Form A

1. The Cincinnati Enquirer (January 26, 2000) reported the following data from the Ohio Department of Development on the size of the state’s exports to Japan (in billions of dollars).

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</tr>
</thead>
<tbody>
<tr>
<td>Exports</td>
<td>1.8</td>
<td>2.3</td>
<td>2.4</td>
<td>2.6</td>
<td>2.1</td>
<td>1.4</td>
</tr>
</tbody>
</table>

(a) [10 points] Using a symmetric difference quotient, estimate the rate at which exports from Ohio to Japan were changing in 1997.

Using the data points for the years 1996 and 1998, we get \( \frac{1.4 - 2.3}{1998 - 1996} = -0.6 \text{ billion per year} = -600 \text{ million per year} \) (rate of decline).

(b) [10 points] Write down the cubic regression model for the function \( E(t) \) that tracks the exports (in billions of dollars) as a function of time \( t = \) number of years since 1990. Round all coefficients to two decimal places. Then use derivative rules to compute the derivative function \( E'(t) \).

The regression model is \( E(t) = -0.01t^3 + 0.06t^2 + 0.47t + 0.17 \). The derivative is \( E'(t) = -0.03t^2 + 0.12t + 0.47 \).

(c) [15 points] Compute \( E'(7) \) from your formula in part (b) and interpret this value in the context of the problem.

Since \( E'(7) = -0.16 \), we infer that in 1997, Ohio exports to Japan were falling at a rate of $160 million per year.

(d) [5 points] What is the percentage rate of change in the size of exports to Japan in 1997?

The percentage rate of change is \( \frac{E'(7)}{E(7)} = \frac{-0.16}{2.10} = -0.076 = -7.6\% \text{ per year} \).

2. Suppose that \( C(t) = 42.5(0.989)^t \) represents a model for the total enrollment (in thousands of students) in the Cincinnati public school district for the coming decade: \( t = \) years since 2000.

(a) [15 points] Determine the slopes of the secant lines through the graph of \( C(t) \) at \( t = 2 \) and \( t = 2 + h \) for the values of \( h \) in the table below. Record your answers in the table to three decimal place accuracy. What conclusion can you make from this about the enrollment of students in Cincinnati Public Schools?
In 2002, enrollment in Cincinnati Public Schools is falling at a rate of 460 students per year (=0.460 thousand per year).

(b) [15 points] Use derivative rules to determine $C'(t)$ and use this to calculate $C'(2)$. Does this confirm your work in part (a)?

$$C'(t) = 42.5(\ln 0.989)(0.989^t) = -0.47(0.989^t) \text{ so } C'(2) = -0.4598. \text{ This does confirm our work.}$$

3. [15 points] Compute the derivative of the function $f(x) = 3x^2$ by finding the limiting value as $h$ approaches 0 of the slopes of secant lines through the graph of the function at $x$ and at the nearby point $x+h$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{(x+h) - x}$$

$$= \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) - 3x^2}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 3x^2}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2}{h}$$

$$= \lim_{h \to 0} 6x + 3h$$

$$= 6x$$

4. [15 points] Differentiate the function $F(x) = (x^2 + x + 1)e^x$.

By the product rule, $F'(t) = (2x + 1)e^x + (x^2 + x + 1)e^x = (x^2 + 3x + 2)e^x$. 
Form B

1. Comair, a regional airline based in Cincinnati, steadily expanded the size of its fleet of airplanes during the 1990s. The data below from the Cincinnati Enquirer (October 2, 1998) indicates the number of planes in its fleet (the last two figures were estimates made in 1998).

<table>
<thead>
<tr>
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<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Fleet</td>
<td>61</td>
<td>69</td>
<td>75</td>
<td>84</td>
<td>88</td>
<td>92</td>
<td>93</td>
<td>97</td>
<td>105</td>
</tr>
</tbody>
</table>

(a) [10 points] Using a symmetric difference quotient, estimate the rate at which the size of the fleet was changing in 1997.

Using the data points for the years 1996 and 1998, we get \((93 - 88)/(1998 - 1996) = 2.5\) planes per year.

(b) [10 points] Write down the cubic regression model for the function \(F(t)\) that tracks the size of the Comair fleet as a function of time: \(t = \text{number of years since 1990}.\) Round all coefficients to one decimal place. Then use derivative rules to compute the derivative function \(F'(t)\).

The regression model is \(F(t) = 0.1t^3 - 2.0t^2 + 18.0t + 31.5\), so \(F'(t) = 0.3t^2 - 4.0t + 18.0\).

(c) [15 points] Compute \(F'(7)\) from your formula in part (b) and interpret this value in the context of the problem.

\(F'(7) = 4.7\), so we infer that in 1997, the fleet grew at a rate of 4.7 planes per year.

(d) [5 points] What is the percentage rate of change in the size of the Comair fleet in 1997?

The percentage rate of change was \(F'(7)/F(7) = 4.7/92 = 0.051 = 5.1\%\) per year.
2. Suppose that $M(t) = 12.9(1.0269)^t$ represents a model for the population of Mexico (in millions of persons) where $t =$ years since 1920.

(a) [15 points] Determine the slopes of the secant lines through the graph of $M(t)$ at $t = 40$ and $t = 40 + h$ for the values of $h$ in the table below. Record your answers in the table to three decimal place accuracy. What conclusion can you make from this about the population of Mexico?

<table>
<thead>
<tr>
<th>$h$</th>
<th>slope of secant line</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>1.003</td>
</tr>
<tr>
<td>0.1</td>
<td>0.991</td>
</tr>
<tr>
<td>0.01</td>
<td>0.990</td>
</tr>
<tr>
<td>0.001</td>
<td>0.990</td>
</tr>
<tr>
<td>0.0001</td>
<td>0.990</td>
</tr>
</tbody>
</table>

Mexico's population grew at a rate of (0.990 million people per year) = 990,000 per year in 1960.

(b) [15 points] Use derivative rules to determine $M'(t)$ and use this to calculate $M'(40)$. Does this confirm your work in part (a)?

$M'(t) = 12.9(\ln 1.0269)(1.0269^t) = 0.3424(1.0269^t)$, so $M'(40) = 0.990$. This does provide confirmation of our calculations above.

3. [15 points] Compute the derivative of the function $f(x) = 4x^2$ by finding the limiting value as $h$ approaches 0 of the slopes of secant lines through the graph of the function at $x$ and at the nearby point $x + h$. 
\[ f'(x) = \lim_{h \to 0} \frac{4(x+h)^2 - 4x^2}{(x+h) - x} \]
\[ = \lim_{h \to 0} \frac{4(x^2 + 2xh + h^2) - 4x^2}{h} \]
\[ = \lim_{h \to 0} \frac{4x^2 + 8xh + 4h^2 - 4x^2}{h} \]
\[ = \lim_{h \to 0} \frac{8xh + 4h^2}{h} \]
\[ = \lim_{h \to 0} 8x + 4h \]
\[ = 4x \]

4. [15 points] Differentiate the logistic function \( F(x) = \frac{49}{1 + 36.07e^{-0.207x}} \).
\[
F'(x) = \frac{(36.07)(0.207)(49)e^{0.207x}}{(1 + 36.07e^{0.207x})^2} = \frac{365.858e^{0.207x}}{(1 + 36.07e^{0.207x})^2}
\]