

# The Hopf Rings for $KO$ and $KU$

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## Abstract

In this work we compute the Hopf rings for the spectra representing orthogonal and unitary  $K$ -theory,  $KO$  and  $KU$ . Specifically, we use Hopf ring properties to find and simplify the ordinary mod 2 homology groups for these spectra. Our main tool is the bar spectral sequence, which allows us to advance from one space to the next in the spectrum.

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## 1 Introduction

The object of this paper is to compute the mod 2 Hopf rings,  $H_*(\underline{KO}_*; \mathbf{Z}/2)$  and  $H_*(\underline{KU}_*; \mathbf{Z}/2)$ , for the periodic Bott spectra  $KO$  and  $KU$ . (For the remainder of this paper, we use the notation  $H_*X$  to mean  $H_*(X; \mathbf{Z}/2)$ .) The spaces in these spectra are the infinite classical groups and their coset spaces, and their homology was first calculated in [1], but the Hopf ring structure was first determined in the second author's unpublished thesis [8]. The presentation given here serves as an introduction

to the first author's much more intricate work on the connective spectrum  $bo$ . The Hopf ring viewpoint turns out to be very convenient for understanding the homological effect of various maps between classical groups and fibrations of their connective covers, for example in transferring the results of [3] from a complex context to a real context; details will be given elsewhere.

We next give a brief statement of our results (using some standard Hopf ring notation that will be recalled in section 2). We push forward the usual generators of  $H_*\mathbf{R}P^\infty$  along the inclusion  $\mathbf{R}P^\infty = BO(1) \subseteq \{1\} \times BO \subseteq \mathbf{Z} \times BO = \underline{KO}_0$  to get elements  $z_k \in H_k\underline{KO}_0$ . We also write  $z_k$  for the image of this element in  $H_k\underline{KU}_0$  (which is zero when  $k$  is odd). We put  $\bar{z}_k = z_k/z_0 = z_k * z_0^{-1} = z_k * [-1]$ . It is well-known that

$$KO_* = \mathbf{Z}[\eta, \beta, \lambda^{\pm 1}] / (\eta^3, 2\eta, \eta\beta, \beta^2 - 4\lambda),$$

where  $\deg(\eta) = 1$ ,  $\deg(\beta) = 4$ , and  $\deg(\lambda) = 8$ , and that

$$KU_* = \mathbf{Z}[\nu^{\pm 1}],$$

where  $\deg(\nu) = 2$ . This gives the ring-ring elements

$$[\eta] \in H_0\underline{KO}_{-1}, [\beta] \in H_0\underline{KO}_{-4}, [\lambda] \in H_0\underline{KO}_{-8}, \text{ and } [\nu] \in H_0\underline{KU}_{-2}.$$

(For more information about ring-ring elements of the form above, see section 3.)

We also have the element  $[1] = z_0 \in H_0\underline{KO}_0$ .

For each of the spaces  $X$  under consideration, the ring  $H_*X$  is either polynomial or exterior on countably many generators, possibly with a polynomial generator inverted. The generators are indexed by a parameter  $i$ , which always runs from 0 to  $\infty$ . If the generator to be inverted has the form  $[x]$ , then the inverse is  $[-x]$ . The map  $x \mapsto [\lambda] \circ x$  gives an isomorphism  $H_*\underline{KO}_n \rightarrow H_*\underline{KO}_{n-8}$ , so we need only describe  $H_*\underline{KO}_n$  for  $0 \leq n < 8$ . Similarly, we have  $H_*\underline{KU}_n \simeq H_*\underline{KU}_{n-2}$ .

Our detailed answer for  $KO$  is as follows.

$$\begin{aligned} H_*\underline{KO}_0 &= H_*(\mathbf{Z} \times BO) &= P(\bar{z}_i, [-1]) \\ H_*\underline{KO}_1 &= H_*(U/O) &= P(e \circ z_{2i}) \\ H_*\underline{KO}_2 &= H_*(Sp/U) &= P(e^{\circ 2} \circ z_{4i}) \\ H_*\underline{KO}_3 &= H_*(Sp) &= E(e^{\circ 3} \circ z_{4i}) \\ H_*\underline{KO}_4 &= H_*(\mathbf{Z} \times BSp) &= P(\bar{z}_{4i} \circ [\beta\lambda^{-1}], [-\beta\lambda^{-1}]) \\ H_*\underline{KO}_5 &= H_*(U/Sp) &= E(e \circ z_{4i} \circ [\beta\lambda^{-1}]) \\ H_*\underline{KO}_6 &= H_*(O/U) &= E(\bar{z}_{2i} \circ [\eta^2\lambda^{-1}]) \\ H_*\underline{KO}_7 &= H_*(O) &= E(\bar{z}_i \circ [\eta\lambda^{-1}]) \\ H_*\underline{KO}_8 &= H_*(\mathbf{Z} \times BO) &= P(\bar{z}_i \circ [\lambda^{-1}], [-\lambda^{-1}]) \cong H_*\underline{KO}_0 \end{aligned}$$

The mod 2 homology for the first term,  $\underline{KO}_0 = \mathbf{Z} \times BO$ , is well known; accordingly, we will take  $\underline{KO}_0$  to be our starting point, and deloop to  $\underline{KO}_1$  using the bar spectral sequence. This process will continue, from  $\underline{KO}_1$  to  $\underline{KO}_2$ , etc., throughout the 8-space cycle until we end with  $\underline{KO}_8 = \underline{KO}_0$ . We will prove that our spectral sequences collapse and solve the extension problems using Hopf ring relations in conjunction with the Frobenius and Verschiebung maps.

We also record the answer for  $KU$ ; this is essentially well-known, but it is convenient to have it stated in a way that allows easy comparison with the case of  $KO$ .

$$\begin{aligned} H_*\underline{KU}_0 &= H_*(\mathbf{Z} \times BU) = P(z_{2i}, [-1]) \\ H_*\underline{KU}_1 &= H_*(U) = E(e \circ z_{2i}) \\ H_*\underline{KU}_2 &= H_*(\mathbf{Z} \times BU) = P(z_{2i} \circ [\nu^{-1}], [-\nu^{-1}]) \end{aligned}$$

The Hopf ring relations that we need are as follows:

1.  $z_j \circ z_k = \frac{(j+k)!}{j!k!} z_{j+k}$
2.  $e^2 = e \circ z_1$
3.  $(e^{\circ 2})^2 = e^{\circ 2} \circ z_2$
4.  $(e^{\circ 3})^2 = 0$
5.  $e^{\circ 4} \circ [\lambda] = [\beta] \circ \bar{z}_4$
6.  $e \circ [\eta] = \bar{z}_1$
7.  $e^{\circ 2} \circ [\beta] = \bar{z}_2 \circ [\eta^2]$
8.  $\bar{z}_1 \circ [\beta] = \bar{z}_2 \circ [\beta] = 0$
9.  $\bar{z}_1 \circ \bar{z}_{2i+1} = \bar{z}_1^2 \circ \bar{z}_{2i}$

Proofs are distributed throughout the paper.

We conclude our introduction with a brief discussion of various maps of spectra. There is a complexification map  $m : KO \rightarrow KU$ , a complex conjugation map  $c : KU \rightarrow KU$ , and a map  $f : KU \rightarrow KO$  that forgets the complex structure. These satisfy

1.  $cm = m, fc = f, c^2 = 1, fm = 2, mf = 1 + c$
2.  $m_*(\eta) = 0, m_*(\beta) = 2\nu^2, m_*(\lambda) = \nu^4$
3.  $c_*(\nu) = -\nu$
4.  $f_*(1) = 2, f_*(\nu) = \eta^2, f_*(\nu^2) = \beta, f_*(\nu^3) = 0.$

The maps  $m$  and  $c$  are ring maps and thus induce maps of Hopf rings. The map  $f$  is a  $KO$ -module map, so it satisfies  $f_*(b * c) = f_*(b) * f_*(c)$  and  $f_*(m_*(a) \circ b) = a \circ f_*(b)$  (for  $a \in H_*\underline{KO}_*$  and  $b, c \in H_*\underline{KU}_*$ ). Using these properties, one can determine the effects of  $m_*$ ,  $f_*$ , and  $c_*$  on all the elements of our Hopf rings.

## 2 Hopf rings

Let  $R$  be a graded associative commutative ring with unit and let  $CoAlg_R$  denote the category of graded cocommutative coassociative coalgebras with counit over  $R$ . Then a Hopf ring is a graded ring object in the category  $CoAlg_R$ . A Hopf ring includes a coproduct  $\psi$ , two products - the  $*$ -product and the  $\circ$ -product, conjugation  $\chi$ , counit  $\varepsilon$ , and relationships interlocking each of these maps.

The primary example of Hopf rings is  $F_*\underline{E}_*$ , where  $F_*$  is a multiplicative homology theory and  $\underline{E}_*$  is an  $\Omega$ -spectrum. For more information about maps and properties of Hopf rings, please see [6].

## 3 The Homotopy Elements $[x]$

The computations of  $H_*\underline{KO}_0$  and  $H_*\underline{KU}_0$  will require the use of elements from the homotopy groups of  $KO$  and  $KU$ , as ring-ring elements. In particular, we will make heavy use of the homotopy elements

$$[\eta], [\beta], [\lambda], \text{ and } [\nu].$$

More information is available in [9] and [8].

Following [6], suppose  $\mathcal{S}$  is a homotopy category of topological spaces (with certain properties), and  $F_*(-)$  is an associative commutative multiplicative unreduced generalized homology theory with unit defined on  $\mathcal{S}$ . If we let  $G^*(-)$  be a similar cohomology theory (also defined on  $\mathcal{S}$ ), then  $G^*(-)$  has a representing  $\Omega$ -spectrum

$$\underline{E}_* = \{\underline{E}_n\}_{n \in \mathbf{Z}} \in \text{gr}\mathcal{S},$$

i.e.  $G^n(X) \simeq [X, \underline{E}_n]$  and  $\Omega \underline{E}_{n+1} \simeq \underline{E}_n$  (with  $\text{gr}\mathcal{S}$  the category of graded objects of  $\mathcal{S}$ ). Denoting the two coefficient rings by  $F_*$  and  $G^*$ , we let  $x \in G^n$  have degree  $-n$  in the coefficient ring. Then  $x \in G^n \simeq [\text{point}, \underline{E}_n]$  and so we have a map in homology  $x_* : F_* \rightarrow F_*\underline{E}_n$ . We define  $[x] \in F_0\underline{E}_n$  to be the image of  $1 \in F_*$  under this map.

If we take  $z \in G^m$  and  $x, y \in G^n$ , then

1.  $[z] \circ [x] = [zx] = [-1]^{\text{onk}} \circ [x] \circ [z]$ .
2.  $[x] * [y] = [x + y] = [y + x] = [y] * [x]$ .
3.  $\psi[z] = [z] \otimes [z]$ .
4. The sub-Hopf algebra of  $F_*\underline{E}_*$  generated by all  $[x]$  with  $x \in G^*$  is the ring-ring of  $G^*$  over  $F_*$ , i.e.  $F_*[G^*]$ .

## 4 Properties of $H_*\underline{KO}_0 = H_*(\mathbf{Z} \times BO)$

In this section, we record the known mod 2 homology for  $\underline{KO}_0$  and introduce Hopf ring properties for the elements in homology. We will compute  $H_*\underline{KO}_0 = H_*(\mathbf{Z} \times BO) \cong H_*(\mathbf{Z}) \otimes H_*(BO)$ , where  $H_*(\mathbf{Z})$  is concentrated in deg 0 and  $\mathbf{Z}$  is the set of integers with the discrete topology.

To understand  $H_*(BO)$ , we examine

$$\mathbf{R}P^\infty = 1 \times BO(1) \subset 1 \times BO \subset \mathbf{Z} \times BO = \underline{KO}_0.$$

Recall that the homology of real projective space is given by

$$H_*(\mathbf{R}P^\infty) = \mathbf{Z}/2\{b_i : i \geq 0\},$$

the free  $\mathbf{Z}/2$ -module on the elements  $b_i$  with  $\deg(b_i) = i$ . The map

$$\mathbf{R}P^\infty \rightarrow 1 \times BO,$$

which classifies the unreduced canonical line bundle, induces an embedding in homology. We denote the image of  $b_i$  under this map as  $z_i \in H_*(1 \times BO)$ . The classical result is that

$$H_*(1 \times BO) = P(z_i : i > 0),$$

and therefore

$$H_*(BO) = P(\bar{z}_i : i > 0).$$

We may gain information about the elements in  $BO$  by considering their corresponding behavior in  $\mathbf{R}P^\infty$ :

1. The coproduct in  $H_*(\mathbf{R}P^\infty)$  is given by  $\psi(b_i) = \sum_{j+k=i} b_j \otimes b_k$ , so the same holds true in  $H_*(BO)$ :

$$\psi(z_i) = \sum_{j+k=i} z_j \otimes z_k.$$

2. If we define  $z(a) = \sum z_i a^i$ , we see that consistent with the coproduct is the equality

$$z(a) \circ z(b) = z(a + b).$$

By comparing coefficients, we obtain

$$z_j \circ z_k = \frac{(j+k)!}{j!k!} z_{j+k}.$$

Looking to  $\mathbf{Z}$ , we note that its mod 2 homology is given by the ring-ring  $\mathbf{Z}/2[\mathbf{Z}]$ , whose polynomial generators are of the form  $[b]$ , for  $b \in \mathbf{Z}$ . The Hopf ring properties for these elements can be found in Section 3. Note that there are two equivalent ways to express the same element,  $1 = [0_0]$ , in  $H_0\underline{KO}_0$ . Additionally,  $z_0$  is the basis element for  $H_0(\mathbf{Z} \times BO) = \mathbf{Z}/2[\pi_0(\mathbf{Z} \times BO)]$ , corresponding to the component  $1 \times BO$ , and will also be denoted by the ring-ring element  $[1]$ . Accordingly,  $\mathbf{Z}/2[\mathbf{Z}]$  can be written as  $P([1], [-1]) = P(z_0^{\pm 1})$ .

We denote the element  $z_i * [-1] \in H_*\underline{KO}_0$  by  $\bar{z}_i = z_i/z_0$ , allowing us to prove:

**Lemma 4.1.** For any element  $[x]$ ,

$$[x] \circ \bar{z}_i = 0 \text{ if and only if } [x] \circ z_i = 0.$$

*Proof.* Suppose that  $[x] \circ \bar{z}_i = 0$ . Then the distributive property shows

$$[x] \circ z_i = [x] \circ (\bar{z}_i * [1]) = ([x] \circ \bar{z}_i) * ([x] \circ [1]) = 0.$$

Conversely, suppose  $[x] \circ z_i = 0$ . Then

$$[x] \circ \bar{z}_i = [x] \circ (z_i * [-1]) = ([x] \circ z_i) * ([x] \circ [-1]) = 0.$$

□

In summary, our Hopf ring relations thus far are given by

1.  $z_0 = [1]$
2.  $1 = [0_0]$
3.  $\psi(z_i) = \sum_{j+k=i} z_j \otimes z_k$  and  $\psi(z(t)) = z(t) \otimes z(t)$
4.  $z_j \circ z_k = \frac{(j+k)!}{j!k!} z_{j+k}$
5.  $\psi[z] = [z] \otimes [z]$
6.  $[x] * [y] = [x + y]$
7.  $[z] \circ [x] = [zx]$

and we will begin our calculations with

$$\begin{aligned} H_* \underline{KQ}_0 &= H_*(\mathbf{Z} \times BO) = \mathbf{Z}/2[\mathbf{Z}] \otimes P(\bar{z}_i : i > 0) = P(z_0^{\pm 1}) \otimes P(\bar{z}_i : i > 0) = \\ &P(\bar{z}_i, [-1]). \end{aligned}$$

## 5 Machinery

Before delving into our calculations, we quickly review the tools used throughout the rest of this article, namely the bar spectral sequence and the Frobenius and Verschiebung maps.

## 5.1 The Bar Spectral Sequence

Let  $E$  be an  $\Omega$ -spectrum, and let  $\underline{E}'_n$  be the component of the basepoint in the  $n$ 'th space of  $E$ . The bar spectral sequence (also called the Rothenberg-Steenrod spectral sequence) is really the homology Eilenberg-Moore spectral sequence for the path-loop fibration on the space  $\underline{E}'_{n+1}$ ,

$$\begin{array}{ccccc} \Omega \underline{E}'_{n+1} = \Omega \underline{E}_{n+1} & \longrightarrow & P \underline{E}'_{n+1} & \longrightarrow & \underline{E}'_{n+1} \\ \cong \downarrow & & \cong \downarrow & & \\ \underline{E}_n & & \star & & \end{array}$$

Based on the bar resolution, it is a spectral sequence of Hopf algebras with the basic property that

$$E_{s,t}^2 = \mathrm{Tor}_{s,t}^{H_*(\underline{E}_n)}(\mathbf{F}, \mathbf{F}) \Rightarrow H_{s+t}(\underline{E}'_{n+1}; \mathbf{F}),$$

where  $\mathbf{F} = \mathbf{Z}/2$ . Its differentials are given by

$$d_r : E_{s,t}^r \rightarrow E_{s-r,t+r-1}^r.$$

For more information, see [4].

When  $R$  is a bicommutative Hopf algebra over  $\mathbf{F} = \mathbf{Z}/2$ , then  $\mathrm{Tor}_{*,*}^R(\mathbf{Z}/2, \mathbf{Z}/2) = \mathrm{Tor}_{**}^R(\mathbf{F}, \mathbf{F})$  is again a bicommutative Hopf algebra over  $\mathbf{F}$ . If  $I$  is the augmentation ideal, there is a natural isomorphism  $\sigma : I/I^2 \rightarrow \mathrm{Tor}_{1,*}^R(\mathbf{F}, \mathbf{F})$ . Moreover, if  $y \in I_k$  and  $y^2 = 0$  there are naturally defined divided powers  $\gamma_j(\sigma(y)) \in \mathrm{Tor}_{j,jk}^R$ . The following computations for  $\mathrm{Tor}_{*,*}^R(\mathbf{Z}/2, \mathbf{Z}/2)$  are well known (see [7]):

1.  $\mathrm{Tor}_{*,*}^{P(x)}(\mathbf{Z}/2, \mathbf{Z}/2) = E[\sigma(x)]$ , the exterior algebra on  $\sigma(x) \in \mathrm{Tor}_{1,*}$ .
2.  $\mathrm{Tor}_{*,*}^{E(x)}(\mathbf{Z}/2, \mathbf{Z}/2) = \Gamma[\sigma(x)]$ , the divided power algebra on  $\sigma(x)$ , which is spanned by the elements  $\gamma_j(\sigma(y))$ .
3.  $\mathrm{Tor}_{*,*}^{\mathbf{Z}/2[\mathbf{Z}]}(\mathbf{Z}/2, \mathbf{Z}/2) = E[\sigma(x)]$ , where  $\mathbf{Z}/2[\mathbf{Z}] = \mathbf{Z}/2(x, x^{-1})$  is the ring-ring.
4.  $\mathrm{Tor}^A \otimes \mathrm{Tor}^B \xrightarrow{\cong} \mathrm{Tor}^{A \otimes B}$ .

## 5.2 Frobenius and Verschiebung Maps

Given  $A$ , a bicommutative Hopf algebra over  $\mathbf{Z}/p$  ( $p$  prime), the Frobenius map  $F : A \rightarrow A$  is described by  $F(x) = x^p$ . The Verschiebung map  $V$  is defined to be the dual of  $F$  on  $A^*$ , the dual of  $A$ . In other words, if  $F' : A^* \rightarrow A^*$  is defined by  $x \mapsto x^p$ , then  $(F')^* = V : A \rightarrow A$ .

From [7], these maps have the following properties (for  $p = 2$ ):

1. With a shift of grading,  $V$  and  $F$  are Hopf algebra maps.

2.  $VF = FV$  :

$$V(x^2) = VF(x) = FV(x) = [V(x)]^2.$$

3.  $V(x \circ y) = V(x) \circ V(y)$

4.  $F(x \circ V(y)) = F(x) \circ y$

5. For the coalgebra  $\Gamma(x)$ ,

$$V(\gamma_{2q}(x)) = \gamma_q(x)$$

$$V(\gamma_q(x)) = 0 \text{ if } q \neq 0 \text{ mod } 2$$

As a consequence, for the elements  $z_i \in H_*\underline{KO}_0$ , we have both  $V(z_{2i}) = z_i$  and  $V(z_{2i+1}) = 0$ .

## 6 The Calculation of the Hopf Ring for $H_*\underline{KO}_*$

We are ready to proceed to the calculation. Each of the following sections contains the computation of the mod 2 homology for one of the spaces in the spectrum  $KO$ . As we continue, we will introduce various Hopf ring relations; we defer the proofs of these relations until the ends of their corresponding sections.

### 6.1 $H_*\underline{KO}_1 = H_*(U/O)$

We start by inputting  $H_*\underline{KO}_0 = H_*(\mathbf{Z} \times BO) = P(z_0^{\pm 1}) \otimes P(\bar{z}_i : i > 0)$  into the bar spectral sequence.

$$\begin{aligned} E_{*,*}^2 &= \text{Tor}_{*,*}^{H_*(\mathbf{Z} \times BO)}(\mathbf{Z}/2, \mathbf{Z}/2) = \text{Tor}_{*,*}^{P(z_0^{\pm 1}) \otimes P(\bar{z}_i : i > 0)}(\mathbf{Z}/2, \mathbf{Z}/2) \\ &= E(\sigma(z_0)) \otimes E(\sigma(\bar{z}_i : i > 0)) = E(\sigma(z_i)) \Rightarrow H_*\underline{KO}_1 = H_*(U/O). \end{aligned}$$

Since the elements  $\sigma(z_i)$  are all in the first filtration, the bar spectral sequence collapses at the  $E^2$ -term, and the  $E^\infty$ -term is  $E(\sigma(z_i))$ .

For any  $x \in H_*\underline{E}_n$ ,  $e \circ x \in H_*\underline{E}_{n+1}$  detects (the image in  $E^\infty$  of)  $\sigma(x)$ , where  $e$  is the fundamental class in  $H_1\underline{KO}_1$ . As such, the element  $e \circ z_i$  detects  $\sigma(z_i)$ . The coproduct for  $e$  is given by

$$\psi(e) = e \otimes 1 + 1 \otimes e = e \otimes [0] + [0] \otimes e.$$

It is useful to note the following:

**Lemma 6.1.** *For all elements  $a$  with  $\varepsilon a = 0$ ,*

$$e \circ (a * [1]) = e \circ a.$$

*In particular, for any  $i > 0$ ,  $e \circ \bar{z}_i = e \circ z_i$ .*

*Proof.* Using distributivity we obtain

$$\begin{aligned} e \circ (a * [1]) &= (e \circ a) * (1 \circ [1]) + (1 \circ a) * (e \circ [1]) \\ &= (e \circ a) * [0] + 0 * (e \circ [1]) = e \circ a. \end{aligned}$$

Thus  $e \circ z_i = e \circ (\bar{z}_i * [1])$  is equivalent to  $e \circ \bar{z}_i$  for all  $i > 0$ .  $\square$

Returning to our calculation, we wish to simplify the  $E^\infty$ -term as a Hopf ring. To do so, we use the relation

$$F(e^{\circ n}) = (e^{\circ n})^2 = e^{\circ n} \circ z_n \quad (1)$$

(to be proved below) with  $n = 1$ . Since  $V(z_{2i}) = z_i$  and  $F(x \circ V(y)) = F(x) \circ y$ , we may write

$$(e \circ z_i)^2 = F(e \circ z_i) = F(e) \circ z_{2i} = (e \circ z_1) \circ z_{2i} = e \circ z_{2i+1}.$$

Note that we have applied the  $\circ$ -product relation for the elements  $z_i$ , from section 4.

We apply this calculation as often as possible, by using the fact that any positive integer  $m$  can be expanded in its binary form;  $m = 2^q(2i + 1) - 1 = 1 + 2 + 2^2 + \dots + 2^{q-1} + 2^{q+1}i$  with  $q \geq 0, i \geq 0$ , both integers. Hence, every element  $e \circ z_m$  is equivalent to the product  $F^q(e \circ z_{2i})$ , and so each element  $e \circ z_{2i}$  is a polynomial generator.

Consequently,  $H_*\underline{KQ}_1 = H_*(U/O) = P(e \circ z_{2i})$ .

### Proofs of Relations for $H_*\underline{KQ}_1$

*Relation (1):*  $F(e^{\circ n}) = (e^{\circ n})^2 = e^{\circ n} \circ z_n$ .

*Proof.* Let  $X$  be an infinite loop space. For each  $n \geq 0$ , the Dyer-Lashof operations  $Q^n : H_q(X) \rightarrow H_{q+n}(X)$  are defined. For our purposes, we need the following properties (from [2] and [5]):

1.  $Q^n([1]) = z_n * [1]$
2.  $Q^n(u) = u^2$  if  $\dim(u) = n$ .
3. The homology suspension homomorphism  $\sigma : H_*(\Omega X) \rightarrow H_{(*+1)}(X)$  yields  $\sigma(Q^n(u)) = Q^n(\sigma(u))$ .

In our case,  $\sigma(x) = e \circ x$ , so

$$F(e^{\circ n}) = (e^{\circ n})^2 = Q^n(e^{\circ n}) = Q^n(e^{\circ n} \circ [1]) = e^{\circ n} \circ Q^n([1]) = e^{\circ n} \circ (z_n * [1]).$$

By lemma 6.1,  $e^{\circ n} \circ (z_n * [1]) = e^{\circ n} \circ z_n$ .  $\square$

## 6.2 $H_*\underline{KO}_2 = H_*(Sp/U)$

The bar spectral sequence produces

$$\begin{aligned} E_{*,*}^2 &= \mathrm{Tor}_{*,*}^{H_*(U/O)}(\mathbf{Z}/2, \mathbf{Z}/2) = \mathrm{Tor}_{*,*}^{P(e \circ z_{2i})}(\mathbf{Z}/2, \mathbf{Z}/2) = E(\sigma(e \circ z_{2i})) \\ &\Rightarrow H_*(Sp/U). \end{aligned}$$

The elements  $\sigma(e \circ z_{2i})$  are all in the first filtration, collapsing the bar spectral sequence at the  $E^2$ -term. Accordingly, the  $E^\infty$ -term is  $E(\sigma(e \circ z_{2i}))$ .

The rest of the proof is the same as the proof of 6.1, except we use property 3 of page 3 instead of property 2. Then we have  $H_*\underline{KO}_2 = H_*(Sp/U) = P(e^{\circ 2} \circ z_{4i})$ .

## 6.3 $H_*\underline{KO}_3 = H_*(Sp)$

The bar spectral sequence gives

$$\begin{aligned} E_{*,*}^2 &= \mathrm{Tor}_{*,*}^{H_*(Sp/U)}(\mathbf{Z}/2, \mathbf{Z}/2) = \mathrm{Tor}_{*,*}^{P(e^{\circ 2} \circ z_{4i})}(\mathbf{Z}/2, \mathbf{Z}/2) = E(\sigma(e^{\circ 2} \circ z_{4i})) \\ &\Rightarrow H_*(Sp) \end{aligned}$$

Again, each generator is located in the first filtration. Accordingly, the bar spectral sequence collapses at the  $E^2$ -term, and the  $E^\infty$ -term is given by  $E(\sigma(e^{\circ 2} \circ z_{4i}))$ .

The rest of the proof is the same as the proof of 6.1, except we use the property

$$(e^{\circ 3})^2 = 0 \tag{2}$$

instead of property 2. We have  $H_*\underline{KO}_3 = H_*(Sp) = E(e^{\circ 3} \circ z_{4i})$ .

### Proofs of Relations for $H_*\underline{KO}_3$

*Relation (2):*  $(e^{\circ 3})^2 = 0$ .

*Proof.* For any elements  $a, b$  with  $\varepsilon a = \varepsilon b = 0$ , we may use the distributive property to obtain  $e \circ (a * b) = (e \circ a) * (1 \circ b) + (1 \circ a) * (e \circ b) = 0$ , using the coproduct for  $e$ . Thus  $e \circ (\text{decomposables}) = 0$ . We use this fact and consequences of relation (1) with  $n = 1, 3$  to compute

$$(e^{\circ 3})^2 = e^{\circ 3} \circ z_3 = e^{\circ 2} \circ e \circ z_1 \circ z_2 = e^{\circ 2} \circ (e)^2 \circ z_2 = 0.$$

□

## 6.4 $H_*\underline{KO}_4 = H_*(\mathbf{Z} \times BSp)$

Since the connected part of  $\underline{KO}_4$  is  $BSp$ , the bar spectral sequence gives

$$\begin{aligned} E_{*,*}^2 &= \text{Tor}_{*,*}^{H_*(Sp)}(\mathbf{Z}/2, \mathbf{Z}/2) = \text{Tor}_{*,*}^{E(e^{\circ 3} \circ z_{4i})}(\mathbf{Z}/2, \mathbf{Z}/2) = \Gamma(\sigma(e^{\circ 3} \circ z_{4i})) \\ &\Rightarrow H_*(BSp). \end{aligned}$$

The bar spectral sequence collapses at the  $E^2$ -term, as each element has even total degree. The  $E^\infty$ -term is therefore given by  $\Gamma(\sigma(e^{\circ 3} \circ z_{4i}))$ , or equivalently by  $E(\gamma_{2j}(\sigma(e^{\circ 3} \circ z_{4i})))$ .

We will simplify in stages, starting with the elements in the first filtration, that is, the elements  $\sigma(e^{\circ 3} \circ z_{4i})$ . As usual,  $e^{\circ 4} \circ z_{4i}$  detects  $\sigma(e^{\circ 3} \circ z_{4i})$ . We use relation (1) with  $n = 4$  and the equality  $V(z_{8i}) = z_{4i}$  to determine the algebra structure:

$$(e^{\circ 4} \circ z_{4i})^2 = F(e^{\circ 4} \circ z_{4i}) = F(e^{\circ 4}) \circ z_{8i} = (e^{\circ 4} \circ z_4) \circ z_{8i} = e^{\circ 4} \circ z_{8i+4}.$$

Again, expanding every positive integer  $m$  in its binary form  $2^q(2i+1) - 1$  yields  $e^{\circ 4} \circ z_{4m} = F^q(e^{\circ 4} \circ z_{8i})$ . Thus the elements in the first filtration are polynomial generators given by  $e^{\circ 4} \circ z_{8i}$ .

Next, we examine the elements of  $\gamma_{2j}$ ,  $j > 0$ . As this calculation will make repeated use of the Verschiebung map, we record the following:

**Proposition 6.2.** *The Verschiebung map has the properties*

1. *the element  $V^j(\gamma_{2j}(\sigma(z_i)))$  is detected by the element  $e \circ z_i$ .*
2. *if  $V^j(x) = z_i$ , then  $x = z_{2^j \cdot i}$ , mod decomposable elements.*

*Proof.* 1. By virtue of the fact that  $V(\gamma_{2q}(x)) = \gamma_q(x)$ , we obtain  $V^j(\gamma_{2j}(\sigma(z_i))) = V^{j-1}(\gamma_{j-1}(\sigma(z_i))) = \dots = \sigma(z_i) = e \circ z_i$ .

2. The facts that  $V(z_{2i}) = z_i$  and  $V(z_{2i+1}) = 0$  may be used to prove this equation.  $\square$

Suppose  $x$  detects  $\gamma_{2j}(\sigma(e^{\circ 3} \circ z_{8i}))$ . Then by proposition 6.2,

$$V^j x = e^{\circ 4} \circ z_{8i}.$$

Further,

$$V^j F^q x = F^q V^j x = F^q(e^{\circ 4} \circ z_{8i}) = e^{\circ 4} \circ z_{4m}$$

detects  $\sigma(e^{\circ 3} \circ z_{4m})$  and thus  $F^q x$  detects  $\gamma_{2j}(\sigma(e^{\circ 3} \circ z_{4m}))$ . Thus the elements  $x$  (as  $i$  and  $j$  vary) are polynomial generators for  $H_*(BSp)$ .

All that remains is to identify the elements  $x$ . To do so, we map down from  $\underline{KO}_4$  to  $\underline{KO}_{-4} = \underline{KO}_4$  via the map  $\circ[\lambda]$ , and seek instead to simplify  $x \circ[\lambda]$ . Application of the relation

$$e^{\circ 4} \circ[\lambda] = \bar{z}_4 \circ[\beta] \tag{3}$$

produces the following, modulo decomposable elements:

$$V^j(x \circ [\lambda]) = e^{\circ 4} \circ z_{8i} \circ [\lambda] = \bar{z}_4 \circ z_{8i} \circ [\beta] = \bar{z}_{8i+4} \circ [\beta].$$

Accordingly, proposition 6.2 shows that

$$x \circ [\lambda] = \bar{z}_{2^j(8i+4)} \circ [\beta] + \text{decomposables.}$$

By virtue of the relations

$$\bar{z}_1 \circ [\beta] = z_1 \circ [\beta] = \bar{z}_{2i+1} \circ [\beta] = 0 \quad (4)$$

$$\bar{z}_2 \circ [\beta] = z_2 \circ [\beta] = \bar{z}_{4i+2} \circ [\beta] = 0 \quad (5)$$

and by the property that every positive number, divisible by 4, has the unique form  $2^j(8i+4)$ , we obtain  $H_*(BSp) = P(\bar{z}_{4i} \circ [\beta] : i > 0) \subset H_*\underline{KQ}_{-4}$ . Thus

$$\begin{aligned} H_*\underline{KQ}_{-4} &= H_*(\mathbf{Z}) \otimes H_*(BSp) = P([\beta][-\beta]) \otimes P(\bar{z}_{4i} \circ [\beta] : i > 0) = \\ &P(\bar{z}_{4i} \circ [\beta], [-\beta]). \end{aligned}$$

To find  $H_*\underline{KQ}_4$ , we use  $\circ[\lambda^{-1}]$  to map up 8 spaces. We obtain  $H_*\underline{KQ}_4 = H_*(\mathbf{Z} \times BSp) = P(\bar{z}_{4i} \circ [\beta\lambda^{-1}], [-\beta\lambda^{-1}])$ .

### Proofs of Relations for $H_*\underline{KQ}_4$

*Relation (3):*  $e^{\circ 4} \circ [\lambda] = \bar{z}_4 \circ [\beta]$ .

*Proof.* There is only one nontrivial element in the bidegree of  $\bar{z}_4 \circ [\beta]$  so either this is zero or it is as claimed above. If it is zero then if we take circle multiplication by  $e^{\circ 4}$  it should still be zero. However,

$$\begin{aligned} e^{\circ 4} \circ [\beta] \circ \bar{z}_4 &= e^{\circ 2} \circ e^{\circ 2} \circ [\beta] \circ \bar{z}_4 \\ &= e^{\circ 2} \circ [\eta^2] \circ \bar{z}_2 \circ \bar{z}_4 && \text{by 7 page 3} \\ &= (e \circ [\eta])^{\circ 2} \circ \bar{z}_2 \circ \bar{z}_4 \\ &= (\bar{z}_1)^{\circ 2} \circ \bar{z}_2 \circ \bar{z}_4 && \text{by 6 page 3} \\ &= \bar{z}_1 \circ \bar{z}_7 && \text{by 1 page 3} \\ &= \bar{z}_1^2 \circ \bar{z}_6 && \text{by 9 page 3} \\ &= (\bar{z}_1 \circ \bar{z}_3)^2 && \text{by distributivity} \\ &= (\bar{z}_1^2 \circ \bar{z}_2)^2 && \text{by 9 page 3} \\ &= (\bar{z}_1 \circ \bar{z}_1)^4 && \text{by distributivity} \\ &= (\bar{z}_1)^8 && \text{by 9 page 3,} \end{aligned}$$

which we know to be non-zero. Thus the relation has to hold. □

*Relation (4):*  $\bar{z}_1 \circ [\beta] = z_1 \circ [\beta] = \bar{z}_{2i+1} \circ [\beta] = 0$ .

*Proof.* We first prove

$$e \circ [\eta] = \bar{z}_1. \quad (6)$$

By definition of  $\eta \in \pi_1(BO)$ , the homotopy element  $\eta : S^1 \rightarrow BO$  classifies the reduced canonical line bundle. Thus  $e \circ [\eta] \in H_1(BO)$  is the image under  $\eta$  of the fundamental class  $e$ , which is nonzero by the Hurewicz theorem. The only possible element in this dimension (in  $0 \times BO$ ) is  $\bar{z}_1$ , proving relation (6).

We return to the proof of relation (4). Using relation (6) and the fact that  $\eta\beta = 0$  gives

$$\bar{z}_1 \circ [\beta] = e \circ [\eta] \circ [\beta] = e \circ [\eta\beta] = e \circ [0] = 0.$$

By lemma 4.1,  $z_1 \circ [\beta] = 0$ . The distributive property completes the proof:

$$\bar{z}_{2i+1} \circ [\beta] = (z_{2i+1} * [-1]) \circ [\beta] = (z_{2i+1} \circ [\beta]) * [-\beta] = (z_{2i} \circ z_1 \circ [\beta]) * [-\beta] = 0.$$

□

*Relation (5):*  $\bar{z}_2 \circ [\beta] = z_2 \circ [\beta] = \bar{z}_{4i+2} \circ [\beta] = 0$ .

*Proof.* We refer to  $KU$ , the unitary Bott spectrum. As we will prove in section 8,

$$H_*\underline{KU}_0 = H_*(\mathbf{Z} \times BU) = P(\bar{z}_{2i}, [-1]).$$

We start by proving the relation

$$e^{\circ 2} \circ [\nu] = \bar{z}_2, \quad (7)$$

where  $e$  is the fundamental class in  $H_*\underline{KU}_1$ . Since  $KU_* = \mathbf{Z}[\nu^{\pm 1}]$ , with  $\deg(\nu) = 2$ , relation (7) is proved using the same argument as relation (6) above.

Relation (5) may now be proven using the forgetful map,  $f : KU \rightarrow KO$ . We build on relation (7) to yield  $e^{\circ 2} \circ [\nu] \circ [\nu^2] = e^{\circ 2} \circ [\nu^3] = \bar{z}_2 \circ [\nu^2]$ . Applying the forgetful map gives

$$0 = e^{\circ 2} \circ [0] = f(e^{\circ 2} \circ [\nu^3]) = f(\bar{z}_2 \circ [\nu^2]) = \bar{z}_2 \circ [\beta].$$

In much the same way as the proof of relation (4), we may use Hopf ring relations to obtain the remaining equalities

$$z_2 \circ [\beta] = 0 \text{ and } \bar{z}_{4i+2} \circ [\beta] = 0.$$

□

## 6.5 $H_*\underline{KQ}_5 = H_*(U/Sp)$

The bar spectral sequence gives

$$E_{*,*}^2 = \mathrm{Tor}_{*,*}^{H_*(\mathbf{Z} \times BSp)}(\mathbf{Z}/2, \mathbf{Z}/2) = \mathrm{Tor}_{*,*}^{P(\bar{z}_{4i} \circ [\beta\lambda^{-1}], [-\beta\lambda^{-1}])}(\mathbf{Z}/2, \mathbf{Z}/2) = \\ E(\sigma(\bar{z}_{4i} \circ [\beta\lambda^{-1}])) \Rightarrow H_*(U/Sp).$$

Due to the fact that every generator is once again located in the first filtration, the bar spectral sequence collapses at the  $E^2$ -term. The  $E^\infty$ -term is thus given by  $E(\sigma(\bar{z}_{4i} \circ [\beta\lambda^{-1}]))$ .

The element  $e \circ \bar{z}_{4i} \circ [\beta\lambda^{-1}]$  detects  $\sigma(\bar{z}_{4i} \circ [\beta\lambda^{-1}])$ . Utilization of relation (4) and lemma 6.1 yields

$$F(e \circ \bar{z}_{4i} \circ [\beta\lambda^{-1}]) = F(e \circ z_{4i} \circ [\beta\lambda^{-1}]) = F(e) \circ z_{8i} \circ [\beta\lambda^{-1}] = \\ (e \circ z_1) \circ z_{8i} \circ [\beta] \circ [\lambda^{-1}] = 0.$$

As there are no further simplifications,  $H_*\underline{KQ}_5 = H_*(U/Sp) = E(e \circ z_{4i} \circ [\beta\lambda^{-1}])$ .

## 6.6 $H_*\underline{KQ}_6 = H_*(O/U)$

Since the connected portion of  $\underline{KQ}_6$  is  $SO/U$ , the bar spectral sequence produces

$$E_{*,*}^2 = \mathrm{Tor}_{*,*}^{H_*(U/Sp)}(\mathbf{Z}/2, \mathbf{Z}/2) = \mathrm{Tor}_{*,*}^{E(e \circ z_{4i} \circ [\beta\lambda^{-1}])}(\mathbf{Z}/2, \mathbf{Z}/2) = \\ \Gamma(\sigma(e \circ z_{4i} \circ [\beta\lambda^{-1}])) \Rightarrow H_*(SO/U).$$

We express  $\Gamma(\sigma(e \circ z_{4i} \circ [\beta\lambda^{-1}]))$  in its equivalent form  $E(\gamma_{2j}(\sigma(e \circ z_{4i} \circ [\beta\lambda^{-1}])))$ , and note that since every element is located in even total degree, the bar spectral sequence collapses at the  $E^2$ -term. Thus the  $E^\infty$ -term is given by  $E(\gamma_{2j}(\sigma(e \circ z_{4i} \circ [\beta\lambda^{-1}])))$ .

As in  $H_*\underline{KQ}_4$ , we simplify in stages, starting with elements of the first filtration,  $\sigma(e \circ z_{4i} \circ [\beta\lambda^{-1}])$ . As usual,  $e^{\circ 2} \circ z_{4i} \circ [\beta\lambda^{-1}]$  detects  $\sigma(e \circ z_{4i} \circ [\beta\lambda^{-1}])$ . The relation

$$e^{\circ 2} \circ [\beta] = \bar{z}_2 \circ [\eta^2] \tag{8}$$

allows us to equate

$$e^{\circ 2} \circ z_{4i} \circ [\beta\lambda^{-1}] = \bar{z}_2 \circ z_{4i} \circ [\eta^2\lambda^{-1}] = \bar{z}_{4i+2} \circ [\eta^2\lambda^{-1}],$$

modulo decomposable elements. Therefore, in the first filtration we have an exterior algebra with generators  $\bar{z}_{4i+2} \circ [\eta^2\lambda^{-1}]$ .

Next, we proceed to the simplification of elements in  $\gamma_{2j}$ . Let  $x$  detect the exterior algebra generator  $\gamma_{2j}(\sigma(e \circ z_{4i} \circ [\beta\lambda^{-1}]))$ . Then by proposition 6.2,

$$V^j x = e^{\circ 2} \circ z_{4i} \circ [\beta\lambda^{-1}] = \bar{z}_{4i+2} \circ [\eta^2\lambda^{-1}] + \text{decomposables}.$$

Thus, modulo decomposables,

$$x = \bar{z}_{2j(4i+2)} \circ [\eta^2\lambda^{-1}].$$

Since every even positive number can be uniquely written as  $2^j(4i+2)$ , we must have  $x = \bar{z}_{2i+2} \circ [\eta^2 \lambda^{-1}] + \text{decomposables}$ . Since

$$\bar{z}_{2i+1} \circ [\eta^2] = 0, \quad (9)$$

we have  $H_*(SO/U) = E(\bar{z}_{2i+2} \circ [\eta^2 \lambda^{-1}])$ .

$$\text{Thus } H_*\underline{KO}_6 = H_*(O/U) = H_*(SO/U) \otimes \mathbf{Z}[(\mathbf{Z}/2)\eta^2 \lambda^{-1}] = E(\bar{z}_{2i} \circ [\eta^2 \lambda^{-1}])$$

### Proofs of Relations for $H_*\underline{KO}_6$

*Relation (8):*  $e^{\circ 2} \circ [\beta] = \bar{z}_2 \circ [\eta^2]$ .

*Proof.* We use the forgetful map  $f : KU \rightarrow KO$ , again building on relation (7) to yield

$$e^{\circ 2} \circ [\beta] = f(e^{\circ 2} \circ [\nu] \circ [\nu]) = f(\bar{z}_2 \circ [\nu]) = \bar{z}_2 \circ [\eta^2].$$

□

*Relation (9):*  $\bar{z}_{2i+1} \circ [\eta^2] = 0$ .

*Proof.* By relation (6) and by the fact that  $\eta^3 = 0$  we have

$$\bar{z}_1 \circ [\eta^2] = e \circ [\eta] \circ [\eta^2] = e \circ [\eta^3] = e \circ [0] = 0.$$

The argument to show  $\bar{z}_{2i+1} \circ [\eta^2] = 0$  is much the same as in relation (4). □

## 6.7 $H_*\underline{KO}_7 = H_*(O)$

The connected portion of  $\underline{KO}_7$  is  $SO$ , so the bar spectral sequence gives

$$E_{*,*}^2 = \text{Tor}_{**}^{H_*(O/U)}(\mathbf{Z}/2, \mathbf{Z}/2) = \text{Tor}_{**}^{E(\bar{z}_{2i} \circ [\eta^2 \lambda^{-1}])}(\mathbf{Z}/2, \mathbf{Z}/2) =$$

$$\Gamma(\sigma(\bar{z}_{2i} \circ [\eta^2 \lambda^{-1}])) \Rightarrow H_*(SO)$$

We rewrite  $\Gamma(\sigma(\bar{z}_{2i} \circ [\eta^2 \lambda^{-1}]))$  as  $E(\gamma_{2^j}(\sigma(\bar{z}_{2i} \circ [\eta^2 \lambda^{-1}])))$ .

The elements  $\gamma_{2^j}(\sigma(\bar{z}_{2i} \circ [\eta^2 \lambda^{-1}])) \in E_{2^j, 2^j(2i)}^2$  have total degree  $2^j(2i+1)$ . The elements

$$\gamma_1(\sigma(\bar{z}_{2i} \circ [\eta^2 \lambda^{-1}])) = \sigma(\bar{z}_{2i} \circ [\eta^2 \lambda^{-1}])$$

must survive to  $E^\infty$ , since any differential which originates at one of these elements must end below the  $x$ -axis. The images of these elements in  $E^\infty$  represent the elements  $e \circ \bar{z}_{2i} \circ [\eta^2 \lambda^{-1}]$ .

We now compare our spectral sequence with the bar spectral sequence for  $\mathbf{Z}/2$ . Define the reflection map  $r' : B\mathbf{Z}/2 = \mathbf{R}P^\infty \rightarrow O$  as the map which sends a line  $L \in \mathbf{R}P^\infty$  to its reflection in the hyperplane orthogonal to  $L$ . If we fix a standard line  $L_0$ , we may define the map  $r : \mathbf{R}P^\infty \rightarrow SO$  by  $r(L) = r'(L)r'(L_0)$ . Using  $r$ , we obtain the map  $\Omega(r) : \mathbf{Z}/2 \rightarrow O/U$ .

We use  $r$  to map from the spectral sequence

$$\mathrm{Tor}^{\mathbf{Z}/2[\mathbf{Z}/2]}(\mathbf{Z}/2, \mathbf{Z}/2) \Rightarrow H_*\mathbf{R}P^\infty \quad (10)$$

into our spectral sequence for  $\underline{KO}_7$ . The  $E_{*,*}^2$  term of the spectral sequence in (10) is exterior on the elements  $\gamma_{2^j}(\sigma([\eta^2]))$ . Since there is no room for differentials, the elements  $\gamma_{2^j}(\sigma([\eta^2]))$  survive in (10), and therefore the elements  $\gamma_{2^j}(\sigma([\eta^2\lambda^{-1}]))$  also survive in our spectral sequence for  $\underline{KO}_7$ .

Using the familiar properties of the Verschiebung map, we will show that the elements  $\gamma_{2^j}(\sigma(\bar{z}_{2i} \circ [\eta^2\lambda^{-1}]))$  must therefore also survive. Since

$$\gamma_{2^j}(\sigma(\bar{z}_{2i} \circ [\eta^2\lambda^{-1}])) = \gamma_{2^j}(V^j(\bar{z}_{2^j(2i)} \circ \sigma([\eta^2\lambda^{-1}])) = \bar{z}_{2^j(2i)} \circ \gamma_{2^j}(\sigma([\eta^2\lambda^{-1}])),$$

we may use the fact that the differentials respect the  $\circ$ -product, so all of our elements must survive as claimed.

Thus the  $E^\infty$ -term is given by  $E(\gamma_{2^j}(\sigma(\bar{z}_{2i} \circ [\eta^2\lambda^{-1}])))$ .

As in  $H_*\underline{KO}_6$ , we simplify in steps. The element  $e \circ \bar{z}_{2i} \circ [\eta^2\lambda^{-1}] = e \circ z_{2i} \circ [\eta^2\lambda^{-1}]$  detects  $\sigma([\eta^2\lambda^{-1}] \circ \bar{z}_{2i})$ . Relation (6) allows us to rewrite

$$e \circ z_{2i} \circ [\eta^2\lambda^{-1}] = \bar{z}_1 \circ z_{2i} \circ [\eta\lambda^{-1}] = \bar{z}_{2i+1} \circ [\eta\lambda^{-1}],$$

modulo decomposable elements.

Next, we let  $x$  detect any exterior algebra generator  $\gamma_{2^j}(\sigma(z_{2i} \circ [\eta^2\lambda^{-1}]))$ . Utilization of proposition 6.2 shows

$$V^j x = e \circ z_{2i} \circ [\eta^2\lambda^{-1}] = \bar{z}_{2i+1} \circ [\eta\lambda^{-1}] + \text{decomposables.}$$

Hence, modulo decomposable elements,  $x = \bar{z}_{2^j(2i+1)} \circ [\eta\lambda^{-1}]$ . Since every positive number can be uniquely written as  $2^j(2i+1)$ , we must have  $x = \bar{z}_{i+1} \circ [\eta\lambda^{-1}] + \text{decomposables}$ . No further simplification is necessary, so  $H_*(SO) = E(\bar{z}_{i+1} \circ [\eta\lambda^{-1}])$ .

We have therefore found  $H_*\underline{KO}_7 = H_*(O) = H_*(SO) \otimes \mathbf{Z}[(\mathbf{Z}/2)\eta\lambda^{-1}] = E(\bar{z}_i \circ [\eta\lambda^{-1}])$ .

## 6.8 $H_*\underline{KO}_8 = H_*(\mathbf{Z} \times BO)$

We complete the cycle, reaching  $\underline{KO}_8 = \mathbf{Z} \times BO = \underline{KO}_0$ . The bar spectral sequence gives

$$\begin{aligned} E_{*,*}^2 &= \mathrm{Tor}_{*,*}^{H_*(O)}(\mathbf{Z}/2, \mathbf{Z}/2) = \mathrm{Tor}_{*,*}^{E(\bar{z}_i \circ [\eta\lambda^{-1}])}(\mathbf{Z}/2, \mathbf{Z}/2) = \\ &\Gamma(\sigma(\bar{z}_i \circ [\eta\lambda^{-1}])) \Rightarrow H_*(BO). \end{aligned}$$

We rewrite  $\Gamma(\sigma(\bar{z}_i \circ [\eta\lambda^{-1}]))$  in its equivalent form  $E(\gamma_{2^j}(\sigma(\bar{z}_i \circ [\eta\lambda^{-1}])))$ . The bar spectral sequence collapses at the  $E^2$ -term, by comparison with the collapsing bar spectral sequence

$$\mathrm{Tor}_{*,*}^{H_*(O(1))}(\mathbf{Z}/2, \mathbf{Z}/2) = \mathbf{Z}/2 \oplus \mathbf{Z}/2\{\gamma_{k+1}(\overline{[\eta]})\}$$

$$\Rightarrow H_*(BO(1)) = H_*\mathbf{R}P^\infty = \mathbf{Z}/2 \oplus \mathbf{Z}/2\{\bar{z}_{k+1}\}.$$

The differentials for the original spectral sequence are therefore given by

$$d_r(\gamma_k[\bar{\eta}] \circ z(t)) = d_r(\gamma_k[\bar{\eta}]) \circ z(t) = 0,$$

where  $z(t) = \sum_{i \geq 0} z_i t^i$ .

The element  $e \circ \bar{z}_i \circ [\eta\lambda^{-1}]$  detects  $\sigma(\bar{z}_i \circ [\eta\lambda^{-1}])$ . Use of relation (6) allows us to rewrite  $e \circ \bar{z}_i \circ [\eta\lambda^{-1}]$  as  $\bar{z}_1 \circ \bar{z}_i \circ [\lambda^{-1}]$ .

As usual, we start with elements in the first filtration. Using the relation

$$\bar{z}_1 \circ \bar{z}_{2i+1} = \bar{z}_1^2 \circ \bar{z}_{2i}. \quad (11)$$

gives

$$\bar{z}_1 \circ \bar{z}_{2i+1} \circ [\lambda^{-1}] = \bar{z}_1^2 \circ \bar{z}_{2i} \circ [\lambda^{-1}] = F(\bar{z}_1) \circ \bar{z}_{2i} \circ [\lambda^{-1}] = F(\bar{z}_1 \circ \bar{z}_i \circ [\lambda^{-1}]).$$

We apply this calculation as often as possible, by expanding each integer in its binary form,  $m = 2^q(2i+1) - 1$ . Then

$$\bar{z}_1 \circ \bar{z}_m \circ [\lambda^{-1}] = F^q(\bar{z}_1 \circ \bar{z}_{2i} \circ [\lambda^{-1}]) = F^q(\bar{z}_{2i+1} \circ [\lambda^{-1}]) + \text{decomposables}.$$

Therefore in the first filtration  $\gamma_1$ , we obtain a polynomial algebra with generators  $\bar{z}_{2i+1} \circ [\lambda^{-1}]$ .

We now look to simplify the remaining elements in  $\gamma_{2^j}$ . By the preceding argument, we have a polynomial algebra with generators given by  $x = \gamma_{2^j}(\bar{z}_{2i+1} \circ [\lambda^{-1}])$ . Proposition 6.2 proves that since  $V^j(x) = \bar{z}_{2i+1} \circ [\lambda^{-1}]$ ,  $x$  may be expressed as  $\bar{z}_{2^j(2i+1)} \circ [\lambda^{-1}]$ , mod decomposables. Since every positive integer can be uniquely expressed in the form  $2^j(2i+1)$ , we have  $H_*(BO) = P(\bar{z}_{i+1} \circ [\lambda^{-1}])$ .

Thus  $H_*\underline{KQ}_8 = H_*(\mathbf{Z} \times BO) = P([\lambda^{-1}][-\lambda^{-1}]) \otimes P(\bar{z}_{i+1} \circ [\lambda^{-1}]) = P(\bar{z}_i \circ [\lambda^{-1}], [-\lambda^{-1}])$ .

Since our answer for  $H_*\underline{KQ}_8$  is equivalent to our answer for  $H_*\underline{KQ}_0$ , the Bott-periodic circle from  $\underline{KQ}_0$  to  $\underline{KQ}_8$  is complete.

### Proofs of Relations for $H_*\underline{KQ}_8$

*Relation (11):*  $\bar{z}_1 \circ \bar{z}_{2i+1} = \bar{z}_1^2 \circ \bar{z}_{2i}$ .

*Proof.* We first find the coproduct for  $\bar{z}_1$ ;

$$\begin{aligned} \psi(\bar{z}_1) &= \psi(z_1 * [-1]) = \psi(z_1) * \psi([-1]) \\ &= ((z_1 \otimes z_0) + (z_0 \otimes z_1)) * ([-1] \otimes [-1]) \\ &= \bar{z}_1 \otimes 1 + 1 \otimes \bar{z}_1. \end{aligned}$$

Thus  $\bar{z}_1 \circ \text{decomposables} = 0$ , as  $\bar{z}_1 \circ (a * b) = (\bar{z}_1 \circ a) * (1 \circ b) + (1 \circ a) * (\bar{z}_1 \circ b) = 0$ .

For any  $i > 0$  we may use distributivity to simplify

$$\bar{z}_1 \circ \bar{z}_i = \bar{z}_1 \circ (z_i * [-1]) = (\bar{z}_1 \circ z_i) * (1 \circ [-1]) + (1 \circ z_i) * (\bar{z}_1 \circ [-1]) = \bar{z}_1 \circ z_i + 0.$$

Thus, for all  $i > 0$ ,  $\bar{z}_1 \circ \bar{z}_i = \bar{z}_1 \circ z_i$ . In particular,  $\bar{z}_1 \circ \bar{z}_{2i} = \bar{z}_1 \circ z_{2i}$ . Next,

$$\begin{aligned}\bar{z}_1 \circ \bar{z}_{2i} &= \bar{z}_1 \circ z_{2i} = (z_1 * [-1]) \circ z_{2i} \\ &= (z_1 \circ z_{2i}) * ([-1] \circ z_0) + \text{decomposables} \\ &= z_{2i+1} * [-1] + \text{decomposables} \\ &= \bar{z}_{2i+1} + \text{decomposables}.\end{aligned}$$

Since  $\bar{z}_1 \circ \bar{z}_{2i} = \bar{z}_{2i+1} + \text{decomposables}$ , and since  $\bar{z}_1 \circ \text{decomposables} = 0$ , we now have

$$\bar{z}_1 \circ \bar{z}_{2i+1} = \bar{z}_1 \circ \bar{z}_1 \circ \bar{z}_{2i} + \bar{z}_1 \circ \text{decomposables} = \bar{z}_1^{\circ 2} \circ \bar{z}_{2i}.$$

To finish, we need only prove  $\bar{z}_1^{\circ 2} = \bar{z}_1^2$ . We have already established that  $\bar{z}_1 \circ \bar{z}_i = \bar{z}_1 \circ z_i$  for all  $i > 0$ , and so  $\bar{z}_1 \circ \bar{z}_1 = \bar{z}_1 \circ z_1$ . Using distributivity and the  $\circ$ -product for the  $z_i$  shows

$$\begin{aligned}\bar{z}_1 \circ \bar{z}_1 &= \bar{z}_1 \circ z_1 = (z_1 * [-1]) \circ z_1 \\ &= (z_1 \circ z_1) * ([-1] \circ z_0) + (z_1 \circ z_0) * ([-1] \circ z_1) \\ &= 0 + z_1 * ([-1] \circ z_1) \\ &= z_1 * (\chi z_1).\end{aligned}$$

To find  $\chi z_1$ , we calculate  $\chi z_0 = \chi[1] = [-1] \circ [1] = [-1]$ . Next, we use the Hopf ring property that  $\sum a' * \chi a'' = \varepsilon a$ . Since  $\varepsilon z_i = \delta_{i0}$ , we have

$$z_1 * \chi z_0 + z_0 * \chi z_1 = \varepsilon z_1 = 0.$$

Thus

$$z_1 * [-1] + [1] * \chi z_1 = 0.$$

We may now solve for

$$\chi z_1 = z_1 * [-2].$$

By virtue of these facts, we have obtained  $\bar{z}_1 \circ \bar{z}_1 = z_1 * (z_1 * [-2]) = \bar{z}_1^2$ , completing our proof.  $\square$

## 7 Properties of $H_*\underline{KU}_0 = H_*(\mathbf{Z} \times BU)$

In this section, we record the known mod 2 homology for  $\underline{KU}_0$  and introduce Hopf ring properties for the elements in homology. We will compute  $H_*\underline{KU}_0 = H_*(\mathbf{Z} \times BU) \cong H_*(\mathbf{Z}) \otimes H_*(BU)$ , where  $H_*(\mathbf{Z})$  is concentrated in deg 0 and  $\mathbf{Z}$  is the set of integers with the discrete topology.

To understand  $H_*(BU)$ , we examine the map

$$\mathbf{R}P^\infty \rightarrow \mathbf{C}P^\infty \rightarrow 1 \times BU \subset \mathbf{Z} \times BU = \underline{KU}_0.$$

Recall that the homology of  $\mathbf{R}P^\infty$  is spanned by generators  $b_i$ . We write  $b_{2i}$  again for the image of  $b_{2i}$  in  $H_*\mathbf{C}P^\infty$  (the image of  $b_{2i+1}$  is zero). The elements  $b_{2i}$  form a basis for  $H_*\mathbf{C}P^\infty$ . We write  $z_{2i}$  for the image of  $b_{2i}$  in  $H_{2i}\underline{KU}_0$ , which is also the image of our previous element  $z_{2i}$  under the map  $m : \underline{KQ}_0 \rightarrow \underline{KU}_0$ . We also write  $\bar{z}_{2i} = z_{2i}/z_0 = z_{2i} * [-1] \in H_{2i}(0 \times BU)$ . The classical result is that

$$H_*(0 \times BU) = P(\bar{z}_{2i} : i > 0),$$

and it follows that  $H_*(\mathbf{Z} \times BU) = P(z_{2i}, [-1])$ .

As the map  $m_* : H_*\underline{KO}_* \rightarrow H_*\underline{KU}_*$  preserves Hopf ring structures, the elements  $z_{2i}$  in  $H_*\underline{KU}_*$  have the same Hopf ring properties as they did in  $H_*\underline{KO}_*$ .

## 8 The Calculation of the Hopf Ring for $H_*\underline{KU}_*$

### 8.1 $H_*\underline{KU}_1 = H_*(U)$

We input  $H_*\underline{KU}_0 = H_*(\mathbf{Z} \times BU) = P([1], [-1]) \otimes P(z_{2i} : i > 0)$  into the bar spectral sequence.

$$\begin{aligned} E_{*,*}^2 &= \text{Tor}_{*,*}^{H_*(\mathbf{Z} \times BU)}(\mathbf{Z}/2, \mathbf{Z}/2) = E(\sigma(\bar{z}_0)) \otimes E(\sigma(\bar{z}_{2i} : i > 0)) = E(\sigma(\bar{z}_{2i})) \\ &\Rightarrow H_*\underline{KU}_1 = H_*(U). \end{aligned}$$

Since the elements  $\sigma(\bar{z}_{2i})$  are all in the first filtration, the bar spectral sequence collapses at the  $E^2$ -term, and the  $E^\infty$ -term is  $E(\sigma(\bar{z}_{2i}))$ .

The element  $e \circ \bar{z}_{2i}$  detects  $\sigma(\bar{z}_{2i})$ . From  $H_*\underline{KO}_*$  we have  $e^2 = e \circ z_1$  and  $z_1$  maps to 0 in  $H_*\underline{KU}_*$  so  $F(e) = e^2 = 0$  in  $H_*\underline{KU}_*$ . Thus

$$F(e \circ \bar{z}_{2i}) = F(e \circ V(\bar{z}_{4i})) = F(e) \circ \bar{z}_{4i} = 0.$$

This solves the extension problem and shows that  $H_*(U) = E(e \circ z_{2i})$  (which is of course well-known by other methods).

### 8.2 $H_*\underline{KU}_2 = H_*(\mathbf{Z} \times BU)$

To finish the cycle of 2 spaces in  $H_*\underline{KU}_*$ , we examine  $H_*\underline{KU}_2$ .

The bar spectral sequence gives

$$E_{*,*}^2 = \text{Tor}_{*,*}^{H_*(U)}(\mathbf{Z}/2, \mathbf{Z}/2) = \Gamma(\sigma(e \circ \bar{z}_{2i})) \Rightarrow H_*(BU),$$

since the connected portion of  $\underline{KU}_2$  is  $BU$ . The bar spectral sequence collapses at the  $E^2$ -term, as each element has even total degree. The  $E^\infty$ -term is therefore given by  $\Gamma(\sigma(e \circ \bar{z}_{2i}))$ , or equivalently by  $E(\gamma_{2i}(\sigma(e \circ \bar{z}_{2i})))$ .

We map down from  $\underline{KU}_2$  to  $\underline{KU}_0$  via the map  $\circ[\nu]$ . The elements  $\gamma_{2i}(\sigma(e \circ \bar{z}_{2i}))$  are detected by  $\gamma_{2i}(e^{\circ 2} \circ \bar{z}_{2i})$ . Applying the map  $\circ[\nu]$  to these elements and using relation (7) yields

$$\gamma_{2i}(e^{\circ 2} \circ \bar{z}_{2i} \circ [\nu]) = \gamma_{2i}(\bar{z}_{2i} \circ \bar{z}_2).$$

A similar argument as in the proof of  $H_*\underline{KO}_8$  shows that these elements form the polynomial algebra  $P(\bar{z}_{2i+2})$ . Mapping back to  $\underline{KU}_2$  via the map  $[\nu^{-1}]$  yields  $P(\bar{z}_{2i+2} \circ [\nu^{-1}]) = H_*(BU) \subset \underline{KU}_2$ . Thus

$$\begin{aligned} H_*\underline{KU}_2 &= H_*(\mathbf{Z} \times BU) = P([\nu^{-1}], [-\nu^{-1}]) \otimes P(\bar{z}_{2i+2} \circ [\nu^{-1}]) \\ &= P(\bar{z}_{2i} \circ [\nu^{-1}], [-\nu^{-1}]), \end{aligned}$$

completing our proof.

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