right]\right)$ and consider the shapes $\gamma, \delta \in \langle\!\langle x, \diamond \rangle\!\rangle$ given by

$$\gamma = \left(x^2 \diamond x\right) \diamond \left(\left(x^2\right)^2 \diamond \omega\right) \qquad \text{and} \qquad \delta = \left(\left(x^2\right)^2 \diamond \left[x^2 \diamond x\right]\right) \diamond \omega.$$

It is straightforward to verify that $\phi_3(\gamma) = 13 = \phi_3(\delta)$ but $\phi_n(\gamma) \neq \phi_n(\delta)$ for $n = 4, 5, 6$. Thus, $f(3) \neq 4, 5, 6$. We have shown that $f(3) = 3$; since 3 is a generator of $\mathcal{A}_2$ by Theorem **??**, it follows that f is the identity map.

Case 3: $s = 0, 1$ and $t \geq 2$
First, let $s = 0$. For any $n \in \mathcal{A}_0$ we have

$$[f(n)]^2 \;=\; f(n^2) \;=\; f(0) \;=\; t.$$

Since $t \geq 2$, this tells us that $f(n) = t$ for all n.

Now let $s = 1$. For any $n \in \mathcal{A}_1$ we have

$$\left([f(n)]^2\right)^2 \;=\; f\left([n^2]^2\right) \;=\; f(1) \;=\; t.$$

Since $t \geq 2$, this tells us that $[f(n)]^2 = t$ and thus $f(n) = t$ for all n.

Case 4: $s \geq 2$ and $t = 0, 1$
First, take $s > 2$. For $m > s$, say, we can see that

$$f(1) \;=\; f\left((m^2)^2\right) \;=\; \left([f(m)]^2\right)^2 \;=\; t.$$

Since 1 is a generator of $\mathcal{A}_s$, we see that f is trivial.

We now take $s = 2$. For a shape $\gamma \in \langle\!| x; \diamond |\!\rangle$ let $|\gamma|$ denote the number of x's appearing in γ.

Sub-Case 4a: $s = 2$ and $t = 0$
Recall that $\psi_n \colon \langle\!| x; \diamond |\!\rangle \to \mathcal{A}_t$ is the evaluation map sending x to n. Because $\mathcal{A}_0$ is associative, for any shape γ and any m, we have

$$f \circ \phi_m(\gamma) \;=\; \psi_{f(m)}(x^{|\gamma|}) \;=\; \begin{cases} f(m) & \text{if } |\gamma| \equiv 1 (\text{mod } 2) \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Now let $\omega \in \langle\!| x; \diamond |\!\rangle$ denote the shape $\omega = \left(x \diamond \left[x^3 \diamond \left(x^2\right)^2\right]\right)$ and let $\gamma, \delta \in \langle\!| x; \diamond |\!\rangle$ denote the shapes

$$\gamma = \left(x^2\right)^2 \diamond \omega \qquad \text{and} \qquad \delta = x \diamond \left(x^2 \diamond \omega\right).$$

Note that $\phi_3(\gamma) = 9 = \phi_3(\delta)$, but $|\gamma| = 12 \neq 11 = |\delta|$. By Equation (**??**) we have

$$0 = f \circ \phi_3(\gamma) = f \circ \phi_3(\delta) = \psi_{f(3)}(\delta) = f(3).$$

Since 3 generates $\mathcal{A}_2$, we see that f is trivial.

Sub-Case 4b: $s = 2$ and $t = 1$
From the relation $f(1) = f(0) * f(0)$ we see that $f(1)$ is either 0 or 1. The observation

$$1 \;=\; f(2) \;=\; f(0 * 1) \;=\; f(0) * f(1)$$

therefore implies that $f(0) = f(1) \in \{0, 1\}$. But the relation $f(1) = f(0) * f(0)$ precludes $f(0) = f(1) = 0$, and thus $f(0) = f(1) = f(2) = 1$. Since, for any $n > 2$, we have

$$1 \;=\; f(0) \;=\; f(n) * f(n)$$

it follows that $\operatorname{im} f \subseteq \{0, 1\}$. Since $\{0, 1\}$ is an associative subloop of $\mathcal{A}_1$, we see that for any shape γ and any m we have

$$f \circ \phi_m(\gamma) \;=\; \psi_{f(m)}(x^{|\gamma|}) \;=\; \begin{cases} f(m) & \text{if } |\gamma| \equiv 1 (\text{mod } 2) \\ 1 & \text{otherwise} \end{cases}$$

We may now complete the proof, *mutatis mutandis*, using the shapes γ and δ as in Subcase 4a.

□

6 Another algebraic property

There is much more to study regarding the algebraic structure of the arrays $\mathcal{A}_s$. For instance, as mentioned in the introduction, we have found the following:

For each of the seeds $s = 0, 1, 2, 3, 5, 7$ *and each pair* i, j *satisfying* $i > s, j > s$, *and* $i * j > s$, *we have* $i * (i * j) = j$.

Note that this algebraic property amounts to a connection between the row index, the column index, and the value of an entry.

This property seems to not be shared by $\mathcal{A}_s$ for any other seeds s.

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