There are two forms of the test; both are included below.

1. [20 points] Find the domain and range of the function \( f(x) = \log(25 - x^2) \).

   Sketch a graph of the curve below (and label it appropriately) to help you identify the range.

   Since the logarithm is only defined for positive inputs, we must require that
   \[
   25 - x^2 > 0 \\
   25 > x^2 \\
   -5 > x > 5
   \]

   We see this clearly in the graph also. The domain is therefore \(-5 > x > 5\). The range is seen in the graph to include \( y \) values smaller than the one that occurs at \( x = 0 \); here, \( y = \log(25) \approx 1.3979 \ldots \) So the range is \( y \leq \log(25) \).
2. [20 points] Solve exactly for \( x \): \( e^{7x+5} = 8 \cdot 3^x \).

Taking natural logarithms,

\[
\ln e^{7x+5} = \ln (8 \cdot 3^x)
\]

\[
7x + 5 = \ln 8 + x \ln 3
\]

\[
x(7 - \ln 3) = \ln 8 - 5
\]

\[
x = \frac{\ln 8 - 5}{7 - \ln 3}
\]

3. (a) [10 points] The annual sales for a particular tire manufacturing company was $229,800 in 1990 and $383,800 in 1995. Assuming exponential growth, give a formula for \( S(t) \), the company’s sales \( t \) years after 1990.

Since the function is exponential, we know that \( S(t) = ab^t \) for certain values of \( a \) and \( b \).

The information given allows us to use the following input/output pairs: (0, 229800) and (5, 383800). Plugging the first pair into the equation gives \( 229800 = ab^0 = a \). (Or just use the fact that \( a \) is the initial output value.) Using the second pair,

\[
383800 = 229800 b^5
\]

\[
(383800/229800) = b^5
\]

\[
b = \sqrt[5]{\frac{383800}{229800}} = 1.10803
\]

So our formula is \( S(t) = 229800(1.10803)^t \).

(b) [5 points] According to your formula model, what were the company’s sales in 2000?

In 2000, we have \( t = 10 \). So sales in 2000 are \( S(10) = $641,003 \).

(c) [5 points] According to the model, how long will it take for sales to double? Does this answer depend on the year you begin counting? Explain.

The answer does not depend on when you start counting: no matter what your sales level starts at, the doubling time is the same. If the sales level is \( y \) at some time \( x \), then \( y = 229800(1.10803)^x \), and the doubling time is found by solving for \( t \) in the equation

\[
2 = \frac{1.10803^t}{1.10803^x}
\]

\[
\log 2 = (t - x) \log 1.10803
\]
\[
t - x = \frac{\log 2}{\log 1.10803} = 6.76
\]
\[
t = x + 6.76 \text{ yrs}
\]
So the doubling time is always 6.76 yrs (it takes 6.76 yrs after whenever you start for the sales to double).

4. [20 points] The half-life of the radioactive isotope carbon-14 is 5,730 years. How old is a piece of human bone which contains just 43% of the amount of carbon-14 expected in a sample of bone from a living person?

Radioactive decay follows the exponential law \( Q = Q_0 e^{kt} \) where \( Q \) is the quantity at time \( t \), \( Q_0 \) is the quantity one starts with, and \( k \) is the decay constant. Here, since the half-life is 5730 yrs, we know that at time \( t = 5730 \), we have \( Q = 0.5Q_0 \):

\[
0.5Q_0 = Q_0 e^{5730k}
\]
\[
0.5 = e^{5730k}
\]
\[
\ln 0.5 = 5730k
\]
\[
k = \frac{\ln 0.5}{5730} = -0.000120968
\]
So the formula is now \( Q = Q_0 e^{-0.0001210968t} \). The bone fragment has 43% of the initial amount of carbon-14, so \( Q = 0.43Q_0 \). Thus,

\[
0.43Q_0 = Q_0 e^{-0.0001210968t}
\]
\[
0.43 = e^{-0.0001210968t}
\]
\[
\ln 0.43 = -0.0001210968t
\]
\[
t = \frac{\ln 0.43}{-0.0001210968} = 6977
\]
The bone is 6977 yrs old.

5. Simplify the following expressions by using properties of logarithms.

(a) [10 points] \( 1000^{\log x} \)

\[
1000^{\log x} = 10^3 \log x = 10^{\log (x^3)} = x^3
\]

(b) [10 points] \( \sqrt{\log x^4} - \log \sqrt{x^4} \)

\[
\sqrt{\log x^4} - \log \sqrt{x^4} = \sqrt{4 \log x} - \log x^2 = 2\sqrt{\log x} - 2 \log x
\]
This is the second form of the test.

1. [20 points] Find the domain and range of the function \( f(x) = \log \left( \frac{x + 1}{x - 1} \right) \).

   Sketch a graph of the curve below (and label it appropriately) to help you identify the range.

Since the logarithm is only defined for positive inputs, we must require that the expression in the parentheses be positive. That is, both \( x + 1 \) and \( x - 1 \) must have the same sign. But \( x + 1 \) is always larger than \( x - 1 \), so for both to be positive, \( x - 1 \) must be positive: so \( x > 1 \). Also, for both to be negative, \( x + 1 \) must be negative: so \( x < -1 \). We see this clearly in the graph also. The domain is therefore \( x < -1 \) and \( x > 1 \). The range is seen in the graph to include only all nonzero \( y \) values. So the range is \( y \neq 0 \).

2. [20 points] Solve exactly for \( x \): \( 9e^{3x+1} = 8e^{x-3} \).

   \[
   \begin{align*}
   \frac{e^{3x+1}}{e^{x-3}} &= \frac{8}{9} \\
   e^{2x+4} &= \frac{8}{9} \\
   2x + 4 &= \ln(\frac{8}{9}) \\
   2x &= \ln(\frac{8}{9}) - 4 \\
   x &= \frac{\ln(\frac{8}{9}) - 4}{2}
   \end{align*}
   \]
3. (a) [10 points] In 1776, a wealthy Pennsylvania merchant named Jacob DeHaven loaned $450,000 to the Continental Congress to rescue the troops at Valley Forge. In 1990, descendants of DeHaven took the U.S. government to court to sue for what they believed they were owed. Since the interest rate in effect in 1776 was 6% compounded annually, how much did the family stand to collect in 1991?

We use \( A = P \left(1 + \frac{r}{n}\right)^{nt} \) with \( P = 450000, \ r = 0.06, \ n = 1, \ t = 1991 - 1776 = 215 \). This gives \( A = 1.241576677E11 = \$124,157,667,700 \). (That's 124 BILLION dollars!)

(b) [10 points] In 2000, General Electric CEO Jack Welch earned $16.7 million in salary and bonuses. Meanwhile, the median family income across the U.S. was $42,148. Suppose you were take this one year's worth of income for the average U.S. worker and invest it at a healthy annual interest rate of 10% compounded continuously. How many years would it take before you earned enough to equal Jack Welch's income in this one year?

We use \( A = Pe^{rt} \) for continuous compounding with \( P = 42148, \ r = 0.10, \ A = 16700000 \):

\[
\frac{16700000}{42148} = e^{0.10t} \\
\ln \left( \frac{16700000}{42148} \right) = 0.10t \\
t = \frac{1}{0.10} \ln \left( \frac{16700000}{42148} \right) = 59.8
\]

It will take almost 60 years of investment for one year's median salary to grow into one year of Jack Welch's salary!
4. (a) [15 points] Which is the better investment, one that offers 14.25% annual interest compounded quarterly, or one that offers 14% annual interest compounded continuously? Suppose you had $10,000 to invest.

Investing $10,000 at 14.25% compounded quarterly gives an amount

\[ A = 10000\left(1 + \frac{.1425}{4}\right)^{4t} = 10000(1.150297)^t \]

while at 14% compounded continuously we get

\[ A = 10000e^{0.14t} = 10000(1.150274)^t. \]

The growth factor for the first investment is slightly larger, so it gives the better investment.

(b) [5 points] Would it make a difference if you had only $1,000 to invest? Explain.

In the formulas in part (a), we just replace 10000 with 1000. This does not affect the answer at all.

5. Simplify the following expressions by using properties of logarithms.

(a) [10 points] \(100^{3\log x}\)

\[ 100^{3\log x} = 10^{2 \cdot 3\log x} = 10^6 \log x = 10^{\log(x^6)} = x^6 \]

(b) [10 points] \(e^{\ln(3x - 7)}\)

\[ e^{\ln(3x - 7)} = 3x - 7 \]